A Macro-Finance Model of Credit Spreads*

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Abstract

We argue that fluctuations in corporate credit spreads arise from the time-varying risk-bearing capacity of financial intermediaries. As motivating evidence, we show that deteriorations in primary dealers' balance sheets coincide with increases in the excess bond premium, the part of spreads that firm-level default risk leaves unexplained. The premium in turn predicts lower investment growth, even after controlling for dealer leverage and default conditions. We then build a continuous-time heterogeneous-agent macro-finance model in which intermediaries face financing constraints and can default on their debt. Three results obtain. (i) The model reproduces the premium, and with it the credit spread puzzle: the premium arises in recessions, when intermediaries are undercapitalized. (ii) Default is non-neutral: it acts as a transfer to intermediaries that rebuilds their capital and expands credit. (iii) When intermediary capital is scarce, bankruptcy law should be lenient and leverage regulation tight.

Keywords: General Equilibrium, Incomplete Markets, Financial Frictions, Endogenous Default, Excess Bond Premium.

JEL Classification: E13, E32, E44, E52, G01, G12, G20

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1 Introduction

Credit spreads are a central gauge of financial conditions, because they price the terms on which the financial sector lends to firms. Yet firm-level default risk explains only part of them, a pattern known as the credit spread puzzle (Collin-Dufresne et al., 2001). In the autumn of 2008, and again in March 2020, the spread between corporate bonds and Treasuries widened by more than the change in the issuers' measured default risk accounts for. Gilchrist and Zakrajšek (2012) isolate this part of the spread, the excess bond premium, as what remains after firm-level default risk has been netted out. It rises in financial distress and it predicts subsequent declines in activity. In this paper, we provide a microfoundation for the premium: it is the price of the risk-bearing capacity of financial intermediaries, charged when their own capital is scarce.

As motivating evidence, we relate intermediary balance sheets to the premium and the premium to capital formation. Using the capital ratio of primary dealers from He et al. (2017), we find that a one-standard-deviation deterioration in the capital ratio is associated with a 9.52-basis-point increase in the premium, after controlling for changes in the default-related component of spreads. The premium also rises with default conditions nonlinearly, the two increasing together in the distressed part of the sample. And it predicts investment: in quarterly regressions that control for leverage and default conditions, 25 basis points higher EBP predicts 1.47 percentage points lower investment growth over the following year. Both results survive excluding the pandemic or the financial-crisis recession. The evidence documents comovement and predictive content rather than causal effects; the model supplies the mechanism.

Our theoretical contribution is to embed default in the sense of Dubey et al. (2005) in a continuous-time macro-finance economy in the tradition of Brunnermeier and Sannikov (2014). Productive intermediaries, whom we call experts, finance capital with debt issued to households and face a leverage ceiling. At each instant they choose what fraction of their obligations to deliver, trading the funds they retain against a default penalty whose harshness the bankruptcy law sets. Default is then a continuous choice that lenders anticipate and price, so repayment, the spread and the allocation of capital are set by the same optimality conditions.

The model provides the microfoundation through the allocation of aggregate risk. Its single state variable is the intermediaries' share of wealth. When intermediaries are well capitalized, they hold all the capital, and households, who only lend, bear no aggregate risk. When intermediaries are undercapitalized, households must operate part of the capital, and the compensation they require for its aggregate risk is the excess bond premium.

The premium is therefore zero when intermediaries hold all the capital and rises with the volatility of the price of capital; the remainder of the spread is the default-related component. Default, in turn, is not neutral, because lenders price it. Wherever repayment is partial, an intermediary withholds a fixed amount per unit of its net worth, set by the harshness of the penalty, and lenders lend until the loss this implies per unit lent equals the gap between the return on capital in intermediaries' hands and the return households require. A more lenient penalty therefore raises credit, leverage and the share of capital in productive hands.

Quantitatively, the calibrated model reproduces the premium where intermediaries are undercapitalized, 70 basis points against 80 in the data, and its absence where they are not. As in the data, almost all of the difference in spreads between recessions and expansions comes from the premium: 70 of 74 basis points in the model, and about 84 of 90 in the data.

Normatively, we compare the default penalty and the margin requirement in the welfare space of Dubey et al. (2005), keeping every initial condition rather than averaging over them. The two instruments differ in kind. The margin requirement matters only when intermediaries are poor: the welfare-maximizing leverage ceiling is about eight when they own a few percent of wealth, against the calibrated thirty, loosens toward ten as they recover, and has no effect once they own about a fifth of wealth. A tight ceiling helps because it limits how much of a scarce stock of inside equity is levered into a volatile asset. The default penalty, by contrast, is best for households at its most lenient admissible value, at every initial condition. Default is a transfer from lenders to the productive sector that lenders price into the contractual rate, so a lenient penalty acts as a standing recapitalization of intermediaries: households pay for it through repayment and gain from it through output and the price of capital. When intermediary capital is scarce, bankruptcy law should therefore be lenient and leverage regulation tight.

Related literature. The paper joins two literatures that have developed apart. The first studies why credit spreads exceed expected default losses. Collin-Dufresne et al. (2001) show that the variables of structural default models explain only a small part of the variation in spreads, and Longstaff et al. (2005) separate the default and non-default components. Gilchrist et al. (2009) and Gilchrist and Zakrajšek (2012) show that spreads and their non-default component predict activity, and they attribute the premium to the risk-bearing capacity of the financial sector without modeling it. The second literature models that capacity: Adrian et al. (2014) and He et al. (2017) price assets with the capital of intermediaries, and Brunnermeier and Pedersen (2009) and Geanakoplos (2010) tie risk-bearing capacity to funding constraints and leverage. The continuous-time economies

of Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2012, 2013) place that capital in general equilibrium, where its dynamics determine the price of risk, and Romanidis (2019) develops the version of that environment on which we build. In these models, however, debt is honored in full.

Default with a penalty on the amount defaulted comes from Dubey et al. (2005), Goodhart et al. (2006), Shubik and Wilson (1977), Tsomocos (2003), and Zame (1993), shows that default improves efficiency when markets are incomplete; none of these has the dynamics of intermediary capital. We bring default of this kind into an economy where intermediary capital is scarce, so that the default penalty becomes an instrument alongside the margin requirement. On the normative side we are closest to Basak and Cuoco (1998) on welfare under limited participation, to Elenev et al. (2021) on macroprudential policy with financially constrained intermediaries, and to Goodhart et al. (2018) and Peiris et al. (2025) on the interaction of debt contracts and capital regulation. The recapitalization channel behind our result on the default penalty is the one through which an equity infusion is the most effective crisis policy in He and Krishnamurthy (2013), and it is related to the ex post case for bailouts in Bianchi (2016) and to the time-inconsistency problem it creates in Chari and Kehoe (2016). Gourio (2013) attributes the behavior of spreads to disaster risk; our source is intermediary wealth and the allocation of aggregate risk, and the allocation channel is the one Moll (2014) studies under collateral constraints.

Outline. Section 2 presents the motivating empirical evidence. Sections 3 and 4 set out the model and characterize its equilibrium. Section 5 calibrates the model, compares it with the data, and analyzes the default penalty and the margin requirement. Section 6 concludes.

2 Motivating Empirical Evidence

This section presents the empirical evidence that motivates our model. It documents three relationships, which the model will connect. First, deteriorations in intermediary capital coincide with a higher excess bond premium, even after accounting for changes in the default-related component of spreads. Second, the premium and default conditions are related nonlinearly, rising together in the distressed part of the sample. Third, the premium predicts lower investment after conditioning on both intermediary leverage and default conditions. Together, these facts link the condition of intermediaries to the pricing of credit, and the pricing of credit to capital formation. The evidence does not identify causal effects; the model supplies the mechanism that connects these facts.

Data and empirical approach. Gilchrist and Zakrajšek (2012) decompose corporate credit spreads into a component explained by firm-level default risk and an excess bond premium (EBP). We denote the default-related component by $d_t = \text{GZ}_t - \text{EBP}_t$. It is the part of the spread that their default-risk model accounts for, so the two components add up to the GZ spread. We combine these series with the intermediary capital ratio of He et al. (2017). The common monthly sample runs from January 1985 to June 2024, yielding 474 observations. Quarterly predictive regressions use forecast origins beginning in 1985Q1 and outcomes observed through 2024Q2.

We report full-sample estimates alongside two sample checks: ending before the pandemic and excluding the financial-crisis recession. Monthly exclusions remove December 2007–June 2009; quarterly forecasting exclusions remove any origin or forward window that touches 2007Q4–2009Q2. Specifications compared within each exercise use the same observations. Standard errors allow for heteroskedasticity and serial correlation, using Bartlett HAC estimates with 12 monthly or four quarterly lags and a finite-sample correction. The forecasting regressions are in-sample and use revised rather than real-time data.

2.1 Intermediary Leverage and the Excess Bond Premium

Intermediary leverage is a natural candidate to explain the premium, because it measures how thin the equity cushion of the institutions that hold corporate debt has become. Let c_t denote the capital ratio of primary dealer holding companies, market equity relative to market equity plus book debt, and let leverage be its reciprocal, $x_t = 1/c_t$. Measured against market equity, leverage rises when equity falls, even as intermediaries shed assets, so it is countercyclical. This differs from the debt-side limits emphasized by Geanakoplos (2010), such as margins and value-at-risk constraints, under which leverage contracts in downturns, and it motivates the restriction in our model that intermediaries cannot issue equity. Empirical leverage is distinct from the model's state variable, the intermediaries' share of aggregate wealth, but both track intermediary equity.

Figure 1 shows that leverage and EBP rise together during major episodes of financial distress. The pattern is especially pronounced during the global financial crisis, when declining market equity coincides with increases in both leverage and the premium. Both series also rise in March 2020 and subsequently retrace. In a monthly regression controlling for the term spread and real federal funds rate, a one-unit increase in leverage is associated with an EBP 5.03 basis points higher. The level relationship is less precisely estimated when the financial-crisis recession is excluded, motivating a complementary analysis of changes in the capital ratio.

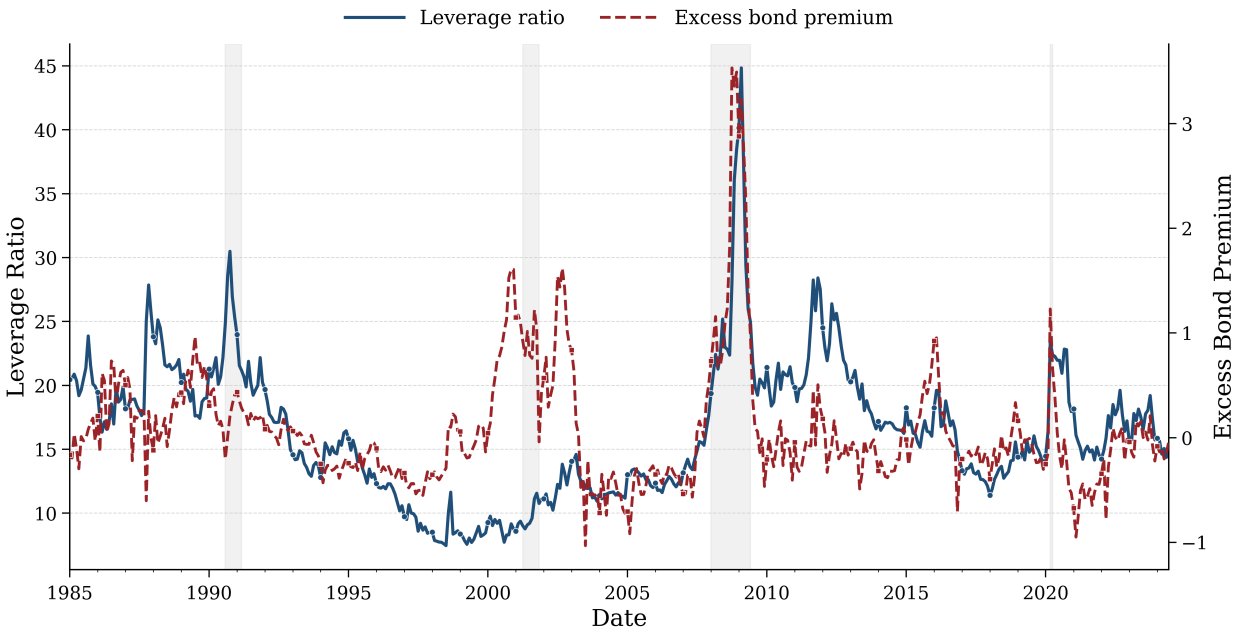


Figure 1: Intermediary leverage ratio (He et al., 2017) and the excess bond premium (Gilchrist and Zakrajšek, 2012). Shaded bars indicate NBER recession dates.

We ask whether a deterioration in intermediary capital accompanies an increase in EBP beyond the contemporaneous change in measured default conditions. Define $a_t = -100\Delta \log c_t / s_m$, where s_m is the full-sample standard deviation of monthly changes in $100 \log c_t$. A one-unit increase in a_t is a one-standard-deviation deterioration in the capital ratio, equivalently a rise in log leverage. We estimate

$$\Delta \text{EBP}_t = \alpha + \beta a_t + \delta \Delta d_t + \rho \text{EBP}_{t-1} + \gamma' R_{t-1} + u_t, \tag{1}$$

where R_{t-1} contains the lagged term spread and real federal funds rate. The coefficient β measures how much the premium rises with balance-sheet deterioration, holding the change in default conditions fixed.

Table 1 reports an increase in EBP of 9.52 basis points per standard deviation of capital-ratio deterioration, with a 95% confidence interval of $[5.17, 13.88]$. The corresponding estimates are 8.07 basis points before the pandemic and 8.12 basis points when the financial-crisis recession is excluded. The positive association therefore survives excluding either episode separately. Using the capital-ratio innovation of He et al. (2017) in place of the observed change yields a similar full-sample association of 9.06 basis points. The two measures are closely related, with a correlation of 0.994, so this is a measurement check rather than independent evidence from a second source.

Table 1: Intermediary Capital Deterioration and the Excess Bond Premium

	Full sample	Pre-COVID	Excluding GFC
EBP change (basis points)	9.52	8.07	8.12
95% confidence interval	[5.17, 13.88]	[3.70, 12.43]	[3.67, 12.58]
Monthly observations	474	420	455

Notes: The dependent variable is the monthly change in the excess bond premium (EBP). Each column reports the association with a decline of one full-sample standard deviation in $100\Delta \log c_t$, where c_t is the intermediary capital ratio. The same standard deviation (6.846 log percentage points) is used in all columns. Controls are the contemporaneous change in the default-related spread component, lagged EBP, the lagged term spread and the lagged real federal funds rate. Estimates use monthly observations from January 1985 to June 2024. The pre-COVID sample ends in December 2019; the final column excludes December 2007–June 2009. Confidence intervals use Bartlett HAC standard errors with 12 monthly lags, a finite-sample correction and a normal reference distribution. Calendar gaps are retained when computing HAC covariances.

Changes in intermediary balance sheets therefore carry information about the part of credit spreads that default risk does not explain: weaker balance sheets accompany a higher price of bearing credit risk, as in the model. Because both variables move within the same month, the regression does not isolate an exogenous shock to intermediary capital.

2.2 Default Conditions and the Excess Bond Premium

The relationship between default conditions and the premium changes with distress, so a single linear coefficient can obscure how the two rise together in adverse states. We allow for this by estimating

$$\text{EBP}_t = \alpha + b_1 \bar{d}_t + b_2 \bar{d}_t^2 + \omega' Z_t + u_t, \qquad \bar{d}_t = d_t - \widetilde{d}, \tag{2}$$

where $\widetilde{d}$ is the full-sample median of the default-related component and Z_t contains successively richer controls.

Figure 2 displays the unconditional relationship on the common monthly sample. The quadratic coefficient is positive, $\hat{b}_2 = 0.559$ (HAC standard error 0.103). The fitted slope is negative at the tenth percentile of default conditions and positive at the ninetieth percentile. Adding intermediary leverage, the term spread and the real federal funds rate preserves positive curvature, 0.363 (0.094), and a positive slope in the high-default part of the distribution (Table 2). The nonlinear pattern therefore contains information beyond a linear leverage control.

The curvature comes from persistent stress episodes rather than from month-to-month changes. Adding lagged EBP makes the full-sample quadratic coefficient approximately

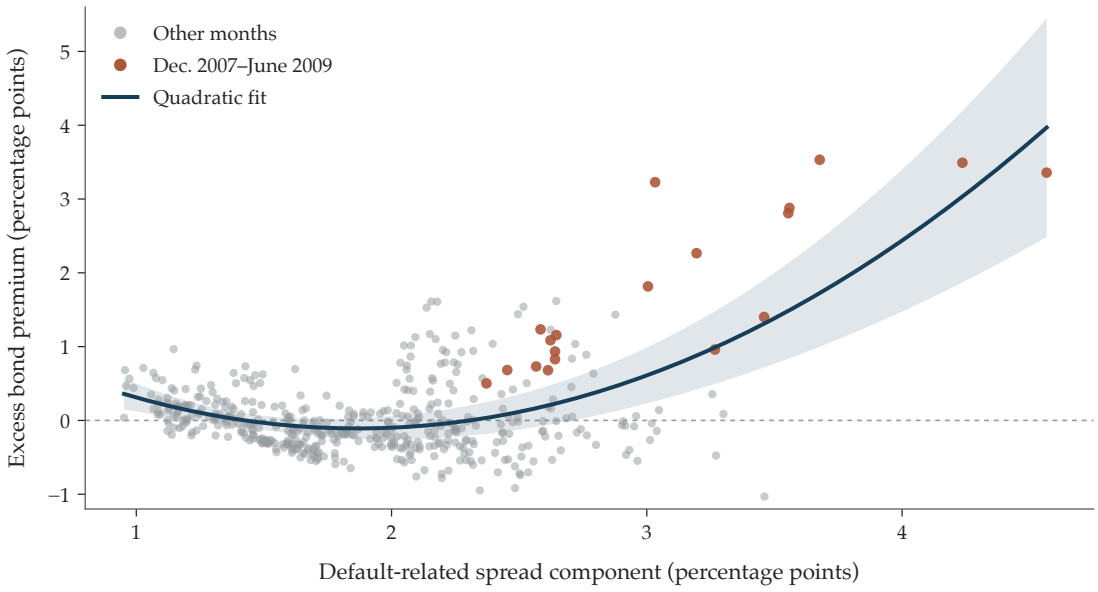


Figure 2: Default conditions and the excess bond premium.

Notes: Monthly observations, January 1985–June 2024. The default-related component is the GZ spread minus EBP. Highlighted observations fall in December 2007–June 2009. The curve is the full-sample unconditional quadratic fit; shading shows a 95% pointwise confidence band for the fitted mean using HAC(12) inference. The fit and band are shown over the full observed range of the default component.

zero, and excluding the financial-crisis recession makes the unconditional curvature imprecise and reverses its sign once lagged EBP is included. That exclusion removes all five months in which the default-related spread exceeds 3.5 percentage points, and with them much of the most distressed variation. We therefore read the nonlinearity as a feature of persistent stress episodes rather than of monthly fluctuations.

To connect default conditions directly to intermediary balance sheets, we interact centered leverage with the centered default component, in a specification that also includes their levels, squared default conditions, lagged EBP, interest rates, and market return and volatility. The premium rises by 0.76 basis points per unit of leverage at the ninetieth percentile of default conditions, against −0.21 at the tenth. The estimates point to state dependence without establishing it: the interaction has $p = 0.061$ and becomes imprecise when the financial-crisis recession is excluded.

In the distressed part of the sample, higher default-related spreads therefore come with a higher premium, which links default conditions to the balance-sheet evidence of Section 2.1. Both components come from the same statistical decomposition, however,

Table 2: Default Conditions and the Excess Bond Premium

	Full sample	Pre-COVID	Excluding GFC
Default component and its square	0.559	0.551	0.194
	(0.103)	(0.108)	(0.154)
+ leverage and rate controls	0.363	0.373	0.076
	(0.094)	(0.096)	(0.149)
+ lagged EBP	−0.002	−0.013	−0.131
	(0.058)	(0.061)	(0.057)
Monthly observations	474	420	455

Notes: Entries are the coefficient on $(d_t - \widetilde{d})^2$ in regressions of EBP, with HAC standard errors in parentheses. The default-related component d_t equals the Gilchrist–Zakrajšek corporate spread less EBP; both components are measured in percentage points. The centering value $\widetilde{d} = 1.861$ is the fixed full-sample median. All regressions include an intercept and $d_t - \widetilde{d}$. Controls are added cumulatively: intermediary leverage, the term spread and the real federal funds rate, followed by lagged EBP. The full sample covers January 1985–June 2024; the pre-COVID sample ends in December 2019; the final column excludes December 2007–June 2009. Bartlett HAC covariances use 12 monthly lags, a finite-sample correction and the original calendar distances.

so their relationship does not directly measure intermediaries' repayment incentives; the model makes that link explicit by letting default and risk-bearing capacity interact in equilibrium.

2.3 Macroeconomic Implications

We next ask whether the premium contains information about real activity after accounting for intermediary leverage and nonlinear default conditions. For real GDP, personal consumption expenditures (PCE) and private nonresidential fixed investment (PNFI), the outcome is annualized growth over the next h quarters, $g^h_{t+h} = (400/h)(\log Y_{t+h} - \log Y_t)$ for $h = 1$ and 4. It excludes growth during quarter t, so that financial conditions at t predict only later growth. The full specification is

$$g^h_{t+h} = \alpha_h + \rho_h g_t + \gamma_h' R_t + \beta_h \mathrm{EBP}_t + \kappa_h x_t + \delta_{1h}\bar{d}_t + \delta_{2h}\bar{d}_t^2 + \varepsilon_{t+h}, \tag{3}$$

where $g_t = 400\Delta \log Y_t$ is current own growth and R_t contains the term spread and real federal funds rate. The default terms are centered at their quarterly full-sample median. We compare this specification with a macroeconomic baseline, EBP alone, leverage alone, and their joint inclusion. Within each outcome and horizon, all specifications use common dates; the full sample contains 157 observations at $h = 1$ and 154 at $h = 4$. Appendix A reports the joint financial coefficients at both horizons.

The investment relationship remains negative and precisely estimated after conditioning on leverage and the nonlinear default terms. Table 3 reports differences in four-quarter growth associated with a 25-basis-point increase in EBP. In the full sample, subsequent investment growth is 1.47 percentage points lower, with a 95% confidence interval of $[-2.18, -0.76]$. The corresponding differences are -0.30 percentage points for GDP and -0.17 for consumption. The investment estimate is similar before the pandemic, -1.50 percentage points, and remains negative when forward windows touching the financial-crisis recession are excluded, -1.22 percentage points. Thus, EBP contains information about subsequent capital formation that neither leverage nor the default-related component alone captures.

Leverage's separate predictive content depends more on the sample: it is imprecisely estimated in the full-sample joint regressions, although some pre-pandemic output and consumption regressions retain it. This does not make balance sheets irrelevant, since leverage and the premium measure different aspects of financial conditions, but the premium's information about investment is the more robust result.

Finally, we ask whether a quarterly deterioration in the capital ratio is followed by changes in EBP, GDP and investment. These dynamic estimates, reported in Appendix A.2, are imprecise: every 95% pointwise interval at horizons of zero to eight quarters includes zero, in both the full and the pre-pandemic samples. On their own, the data therefore do not trace the full chain from weaker intermediary capital through the premium to real activity. The model supplies that chain, showing how intermediary wealth can connect the financial and real patterns documented here.

Table 3: The Excess Bond Premium and Subsequent Real Activity

	Full sample	Pre-COVID	Excluding GFC
GDP	−0.30	−0.26	−0.25
	[−0.473, −0.122]	[−0.428, −0.088]	[−0.418, −0.082]
Consumption	−0.17	−0.11	−0.16
	[−0.319, −0.020]	[−0.215, −0.001]	[−0.312, −0.011]
Investment	−1.47	−1.50	−1.22
	[−2.179, −0.764]	[−2.252, −0.755]	[−1.873, −0.571]
Quarterly observations	154	136	143

Notes: Entries report the association of a 25-basis-point increase in EBP with growth over the next four quarters, in percentage points. Brackets contain 95% confidence intervals. The dependent variable is $100[\log Y_{t+4} - \log Y_t]$, where Y denotes real GDP, real personal consumption expenditures or real private nonresidential fixed investment. Controls are current own growth, $400[\log Y_t - \log Y_{t-1}]$, intermediary leverage, the default-related spread component and its square, the term spread and the real federal funds rate. Forecast origins begin in 1985Q1, and the final outcome quarter is 2024Q2. The pre-COVID column restricts both origins and outcome quarters to 2019Q4 or earlier. The final column excludes every forecast window $[t, t+4]$ that overlaps 2007Q4–2009Q2. Intervals use Bartlett HAC covariances with four quarterly lags, a finite-sample correction, original calendar distances and a normal reference distribution. These are conditional predictive associations.

3 The Model

We extend the continuous-time economy of Brunnermeier and Sannikov (2014) by letting intermediaries choose how much of their debt to repay. Experts, the financial intermediaries of the model, operate capital more productively than households, so intermediation raises output. They borrow from households to expand production, but they cannot issue equity and face a leverage ceiling; when their net worth is too small to finance the entire capital stock, households operate part of it. At each instant experts trade the funds they retain by withholding repayment against a non-pecuniary penalty, and households price the resulting default when deciding whether to lend or to hold capital themselves. The terms of credit and the allocation of capital are therefore determined together. Derivations appear in Appendix B.

3.1 Technology

The economy evolves over $[0, \infty)$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ supporting a standard Brownian motion Z_t. This common shock is the only source of aggregate uncertainty.

Experts produce $y_t = Ak_t$ and households produce $\underline{y}_t = \underline{A}\,\underline{k}_t$, with $\underline{A} < A$. The

productivity gap $A - \underline{A}$ is the source of the gains from intermediation, and it also makes a contraction in intermediation costly: capital transferred from experts to households produces less output for a given aggregate stock.

Capital accumulation. Expert capital follows

$$\frac{dk_t}{k_t} = \Big(\phi(\iota_t) - \delta\Big)\,dt + \sigma\,dZ_t, \tag{4}$$

where δ is depreciation and $\sigma > 0$ is the volatility of the aggregate capital shock. Investing ι_t goods per unit of capital adds $\phi(\iota_t)$ units of new capital, with

$$\phi(\iota_t) = \frac{1}{\theta}\Big(\sqrt{1 + 2\theta\iota_t} - 1\Big). \tag{5}$$

The concavity of ϕ captures adjustment costs in the spirit of Hayashi (1982), and a larger $\theta > 0$ makes investment more costly. Without adjustment costs ($\theta \to 0$), $\phi(\iota_t) \to \iota_t$ and the price of capital is fixed at one; with them, the price of installed capital can move, and investment responds to it. Households use the same technology, with depreciation $\underline{\delta} \geq \delta$.

Asset prices. We conjecture that the price of capital follows

$$\frac{dq_t}{q_t} = \mu_t^q\,dt + \sigma_t^q\,dZ_t, \tag{6}$$

where the drift μ_t^q and volatility σ_t^q are determined in equilibrium as functions of the experts' share of aggregate wealth (Section 4).

Applying Itô's product rule to $q_t k_t$ gives the expert's return on capital:

$$dr_t^k = \underbrace{\frac{A - \iota_t}{q_t}\,dt}_{\text{dividend yield}} + \underbrace{\Big(\phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q\Big)\,dt}_{\text{expected capital gain}} + \underbrace{\Big(\sigma + \sigma_t^q\Big)\,dZ_t}_{\text{capital risk}}. \tag{7}$$

The dividend yield is output net of investment per unit of capital value. The expected capital gain combines the growth of capital, the drift of its price and their covariance $\sigma\sigma_t^q$. Capital risk has two parts, the fundamental shock σ and the endogenous price volatility σ_t^q. When a shock that lowers expert wealth also lowers the price of capital, σ_t^q adds to fundamental risk, and leveraged experts lose on both the quantity and the value of their capital.

3.2 Experts

A representative expert finances capital k_t with net worth n_t and debt d_t:

$$\underbrace{q_t k_t}_{\text{assets}} = \underbrace{d_t}_{\text{debt}} + \underbrace{n_t}_{\text{equity}}. \tag{8}$$

Experts cannot issue new equity: they retain their own wealth in the business and raise external funds only through defaultable debt. This restriction makes past gains and losses matter for current financing capacity.

Investment. Experts invest until the marginal cost of a unit of new capital equals its market price, which gives the Tobin's q rule

$$\iota_t = \underline{\iota}_t = \frac{q_t^2 - 1}{2\theta}. \tag{9}$$

Investment is positive when $q_t > 1$ and rises with the price of capital, by less when adjustment costs are larger. Households use the same rule: their productivity and financing conditions differ, but their installation technology does not.

The default decision. The debt contract permits partial repayment. During $(t, t + dt)$ the expert owes principal and interest $d_t(1 + r_t)\,dt$ and chooses the fraction $\nu_t \in [0, 1]$ of this obligation that it repays: $\nu_t = 1$ is full repayment, $\nu_t = 0$ complete default, and $\nu_t \in (0, 1)$ partial default. The repayment fraction is therefore a continuous choice on outstanding debt, not the probability of a discrete default event.

Following Dubey et al. (2005) and Shubik and Wilson (1977), default carries a non-pecuniary penalty, a reduced-form stand-in for the legal and reputational costs of reneging on debt. We use

$$\Upsilon(\nu_t, x_t, r_t) \triangleq \max\{0,\, 1 - \nu_t\}^2\, \mathbf{1}(x_t \geq 1)\, (x_t - 1)^2\, (1 + r_t)^2. \tag{10}$$

The penalty is zero under full repayment and rises quadratically with the unpaid fraction $1 - \nu_t$, so its marginal cost increases and an interior repayment choice is possible, although boundary choices remain admissible.

The exposure term $(x_t - 1)^2(1 + r_t)^2$ scales the penalty with the square of gross debt obligations per unit of equity, where $x_t \triangleq q_t k_t / n_t$ is leverage, so the same unpaid fraction

carries a larger penalty when debt exposure is larger.[1] The coefficient $\lambda > 0$ that multiplies Υ in the expert's objective measures the harshness of the penalty. It is a parameter of the bankruptcy law and is the same in every state. Work on default under aggregate adversity (Goodhart et al., 2018; Peiris et al., 2025) suggests that enforcement may itself vary with the state.

Net worth dynamics. Using the balance-sheet identity, expert net worth evolves as

$$\frac{dn_t}{n_t} = \underbrace{x_t\, dr_t^k}_{\text{leveraged return on capital}} \underbrace{-\,(x_t - 1)\Big[\nu_t(1 + r_t) - 1\Big]dt}_{\text{debt service after default}}. \tag{11}$$

The first term scales the return on capital by leverage. The second is the cost of debt after repayment: per unit of debt the expert pays the effective rate $\nu_t(1 + r_t) - 1$, which equals the contractual rate r_t under full repayment and falls by $(1 - \nu_t)(1 + r_t)$ under partial repayment. The expert retains these funds, and households bear the corresponding loss.

Leverage constraint. Experts face a margin requirement, a collateral constraint in the spirit of Kiyotaki and Moore (1997): their equity must cover at least a fraction $m \in (0, 1]$ of the value of the capital they hold. Leverage is therefore bounded by

$$x_t \le U_t \triangleq \min\left\{\frac{1}{m},\, \frac{q_t K_t}{n_t}\right\}, \tag{12}$$

where K_t is aggregate capital. The first bound is the leverage ceiling $1/m$; a higher margin lowers the capital an expert can finance per unit of net worth. The second keeps the expert's capital holding within the aggregate stock.

A fall in expert net worth reduces the capital that experts can finance at a given leverage, and the capital they cannot finance passes to households, lowering aggregate productivity. Falling prices can reinforce this contraction, as in the fire-sale mechanism of Brunnermeier and Pedersen (2009). A tighter margin requirement limits leverage, and with it this amplification, but also keeps capital from its most productive users; Section 5.4 studies this trade-off.

[1] An unleveraged expert has no debt to default on and faces no penalty.

Optimization problem. The expert chooses investment ι_t, leverage x_t and repayment ν_t to maximize the expected instantaneous return on net worth, net of the default penalty:

$$\max_{\iota_t, x_t, \nu_t \in [0,1]} \left\{ \frac{\mathbb{E}_t[dn_t]}{n_t\, dt} - \lambda\, \Upsilon(\nu_t, x_t, r_t) \right\}, \tag{13}$$

subject to the law of motion (11) and the leverage constraint (12). Experts consume nothing and reinvest all their wealth.[2]

Optimality conditions. The investment condition is (9). For an interior repayment choice, the funds retained by withholding a little more repayment equal the marginal penalty:

$$\underbrace{(x_t - 1)(1 + r_t)}_{\text{marginal balance-sheet relief}} = \underbrace{2\lambda(1 - \nu_t)(x_t - 1)^2(1 + r_t)^2}_{\text{marginal penalty cost}}, \tag{14}$$

or, rearranging,

$$(1 - \nu_t)(1 + r_t)(x_t - 1) = \frac{1}{2\lambda}. \tag{15}$$

The left-hand side is the amount the expert withholds per unit of its own net worth. Wherever repayment is partial, this amount is therefore the constant $1/(2\lambda)$, whatever the state of the economy. Leverage and the contractual rate determine only what fraction of debt it represents, and that fraction falls as debt exposure rises. A harsher penalty lowers the amount withheld, and repayment becomes full as $\lambda \to \infty$. At a repayment boundary the corresponding inequality replaces (15).

For interior leverage, the expected return on capital equals the contractual rate:

$$\frac{\mathbb{E}_t[dr_t^k]}{dt} = r_t. \tag{16}$$

At the leverage ceiling the expected return can exceed the rate, and the gap measures the marginal value of relaxing the constraint, that is, the scarcity of intermediary financing capacity.

3.3 Households

Households, a unit mass indexed by $j \in (1, 2]$, save by lending to experts and by holding capital directly. Lending gives them a claim on expert productivity but exposes them to

[2] With a continuation value linear in net worth, maximizing lifetime value reduces to maximizing the instantaneous expected return.

default; holding capital exposes them to aggregate risk. Their portfolio choice therefore determines how default affects the funds available to experts.

A household holds capital $\underline{k}_t$ and bonds $\underline{d}_t$, so its net worth is $\underline{n}_t = q_t\underline{k}_t + \underline{d}_t$ and its capital share is $\underline{x}_t \equiv q_t\underline{k}_t/\underline{n}_t$. In equilibrium $\underline{x}_t \leq 1$: households lend rather than borrow (Proposition 1). Their budget constraint is

$$d\underline{n}_t = (\underline{A} - \underline{\iota}_t)\underline{k}_t\,dt + d(q_t\underline{k}_t) + \nu_t\underline{d}_t r_t\,dt - (1-\nu_t)\underline{d}_t\,dt - \underline{c}_t\,dt, \tag{17}$$

where $\underline{c}_t$ is consumption. Dividing by net worth gives

$$\frac{d\underline{n}_t}{\underline{n}_t} = \underline{x}_t\,d\underline{r}_t^k + (1-\underline{x}_t)\Big[\nu_t(1+r_t) - 1\Big]dt - \frac{\underline{c}_t}{\underline{n}_t}\,dt. \tag{18}$$

Per unit of bonds, households earn the effective rate $\nu_t(1+r_t) - 1$ that experts pay in (11). Lenders care about this effective return, so the contractual rate must rise to compensate them for what experts withhold.

Household optimization. Households maximize

$$\mathbb{E}\left[\int_0^\infty e^{-\gamma t}\log(\underline{c}_t)\,dt\right],$$

subject to (18) and $\underline{x}_t \geq 0$. The discount rate $\gamma > 0$ measures impatience. Logarithmic utility gives the consumption rule $\underline{c}_t = \gamma\underline{n}_t$: households consume a fixed fraction of wealth, so financial losses also reduce consumption.

Because utility is logarithmic, the portfolio choice is myopic, and the interior capital share is

$$\underline{x}_t = \frac{\dfrac{\mathbb{E}_t[d\underline{r}_t^k]}{dt} - \nu_t(1+r_t) + 1}{(\sigma + \sigma_t^q)^2}. \tag{19}$$

The numerator is the expected return on capital in excess of the effective bond return, and the denominator is the variance of the return on capital, so the rule is the Merton share of a log investor with the effective bond return in place of a riskless rate. Holding other terms fixed, a lower repayment fraction makes direct capital more attractive, and greater return variance makes it less so. Equilibrium prices adjust to reconcile this portfolio choice with the capital that experts can finance. At zero direct capital holdings, the corresponding inequality replaces the interior equality.

4 Equilibrium

The experts' share of aggregate wealth summarizes the distribution of financing capacity: it determines how much capital experts can operate and how much households must hold directly. We express prices, portfolios and repayment as functions of this share and characterize them using market clearing and the agents' optimality conditions.

4.1 Market Clearing and the State Variable

Capital, debt and goods markets clear (Appendix B.4). Debt is an asset of households and a liability of experts, so it cancels from aggregate net worth, and aggregate wealth equals the value of installed capital:

$$N_t + \underline{N}_t = q_t K_t. \tag{20}$$

The state variable. Define the expert wealth share as

$$\eta_t \triangleq \frac{N_t}{q_t K_t}, \tag{21}$$

with households holding the remaining share $1 - \eta_t$. Because production is linear in capital and decision rules are linear in wealth, the level of aggregate wealth does not matter for prices or portfolio ratios; only its division between the two sectors does. The wealth share is therefore the single state variable, and a lower η_t means that experts have less equity relative to the value of the capital stock.

We conjecture that the wealth share follows

$$\frac{d\eta_t}{\eta_t} = \alpha(\eta_t)\, dt + \beta(\eta_t)\, dZ_t, \tag{22}$$

with drift α and volatility β determined in equilibrium (Proposition 3). In terms of η_t, the leverage constraint becomes $x_t \leq \min\{1/m, 1/\eta_t\}$. As the expert wealth share falls, the bound $1/\eta_t$ rises while the leverage ceiling $1/m$ stays fixed, so at the ceiling experts can operate at most a share η_t/m of the capital stock, which shrinks as they become poorer.

4.2 Equilibrium Characterization

Let $\psi_t \triangleq \int_0^1 k_t^i\, di/K_t$ denote the experts' share of the capital stock. Whenever experts borrow, $\psi_t > \eta_t$: they operate more capital than their equity alone can finance, and the difference

$\psi_t - \eta_t$ is financed by household lending. Market clearing then pins down both sectors' portfolios.

Proposition 1 (Experts borrow, households lend)**.** *In equilibrium, experts borrow and households lend:* $0 \le \underline{x}_t \le 1 \le x_t \le U_t$*, with*

$$x_t = \frac{\psi_t}{\eta_t}, \qquad \underline{x}_t = \frac{1-\psi_t}{1-\eta_t}. \tag{23}$$

Intuitively, an expert who lent would earn the effective bond return, which is no higher than the return on its own capital. The allocation of capital then defines two regions. When $\psi_t = 1$, experts operate all the capital and households hold only bonds; when $\psi_t < 1$, households also operate capital, and aggregate productivity is lower. These are the expansion and recession regions of Section 5.

Proposition 2 (Optimal repayment)**.** *Wherever repayment is partial, the equilibrium repayment fraction satisfies*

$$(1-\nu_t)(1+r_t)(x_t-1) = \frac{1}{2\lambda}, \qquad 0 < \nu_t < 1. \tag{24}$$

The proposition restates (15) as an equilibrium condition. Because the amount withheld per unit of net worth is fixed at $1/(2\lambda)$, the unpaid fraction moves inversely with $(1+r_t)(x_t-1)$, so the cyclical pattern of repayment depends on how leverage and the contractual rate move together. The law of motion of the state variable follows from the experts' budget constraint and market clearing.

Proposition 3 (Expert wealth dynamics)**.** *The evolution of the expert wealth share is*

$$\frac{d\eta_t}{\eta_t} = \frac{\psi_t - \eta_t}{\eta_t}\left(dr_t^k - r_t\,dt + \frac{dt}{2\lambda(x_t-1)} - (\sigma+\sigma_t^q)^2\,dt\right) + \left[\frac{A-\iota_t}{q_t} + (1-\psi_t)(\underline{\delta}-\delta)\right]dt, \tag{25}$$

with drift

$$\alpha(\eta_t) = \frac{\psi_t - \eta_t}{\eta_t}\left(\frac{\mathbb{E}_t[dr_t^k]}{dt} - r_t + \frac{1}{2\lambda(x_t-1)} - (\sigma+\sigma_t^q)^2\right) + \frac{A-\iota_t}{q_t} + (1-\psi_t)(\underline{\delta}-\delta), \tag{26}$$

and diffusion $\beta(\eta_t) = \frac{\psi_t-\eta_t}{\eta_t}(\sigma+\sigma_t^q)$.

The exposure $(\psi_t - \eta_t)/\eta_t = x_t - 1$ amplifies changes in the return on capital relative to the funding rate, so the diffusion is $\beta(\eta_t) = (x_t-1)(\sigma+\sigma_t^q)$: a given capital-return shock

moves the wealth share more when experts are more leveraged, as in the debt-deflation mechanism of Fisher (1933).

Partial repayment enters the drift through the term $1/[2\lambda(x_t - 1)]$, which the exposure $x_t - 1$ turns into the constant $1/(2\lambda)$. This is the amount withheld per unit of net worth: a transfer from household creditors to experts that raises the growth of expert wealth in every state, by more the more lenient the penalty. The variance correction works in the opposite direction, reflecting the risk of a leveraged position relative to aggregate wealth, and output net of investment and the depreciation advantage of expert capital also enter the drift.

Goods-market clearing, together with household consumption and the investment rule, gives the price of capital.

Proposition 4 (Equilibrium capital price)**.**

$$q_t(\eta_t) = -\theta\gamma(1-\eta_t) + \sqrt{\theta^2\gamma^2(1-\eta_t)^2 + 2\theta\Big[x_t(\eta_t)\eta_t A + (1 - x_t(\eta_t)\eta_t)\underline{A}\Big] + 1}. \tag{27}$$

For a given allocation $\psi_t = x_t\eta_t$, a larger expert wealth share leaves less household wealth to finance current consumption and raises the price that clears the goods market; this effect depends on γ, the households' consumption-wealth ratio. A larger expert capital share also raises the price, by raising aggregate productivity in proportion to the productivity gap. Both effects push the price up as expert wealth grows, until experts hold all the capital and only the first remains.

The household's portfolio condition then gives the contractual rate.

Proposition 5 (Equilibrium interest rate)**.**

$$r_t = \frac{\overline{A} - \iota_t}{q_t} + \phi(\iota_t) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q - (\sigma + \sigma_t^q)^2\frac{1 - x_t\eta_t}{1 - \eta_t} + \frac{1}{2\lambda(x_t - 1)}. \tag{28}$$

The rate has three parts. The terms before the variance adjustment are the expected return on household capital. The variance term, capital-return variance times the household capital share, is the extra return households require to hold capital rather than bonds; Section 5 identifies it as the model's excess bond premium. The final term, $1/[2\lambda(x_t - 1)]$, compensates lenders for what experts withhold. The result is the promised rate on *defaultable* debt, valid where both the household's capital position and the expert's repayment choice are interior.

Applying Itô's lemma to $q_t(\eta_t)$ and matching the diffusion terms gives the volatility of the price of capital.

Proposition 6 (Capital price volatility)**.**

$$\sigma_t^q = \frac{(x_t\eta_t - \eta_t)\,\dfrac{q'(\eta_t)}{q_t(\eta_t)}\,\sigma}{1 - (x_t\eta_t - \eta_t)\,\dfrac{q'(\eta_t)}{q_t(\eta_t)}}. \tag{29}$$

Suppose a negative shock lowers expert net worth. Because the price of capital rises with expert wealth, the price falls too, and this second loss lowers expert net worth further; the feedback converges as long as the denominator is positive. Equivalently, total return volatility is $\sigma + \sigma_t^q = \sigma/\left[1 - (x_t - 1)\,\eta_t q'(\eta_t)/q_t\right]$: leverage multiplies the elasticity of the price of capital with respect to intermediary wealth, a measure of how illiquid the market for capital is. As the denominator approaches zero, a small fundamental shock produces a large movement in the price.

Combining the expert's leverage condition with the contractual rate gives equilibrium leverage.

Proposition 7 (Equilibrium leverage)**.** *The optimal leverage $x_t \in (1, U_t)$ solves*

$$\underbrace{\frac{A - \underline{A}}{q_t} + (\underline{\delta} - \delta) + (\sigma + \sigma_t^q)^2 \frac{1 - x_t\eta_t}{1 - \eta_t}}_{\textit{marginal benefit of leverage}} = \underbrace{\frac{1}{2\lambda(x_t - 1)}}_{\textit{marginal default cost}}. \tag{30}$$

The right-hand side is the loss lenders bear per unit lent: experts withhold $1/(2\lambda)$ per unit of net worth and owe $x_t - 1$ per unit of net worth, so the loss per unit lent falls as lending rises. The left-hand side is the gap between the return on capital in expert hands and the return households require on their loans: the productivity advantage $(A - \underline{A})/q_t + (\underline{\delta} - \delta)$ plus the compensation households require for the risk on the capital they hold, which vanishes when they hold none. Lenders extend credit until the two are equal.

Holding prices and portfolios fixed, a more lenient penalty raises the right-hand side, and leverage must rise to restore the equality; this is the partial-equilibrium sense in which leniency expands credit. The same change also moves the price of capital, household exposure and repayment, so we evaluate policies in general equilibrium, imposing the relevant boundary conditions when an interior choice reaches a constraint.

Proposition 8 (Household optimum)**.** *The household's optimality conditions are*

$$\underline{c}_t = \gamma \, \underline{n}_t, \tag{31}$$

$$\underline{\iota}_t = \frac{q_t^2 - 1}{2\theta}, \tag{32}$$

$$\underline{x}_t = \frac{\dfrac{\mathbb{E}[d\underline{r}_t^k]}{dt} - \nu_t(1 + r_t) + 1}{(\sigma + \sigma_t^q)^2}. \tag{33}$$

The proposition collects the household conditions of Section 3.3.

Equilibrium as a fixed point. Propositions 1–8 characterize equilibrium prices and portfolios at each expert wealth share. On the interior branch, substituting (27), (28) and (29) into (30) reduces the problem to a single nonlinear equation in leverage, coupled to the price derivative; on the boundary branches the corresponding portfolio and leverage inequalities replace the interior conditions. Section 5 solves this system numerically and compares the results with the motivating evidence of Section 2.

The leverage constraint and strategic default create scope for a wedge between private decisions and the social value of intermediary wealth, as in economies with incomplete financial markets (Geanakoplos and Polemarchakis, 1985). Section 5.4 asks how the default penalty and the margin requirement act on this wedge through the allocation of capital and the path of consumption.

5 Quantitative Analysis

We now take the model to the data and ask how much of the excess bond premium, and of its comovement with leverage, the mechanism accounts for. In the calibrated economy, the states in which intermediaries are undercapitalized, which we match to recessions, are those with high leverage, high spreads and high volatility, and the premium is concentrated in them. We first calibrate the model to the contrast between recessions and expansions, then follow the equilibrium across states, and close with the welfare analysis of the default penalty and the margin requirement.

5.1 Calibration

Parameters. The model has seven parameters and one numerical setting (Table 4). The margin requirement is anchored to the data: the leverage ceiling $1/m = 30$ is the level that primary dealer leverage reaches in distress in the series of He et al. (2017) shown in Figure 1. The ceiling is slack at the calibrated parameters, so the moments below are the same for any ceiling above the equilibrium leverage of 29. We choose the remaining six parameters to match the means of dealer leverage, the GZ spread, the excess bond premium, the default-related spread and the volatility of capital returns, inside and outside NBER recessions. The default-related spread is the GZ spread less the premium, so eight of these ten moments are independent. We minimize the distance between the model's regional means and their empirical counterparts, subject to the default-related component being no lower in recession states than in expansion states, as it is in the data. Household productivity is close to zero, so capital in household hands produces almost no output; together with the discount rate, it is pinned down by the cyclicality of the default-related component.[3] Table 4 lists each value with what disciplines it.

Table 4: Calibrated Parameters

Symbol	Interpretation	Value	Discipline
A	expert productivity	0.025	Dividend yield on capital of 2.5 percent at a price of one
$\underline{A}$	household productivity	0.00	Near zero; household capital produces almost no output
θ	adjustment cost	0.92	Investment moves about one for one with q (Hayashi, 1982)
λ	default penalty	0.62	Amount withheld per unit of equity 0.81; lowest penalty consistent with equilibrium 0.60
σ	fundamental volatility	0.078	Volatility moments; between output growth volatility and the 9 percent of He and Krishnamurthy (2013)
m	margin requirement	0.033	Leverage ceiling of 30, dealer leverage in distress (He et al., 2017); slack at the calibrated parameters
γ	household discount rate	0.004	Cyclicality of the default-related component
N	number of states	1,000	Equally spaced on [0.001, 0.999]

[3]Taken literally, a discount rate of 0.4 percent is low. We treat γ as a parameter disciplined by the cyclicality of the default-related component rather than as an estimate of impatience.

Regions. Recession states are those in which experts do not hold all the capital, $\psi(\eta) < 1$, which is the case for η below 0.034; expansion states are those in which they hold all of it, $\psi(\eta) = 1$, from 0.034 upward. The names refer to the data periods they are matched with, NBER recessions and expansions; in the model the two regions are defined by the allocation of capital. The comparison with the data uses the states in which the model's leverage lies within the range observed for primary dealers, between about 8 and 45 in Figure 1. Since leverage equals $1/\eta$ once experts hold all the capital, these are the states with η up to 0.12, and the regional means are taken over them.

The numerical system. The interior conditions of Section 4 connect the capital price q_t, the expert capital share ψ_t and the endogenous price volatility σ_t^q to expert wealth η_t.

Proposition 9 (Interior equilibrium conditions)**.** *On the interior branch, the equilibrium triplet satisfies*

$$q_t^2 + 2\theta\gamma(1-\eta_t)q_t - 2\theta\Big[\psi_t(A-\underline{A}) + \underline{A}\Big] - 1 = 0, \tag{34}$$

$$(\sigma + \sigma_t^q)\, q'(\eta_t)\, (\psi_t - \eta_t) - \sigma_t^q\, q_t = 0, \tag{35}$$

$$\begin{aligned} &2\lambda(1-\psi_t)(\psi_t-\eta_t)(\sigma+\sigma_t^q)^2 q_t \\ +2\lambda(\psi_t-\eta_t)(1-\eta_t)&\Big[A-\underline{A}-q_t(\delta-\underline{\delta})\Big] - \eta_t(1-\eta_t)q_t = 0. \end{aligned} \tag{36}$$

The first equation is goods-market clearing: it ties the valuation of capital to aggregate productivity and to household consumption. The second is the feedback between wealth and prices: a price change alters the value of the levered position $\psi_t - \eta_t$, and a change in expert wealth moves the price. The third is the interior leverage condition, which sets the amount lent at the point where the loss lenders bear per unit lent equals the gap between the return on capital in expert hands and the return households require. At the portfolio corners the corresponding inequalities replace it.

We solve the system state by state on the grid of Table 4. Where experts hold all the capital, goods-market clearing gives the price in closed form, and we use it and its analytical derivative directly. Appendix C describes the solution method and the checks of the equilibrium conditions.

Spread decomposition. The spread that intermediaries pay has two parts. Households hold capital directly only when experts cannot hold all of it, and on that capital they bear aggregate risk; the compensation they require is the gap between the return on the capital they hold and the return they require on intermediary debt. By the household's portfolio

rule (19), this gap equals the household capital share times the variance of the return on capital,

$$\mathrm{EBP}(\eta) = \underline{x}(\eta)\Big[\sigma + \sigma^{q}(\eta)\Big]^{2}, \tag{37}$$

where $\underline{x}(\eta) = [1 - \psi(\eta)]/(1 - \eta)$ is the household capital share of Proposition 1. It is positive where intermediaries are undercapitalized and zero where they hold all the capital. This is the model's excess bond premium: the price of bearing risk that intermediaries cannot absorb, in the spirit of the decomposition of Gilchrist and Zakrajšek (2012). The remainder of the promised rate,

$$\underbrace{r(\eta)}_{\text{total spread}} = \underbrace{\Big[r(\eta) - \mathrm{EBP}(\eta)\Big]}_{\text{default-related component}} + \underbrace{\mathrm{EBP}(\eta)}_{\text{model EBP}}, \tag{38}$$

is the default-related component. The model has no default-free bond, so the promised rate itself plays the role of the spread over Treasuries that the GZ spread measures.

Fit. Table 5 compares the model with the data. The model reproduces the ordering of every moment: leverage, total spreads, the premium, the default-related component and volatility are all higher in recessions than in expansions, in the model as in the data. Recession-state leverage is within ten percent of its target, and the premium is 70 basis points against 80 in recessions and zero against about minus 4 in expansions. Total spreads differ across the two regions by 74 basis points, close to the 90 observed, and almost all of that difference comes from the premium, 70 of the 74 basis points, while the default-related component barely moves; in the data the premium accounts for about 84 of the 90. This is the credit spread puzzle in the model: the part of spreads that default risk does not explain is the part that moves with intermediary wealth. The model misses in two places. The default-related component is about 70 basis points too high in both regions, so total spreads are too high by about 60 and 75 basis points. Volatility in expansions is 7.9 percent against 14.2 in the data, because once experts hold all the capital the endogenous component vanishes and return volatility falls to its fundamental level. A single calibration, with a default penalty that is the same in every state, is nevertheless enough to generate the countercyclical pattern; no state-dependent penalty schedule is needed. Appendix C reports the numerical checks.

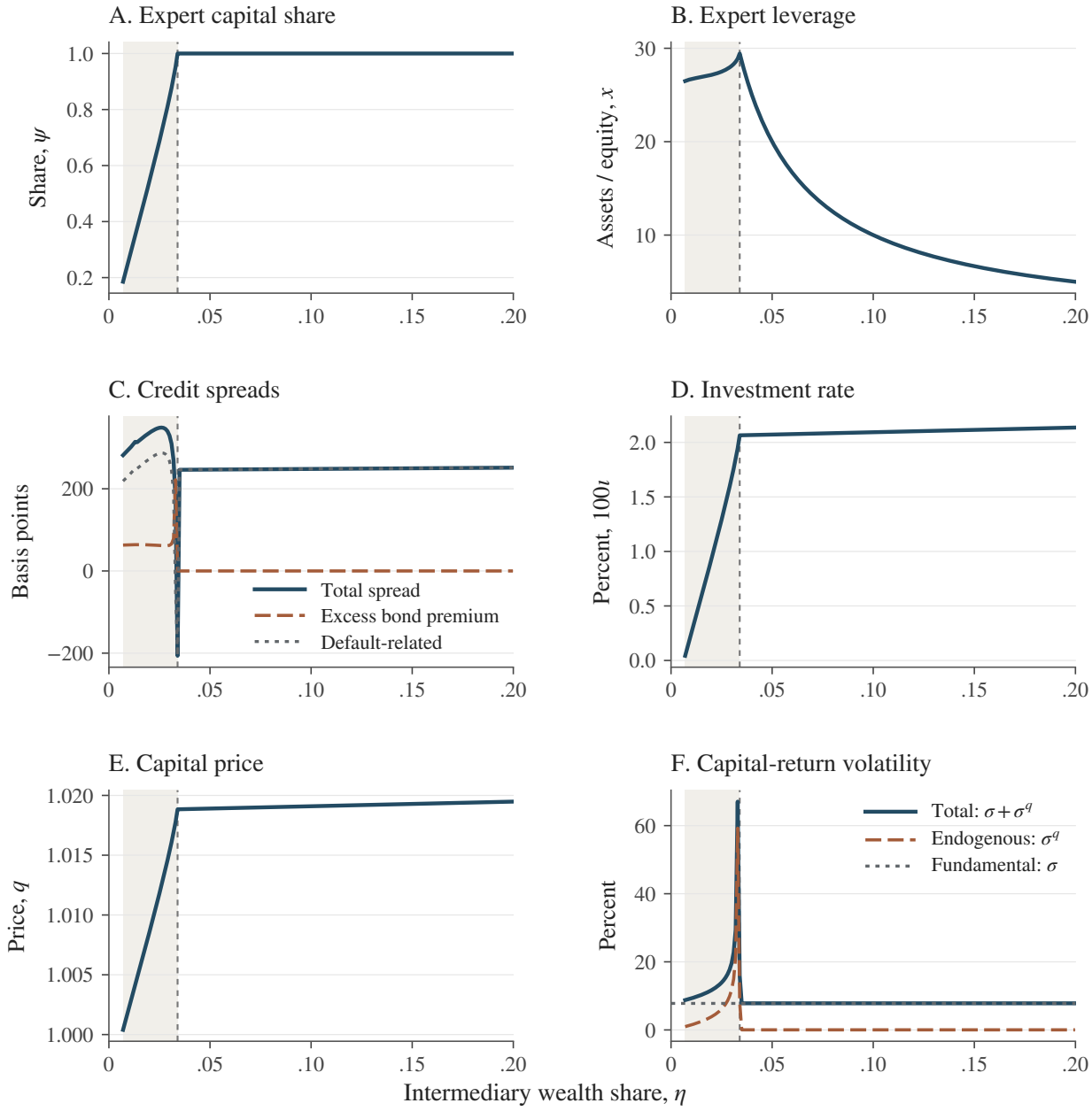


Figure 3: Financial conditions and capital allocation across intermediary wealth. Panels A–F show expert capital share ψ, leverage $x = \psi/\eta$, the spread decomposition, investment 100ι, capital price q, and capital-return volatility. The dashed line marks the first node with $\psi = 1$, at $\eta \simeq 0.034$; light shading denotes the recession region. The display covers η up to 0.20.

Table 5: Calibration Fit: Recessions and Expansions

	Recessions		Expansions	
	Data	Model	Data	Model
Panel A: Spreads (basis points)				
Total spread	256.69	316.43	166.80	242.32
Excess bond premium	80.18	69.57	−3.64	0.00
Default-related component	176.51	246.86	170.44	242.32
Panel B: Balance sheets and risk				
Leverage (assets/equity)	24.85	27.31	17.29	14.73
Capital-return volatility (%)	18.99	15.11	14.22	7.91

Notes: Data: means over NBER recession months and all other months, 1973–2024. Model: means over recession states, in which intermediaries are undercapitalized ($\eta < 0.034$), and expansion states, in which they hold all the capital, within the comparison range $\eta \leq 0.12$. Total spread equals the excess bond premium plus the default-related component. Volatility is $\sigma + \sigma^q$.

5.2 Equilibrium Dynamics

Figure 3 shows the equilibrium functions behind the regional means, from the allocation of capital to the spread decomposition and the volatility of the return on capital. All of them are organized by the switch at $\eta^* \simeq 0.034$, where experts come to hold all the capital.

Capital allocation and leverage. At the left edge of the display experts operate about one fifth of the capital, and their share rises to one at η^*. This is the allocation channel: more expert wealth supports a larger position in the productive technology. Leverage rises from 26.5 to 28.9 over the recession region and, once experts hold all the capital, equals $1/\eta$, falling to 10 at $\eta = 0.10$ and 5 at $\eta = 0.20$; this fall is what makes leverage lower in expansions than in recessions. Displayed leverage stays below the ceiling of 30, so the switch at η^* separates household from expert ownership of capital, not a binding from a slack margin requirement: the condition $\psi = 1$ and the leverage ceiling constrain different objects, a distinction that matters for the policy analysis. The harshness of the default penalty moves the switch: a more lenient penalty lowers the wealth share at which experts come to hold all the capital, and a harsher one raises it.

Credit spreads and volatility. The premium is about 22 percent of the mean total spread in recession states and zero in expansion states, where households hold no capital and bear no aggregate risk; the default-related component supplies the rest of the spread in both regions. These are ratios of regional means, not expected default losses or stationary shares.

Volatility rises sharply before the allocation switch. Total volatility of the return on capital reaches 67 percent at $\eta = 0.033$, of which 59 percent is endogenous, against fundamental volatility of 7.8 percent. Near the switch small movements in wealth coincide with large movements in the price, through the feedback in equation (35). Volatility then falls to its expansion-state level of 7.9 percent. The spread functions change sharply at the same point: the premium rises to 222 basis points at the last node of the recession region and total spreads reach 349 basis points before falling. The regional contrast is more reliable than the shape of the functions at the transition, which the grid does not resolve; the two transition nodes move the recession-state means by six basis points of premium and nearly three percentage points of volatility.

Investment and valuations. Reallocation of capital comes with rising investment. At the left edge investment is close to zero and $q \simeq 1.000$; at the first node of the expansion region it is 2.06 percent of capital and $q \simeq 1.019$. Investment continues to rise slowly in the expansion region, to 2.14 percent at the right edge of the display, while q stays at about 1.019. Investment is $\iota = (q^2 - 1)/(2\theta)$ in goods, while the growth of installed capital before depreciation, which we call capital installation, is $\phi(\iota) = (q - 1)/\theta$: the first uses current goods, the second measures how much physical capital those goods add.

Household capital share and capital installation. The household capital share $\underline{x} = (1 - \psi)/(1 - \eta)$ is the fraction of household wealth held in capital rather than lent to intermediaries. It is about 0.8 at the left edge of the display and falls to zero at the allocation switch, where experts take over all the capital (Figure 4). Capital installation rises with intermediary wealth from close to zero at the left edge to about two percent at the switch and stays there in the expansion region.

Together, these functions trace the balance-sheet mechanism of the model. A loss that moves intermediary wealth below η^* forces experts to cede capital to households, who operate it less productively, so investment and the price of capital fall. Households now bear aggregate risk on the capital they hold, and the compensation they require appears in the spread as the premium. Near the switch the price of capital responds strongly to intermediary wealth, so volatility peaks; since the premium is the household capital share times this variance (37), it peaks there too, just before intermediaries recover the whole capital stock. The channel shuts off once they hold all of it: households then bear no aggregate risk, and the premium is zero.

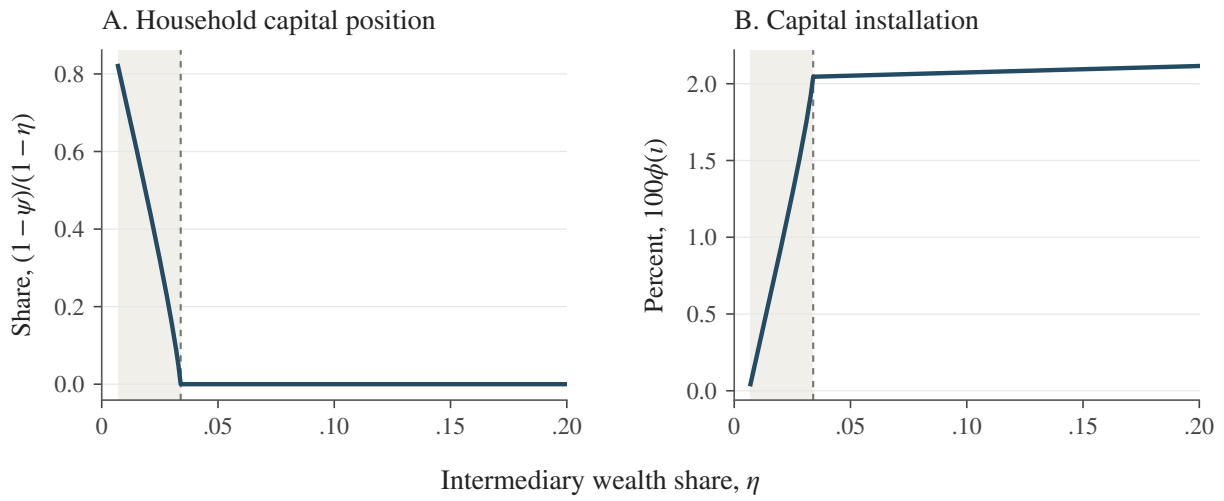


Figure 4: Household capital share and capital installation across intermediary wealth. Left: the household capital share $\underline{x} = (1-\psi)/(1-\eta)$. Right: capital installation $100\phi(\iota)$. The state range and region definitions match Figure 3.

5.3 State Dynamics

The welfare comparisons below depend on the consumption paths reached from an initial financial state, and therefore on the law of motion of intermediary wealth. In levels, the law of motion of Proposition 3 reads

$$d\eta_t = b(\eta_t)\,dt + s(\eta_t)\,dZ_t, \qquad s(\eta) = (\psi(\eta) - \eta)(\sigma + \sigma^q(\eta)), \tag{39}$$

with $b(\eta) = \eta\alpha(\eta)$. Figure 5 reports the computed drift and diffusion. The volatility peak at the switch appears in the diffusion of intermediary wealth as well as in the price of capital. The drift is positive almost everywhere and turns negative only just below the switch, between $\eta \approx 0.031$ and 0.034, where the high volatility of the price of capital lowers the expected growth of leveraged expert wealth. Near the switch the computed drift also differs from the drift implied by the budget identity, because the price derivative changes quickly there; the welfare computations use the computed drift (Appendix C).

We compare welfare at each initial wealth share, with initial capital normalized to one, so that the initial financial condition remains visible and weak balance sheets can be distinguished from strong ones. We do not weight states by the stationary distribution of the computed law of motion. With experts consuming nothing, their wealth share drifts up in almost every state, and the stationary distribution places more than 99.99 percent of its mass above $\eta = 0.99$: in the long run the model economy spends almost no time in the

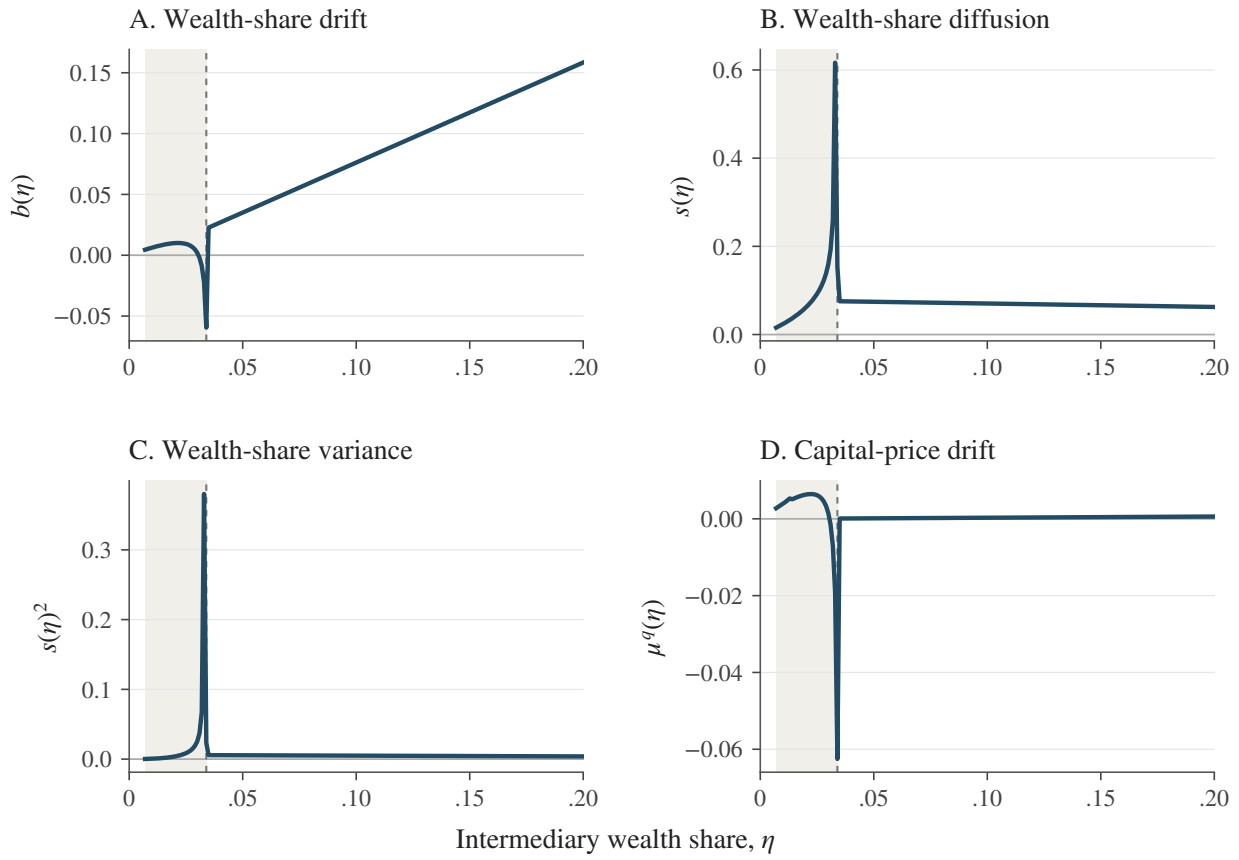


Figure 5: Numerical state and capital-price dynamics. Top row: the level drift b and diffusion s of intermediary wealth. Bottom row: instantaneous variance s^2 and capital-price drift μ^q.

recession region. The model therefore speaks to what happens in recessions rather than to how often they occur, and we treat the two regions as states to compare rather than as frequencies to weight, in the calibration as in the welfare analysis.

5.4 Welfare: the Default Penalty and the Margin Requirement

The bankruptcy law and the margin requirement act on the same economy through different channels. The default penalty sets how much an undercapitalized intermediary withholds and, through the lenders' pricing of that default, how much credit the sector obtains; the margin requirement caps the leverage it can take. We ask what each instrument does to the two agents, holding the other at its calibrated value, and report the answer in the form that Dubey et al. (2005) use for their Example 2: a point in the plane of the two agents' utilities, with the intermediaries' share of wealth at the date the policy is set as a third axis, so that

every initial condition is kept and none is averaged away.

Criteria. Household welfare is the expected discounted utility of consumption from an initial wealth share,

$$W_H(\eta) = \mathbb{E}_\eta\Big[\int_0^\infty e^{-\gamma t}\log \underline{C}_t\, dt\Big], \qquad K_0 = 1, \tag{40}$$

under the law of motion of the previous subsection for the policy in force, which is chosen once and held fixed. Experts consume nothing, so their objective is the flow they maximize in Section 3, the expected growth of net worth net of the penalty they suffer, discounted at the same rate,

$$U_E(\eta) = \mathbb{E}_\eta\Big[\int_0^\infty e^{-\gamma t}\Big(\mu_t^N - \lambda \Upsilon_t\Big)dt\Big], \qquad \mu_t^N = R_t^E + (x_t - 1)(R_t^E - r_t) + D_t, \tag{41}$$

with $D_t = (1-\nu_t)(x_t - 1)(1 + r_t)$ the amount withheld per unit of net worth. Both solve linear equations in the generator of the wealth share on the grid above. Consumption equivalents are $100[\exp(\gamma\,\Delta W_H) - 1]$, the permanent percentage change in consumption that households would accept in exchange for a policy. Because households consume all net output, the logarithm of their consumption separates into the size of the capital stock, the productivity of its allocation between experts and households, and the share of output left after investment (Appendix C); a policy affects household welfare through these three channels.

Admissible penalties. The penalty has a lower bound. The amount withheld per unit of equity is $1/(2\lambda)$, and by Proposition 7 lenders fund debt of up to $1/[2\lambda(\Delta + \mathrm{EBP})]$ per unit of expert equity, where $\Delta = (A - \underline{A})/q + (\underline{\delta} - \delta)$ is the productivity gap and EBP is the households' risk compensation (37). A penalty lenient enough to make this exceed $1/m - 1$, the debt per unit of equity at the leverage ceiling, leaves no rate at which lenders are compensated, and the credit market has no equilibrium. The lowest penalty consistent with equilibrium is therefore

$$\lambda_{\min} = \max_\eta \ \frac{1}{2\,(1/m - 1)\Big[\Delta(\eta) + \mathrm{EBP}(\eta)\Big]}, \tag{42}$$

which is 0.60 at the calibrated margin. The calibrated penalty, 0.62, lies just above this bound. Penalties are compared from $\lambda_{\min}$ upward; Appendix C reports the region of the state space on which the equilibrium conditions are verified for each policy.

The default penalty. Figure 6 shows the welfare space. The solid curve is the calibrated economy across initial conditions: the stronger the position the intermediary sector starts from, the better off households are and the less experts gain, until intermediaries own about a fifth of wealth. Beyond that, household utility is flat up to a wealth share of about one half, and expert utility settles near six because experts who hold all the capital gain and lose the same amount on each unit they withhold. At each marked initial condition the short curve is the menu, the set of utility pairs the law can reach by varying the penalty.

The lowest penalty consistent with equilibrium gives households the highest welfare at every initial condition, and experts the highest welfare whenever they are undercapitalized. The gain from leniency is large in consumption terms. Relative to $\lambda_{\min}$, a penalty of 3 costs households 12 to 15 percent of permanent consumption when intermediaries own less than two percent of wealth and 40 to 48 percent when they own between a tenth and a half; a penalty of 30 costs 44 to 66 percent over the same range (Figure 7, left panel). Experts' utility from a wealth share of 0.005 is 29.5 at $\lambda_{\min}$ against 10.6 at a penalty of 3, and 24.9 against 9.6 from a share of 0.02; from a share of 0.3 upward it is about 6 whatever the penalty.

The reason is the way default is priced. The amount an expert withholds per unit of equity is $1/(2\lambda)$, lenders anticipate it and price it into the contractual rate, and the amount they lend settles where the loss per unit lent equals the gap between the return on capital in expert hands and their own required return. A more lenient penalty raises the withheld amount, spreads the loss over more lending, and raises credit, the expert share of capital and output, which is the allocation channel of household welfare. Default is a transfer from lenders to the productive sector for which lenders are compensated ex ante: a lenient penalty acts as a standing recapitalization of intermediaries, paid for by households through repayment and returned to them through output and the price of capital, because production requires inside equity. This is the channel through which an equity infusion is the most effective crisis policy in He and Krishnamurthy (2013), and through which transfers to the constrained sector raise welfare in Brunnermeier and Sannikov (2014), where the competitive equilibrium is constrained inefficient because experts do not internalize the effect of their net worth on asset prices. It is the mechanism behind the ex post case for bailouts in Bianchi (2016) and Chari and Kehoe (2016), and behind the ex ante efficiency of debt relief under incomplete contracts in Bolton and Rosenthal (2002).

The result is the dynamic counterpart of what Dubey et al. (2005) find in their Section 7 and Zame (1993) proves for incomplete markets: the best penalty lets agents default even when they could pay. The mechanism differs. In their economies default lets each agent

tailor a given promise to its own needs and enlarges the asset span, whereas here the gain is the transfer to the sector whose capital is scarce; the two channels are distinct and can operate together. The model leaves out the enforcement and verification costs that bound leniency from below in their example, to isolate the recapitalization channel; with such costs the optimum would move inside the admissible range, as theirs does.

The margin requirement. Figure 8 shows the welfare space for the margin requirement, drawn as for the penalty: the solid curve is the calibrated economy and each short curve is the set of utility pairs reached by varying the leverage ceiling from a marked initial condition. From a weak initial position, tightening the ceiling from 100 toward 8 moves both agents up and to the right; tightening it further turns the menu back on itself, which is the mark of an interior optimum. From a strong initial position the menu shrinks to a point. The margin requirement matters when intermediaries are poor and not otherwise (Figure 7, right panel).

From a wealth share of 0.005 the ceiling that maximizes the sum of the two utilities is 8 against the calibrated 30, and the gain to households is worth 82 percent of permanent consumption; from 0.02 the ceiling is still 8 and the gain 62 percent; from 0.05 the gain is 32 percent and from 0.10 it is 6 percent. At a wealth share of 0.13 the best ceiling is between 8 and 10, and from about 0.2 upward every ceiling between 5 and 100 gives the same welfare, because the binding limit on leverage is then the intermediaries' ownership of all the capital rather than the rule. The optimal ceiling is state dependent: tight when intermediaries are undercapitalized, loosening as they recover, irrelevant once they are well capitalized. Households gain from the tighter ceiling because it slows the transfer of wealth toward a highly levered sector and lowers the volatility of the price of capital they are exposed to; experts lose the leverage but keep the capital, which is why the two criteria select the same ceiling.

The macroprudential literature argues for this kind of state dependence on other grounds. Geanakoplos (2010) makes leverage the object of regulation over the cycle, Elenev et al. (2021) quantify the trade-off between the safety and the lending capacity of constrained intermediaries, and Bianchi (2011) derives a state-contingent tax on borrowing from the pecuniary externality that levered agents impose through prices. Our result identifies the dimension along which that trade-off is sharpest, the condition of the intermediaries' balance sheets when the rule is set. Brunnermeier and Sannikov (2014) caution that capital requirements set without regard to the state can bind in downturns and harm welfare. Here the ceiling is worth tightening precisely when intermediaries are poor, because it limits how much of a scarce stock of inside equity is levered into a volatile asset, and

so reduces the volatility households bear through the capital they still hold. The two prescriptions answer different questions: Brunnermeier and Sannikov (2014) ask what a requirement fixed across states does in a downturn, while we ask which requirement to set given the state, and the model's answer is that it should move with the state.

The two instruments together. The penalty and the margin requirement differ in kind. The penalty has no interior optimum: households are better off the more lenient it is at every initial condition, experts are too whenever they are undercapitalized, and the calibrated value lies essentially at $\lambda_{\min}$, the lenient end of the range that sustains a credit market. The margin requirement has an interior optimum that moves with the state: a ceiling near 8 when intermediaries own a few percent of wealth, 8 to 10 at an eighth, and no bite above a fifth. The first instrument is a matter of leniency; the second is a matter of timing.

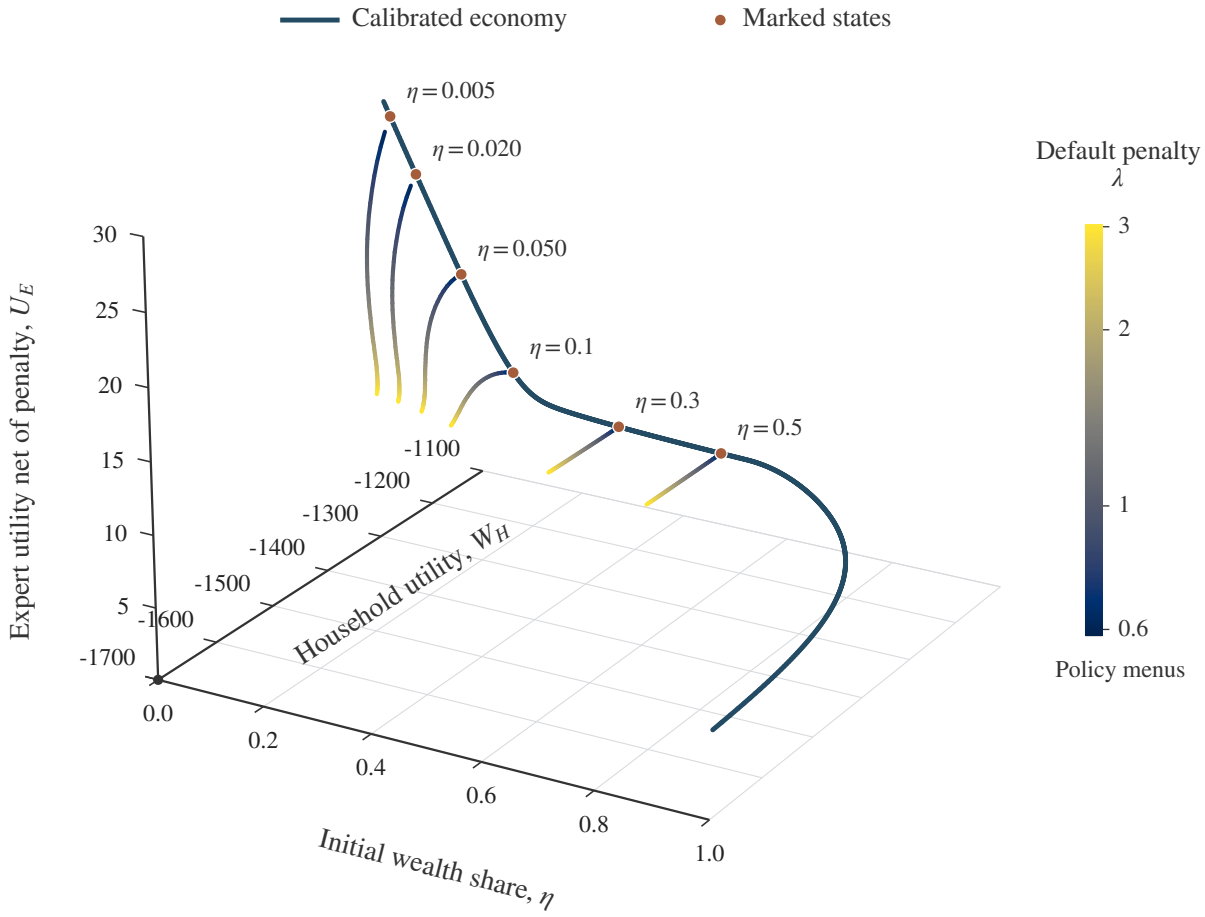


Figure 6: Welfare space at the calibrated margin. The solid navy curve traces the calibrated economy across initial wealth shares; the marked points identify selected initial conditions. At each marked initial condition the short curve is the set of (W_H, U_E) pairs reached by varying the penalty between $\lambda_{\min} = 0.60$ and 3, with color indicating the penalty. The axes share a common origin.

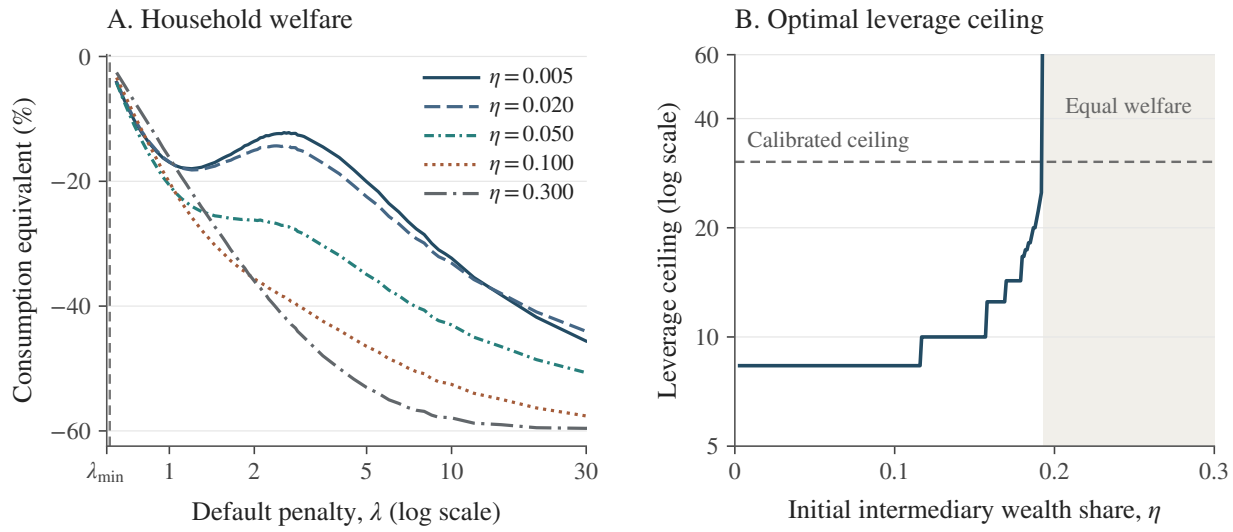


Figure 7: Left: household welfare against the default penalty, as the consumption equivalent relative to $\lambda_{\min}$, essentially the calibrated penalty, at five initial wealth shares; the axis starts at $\lambda_{\min} = 0.60$, the lowest penalty consistent with equilibrium, and the dashed line marks the calibrated penalty, 0.62. Right: the leverage ceiling that maximizes the sum of the two utilities, by initial wealth share; the dashed line is the calibrated ceiling of 30, and in the shaded range every ceiling gives the same welfare.

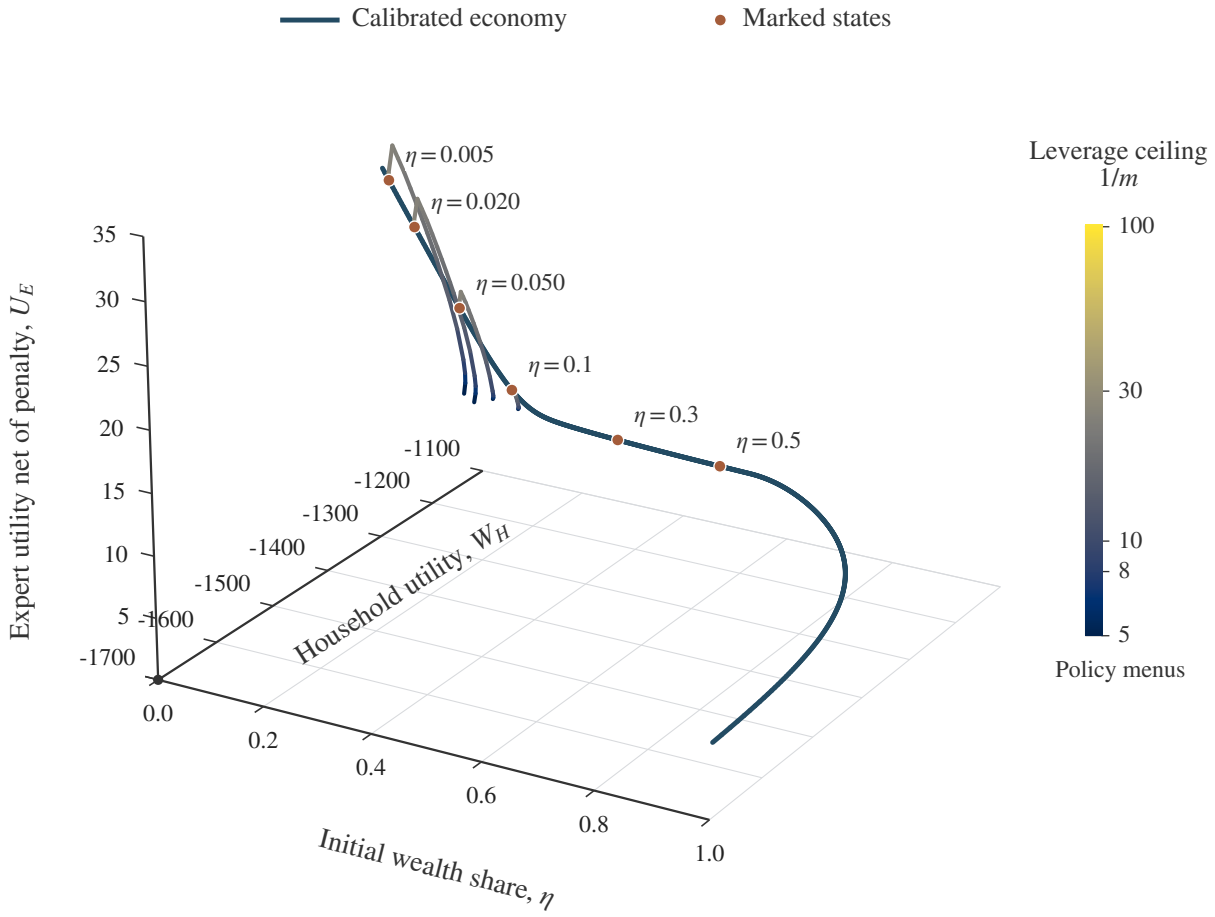


Figure 8: Welfare space at the calibrated penalty. The solid navy curve traces the calibrated economy across initial wealth shares; the marked points identify selected initial conditions. At each marked initial condition the short curve is the set of (W_H, U_E) pairs reached by varying the leverage ceiling between 5 and 100, with color indicating the ceiling. The axes share a common origin.

6 Conclusion

We have argued that the excess bond premium, the part of corporate credit spreads that default risk does not explain, is the price of intermediaries' risk-bearing capacity, and that it rises when their capital is scarce. The motivating evidence is consistent with this reading. When primary dealers' balance sheets weaken, the premium rises, even after accounting for changes in default risk; the premium and default risk rise together in periods of severe stress; and a higher premium predicts weaker investment, beyond what dealer leverage and default conditions predict.

We develop a model to explain these empirical findings. When intermediaries are well capitalized, they hold all of the economy's productive capital and households simply lend to them. When intermediaries are short of capital, households must hold some of that capital themselves and demand compensation for the risk. That compensation is the premium, and it is largest when the price of capital is most volatile. Calibrated to U.S. data, the model reproduces most of the premium in recessions and its absence in expansions, and, as in the data, almost all of the rise in spreads in recessions comes from the premium rather than from default risk.

The model also speaks to policy. Because lenders anticipate default and price it into the interest rate, default works as a transfer to intermediaries rather than as a pure loss, and households are repaid through higher output and asset prices. A lenient bankruptcy law therefore acts as a standing recapitalization of the financial sector, and the most lenient penalty that still sustains a credit market is best for households. Leverage regulation works differently. A tighter leverage ceiling helps only when intermediaries are poor, by limiting how much of their scarce capital is exposed to volatile asset prices; once they are well capitalized, it has no effect. When intermediary capital is scarce, the two rules therefore point in opposite directions: lenient on default, tight on leverage.

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A Additional Empirical Results

A.1 Predictive Regressions and Inference

Table 6 reports the coefficients on EBP and intermediary leverage in equation (3), with and without the nonlinear default controls. All regressions include current own growth, the term spread and the real federal funds rate. The target is strictly forward growth from t to $t + h$, annualized as in Section 2.3. Predictors and outcomes are drawn from revised data, so these are in-sample predictive relationships rather than an evaluation of real-time forecasting performance.

HAC covariance estimates use Bartlett weights and the finite-sample factor $n/(n - k)$, where k includes the intercept. We use 12 lags for the monthly regressions and four for quarterly predictive regressions; confidence intervals use a normal reference distribution. Lags and differences are constructed before sample restrictions. When an interval is excluded, the covariance calculation retains the actual calendar distance between the remaining observations. Quarterly sample exclusions also remove any forecast origin or target window overlapping the financial-crisis recession. Spread components are treated as observed; the aggregate series do not permit us to recover uncertainty from their original bond-level estimation.

Table 6: Predictive Content of EBP and Leverage across Horizons

	One quarter ahead		Four quarters ahead	
	Joint	+ default	Joint	+ default
GDP				
EBP	−3.048	−2.925	−1.207	−1.190
	(1.074)	(1.109)	(0.344)	(0.358)
Intermediary leverage	−0.055	−0.079	−0.042	−0.064
	(0.072)	(0.084)	(0.047)	(0.063)
Consumption				
EBP	−2.250	−2.243	−0.804	−0.677
	(1.200)	(1.255)	(0.311)	(0.305)
Intermediary leverage	−0.113	−0.142	−0.080	−0.079
	(0.072)	(0.089)	(0.053)	(0.074)
Investment				
EBP	−7.196	−6.872	−5.348	−5.887
	(1.509)	(1.472)	(1.231)	(1.444)
Intermediary leverage	−0.067	−0.008	−0.082	−0.146
	(0.136)	(0.148)	(0.137)	(0.155)
Quarterly observations	157	157	154	154

Notes: The dependent variable is strictly forward annualized growth, $(400/h)[\log Y_{t+h} - \log Y_t]$, for $h = 1$ or $h = 4$. Entries are regression coefficients, with Bartlett HAC(4) standard errors in parentheses; the covariance estimator includes a finite-sample correction. EBP coefficients are percentage points of annualized growth per 100 basis points of EBP, and leverage coefficients are percentage points of annualized growth per unit of leverage. Joint specifications include both EBP and leverage, current own growth, $400[\log Y_t - \log Y_{t-1}]$, the term spread and the real federal funds rate. The additional default controls are the default-related spread component and its square. All specifications use the same observations within each outcome and horizon. The first origin is 1985Q1 and the final outcome quarter is 2024Q2.

A.2 Dynamics Following Capital Deterioration

To follow financial and real variables after the same observed balance-sheet movement, we estimate a separate regression for each outcome and horizon $h = 0, \ldots, 8$:

$$v_{t,h}^{j} = \alpha_h^j + \psi_h^j a_t^q + \Gamma_h^{j\prime} W_{t-1} + \chi_h^j r_t^e + u_{t,h}^j. \tag{43}$$

Here $a_t^q = -100\Delta \log c_t / s_q$ is observed quarterly capital-ratio deterioration, normalized by its full-sample standard deviation $s_q = 13.27$ log percentage points. The outcome is $v_{t,h}^{\mathrm{EBP}} = \mathrm{EBP}_{t+h} - \mathrm{EBP}_{t-1}$ for the premium and $v_{t,h}^j = 100(\log Y_{t+h}^j - \log Y_{t-1}^j)$ for GDP and investment. These cumulative changes include the current quarter and therefore differ from the strictly forward forecasting targets.

The vector W_{t-1} contains one lag of EBP, the log capital ratio, the default-related

component, GDP growth, investment growth, inflation, and the federal funds rate; r_t^e is the current CRSP excess market return. HAC bandwidth is $\max\{4, h+1\}$. Figure 9 reports full-sample and pre-pandemic estimates using the same capital-ratio normalization. Dark and light bands denote 68% and 95% pointwise intervals. All 54 outcome–sample–horizon estimates have 95% intervals that include zero. The estimates do not identify an exogenous shock, and the bands are pointwise rather than simultaneous over the whole path.

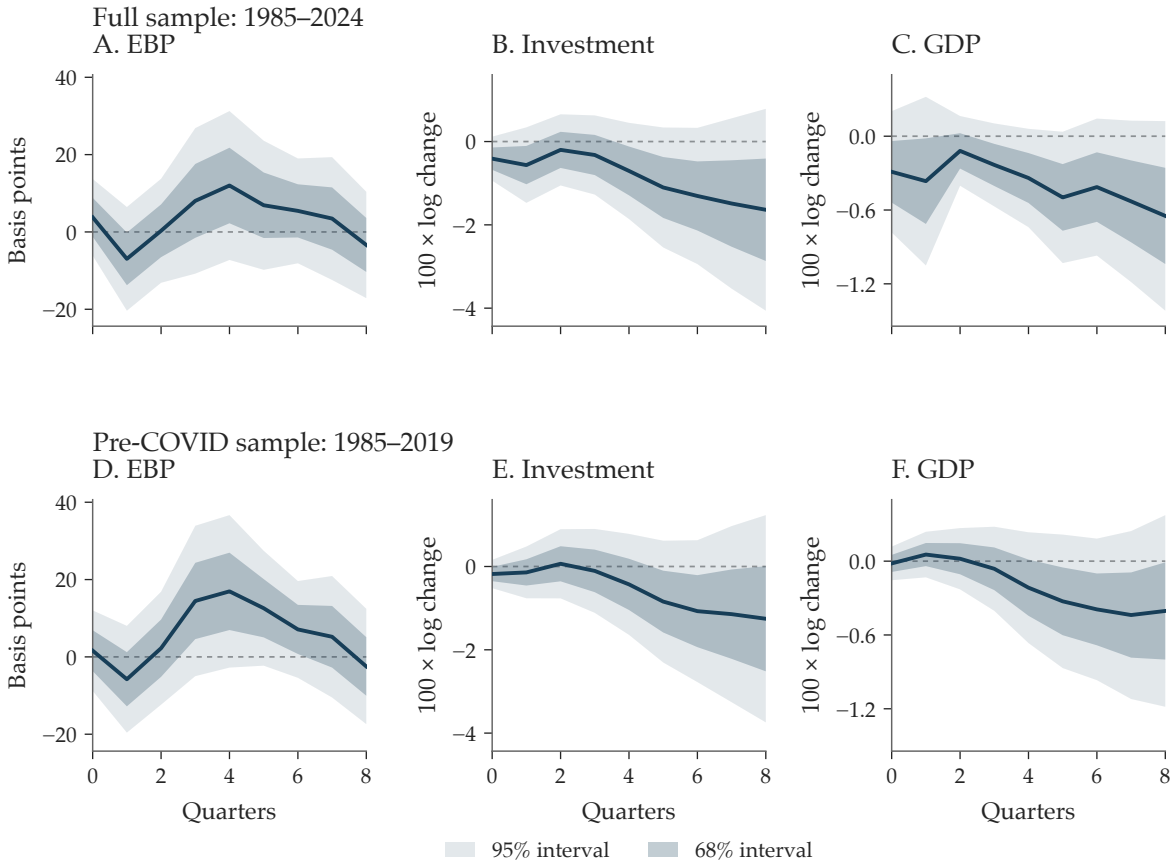


Figure 9: Financial and real activity following capital-ratio deterioration.

Notes: Estimates of ψ_h^j in equation (43). A unit event is a one-standard-deviation decline in the log intermediary capital ratio. Full-sample origins start in 1985Q1 and final outcomes end in 2024Q2; pre-pandemic outcomes end in 2019Q4. EBP changes are in basis points; GDP and investment are 100 times cumulative log changes. Dark and light bands are 68% and 95% normal-reference HAC intervals.

B Model Derivations

B.1 Returns on Capital

We derive the total return on capital for experts; the household case is analogous. Applying Itô's product rule to $q_t k_t$,

$$d(q_t k_t) = q_t\, dk_t + k_t\, dq_t + dq_t\, dk_t.$$

Substituting the capital evolution (4) and the price process (6),

$$\begin{aligned}\frac{d(q_t k_t)}{q_t k_t} &= \frac{dk_t}{k_t} + \frac{dq_t}{q_t} + \frac{dk_t}{k_t}\cdot\frac{dq_t}{q_t} \\ &= \Big(\phi(\iota_t) - \delta\Big)\,dt + \sigma\, dZ_t + \mu_t^q\, dt + \sigma_t^q\, dZ_t + \sigma\sigma_t^q\, dt \\ &= \Big(\phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q\Big)\,dt + (\sigma + \sigma_t^q)\, dZ_t.\end{aligned}$$

Adding the dividend yield $(A - \iota_t)/q_t$ gives the total return on expert capital stated in equation (7):

$$dr_t^k = \frac{A - \iota_t}{q_t}\, dt + \Big(\phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q\Big)\,dt + (\sigma + \sigma_t^q)\, dZ_t.$$

For households, replacing (A, δ) with $(\underline{A}, \underline{\delta})$ yields

$$d\underline{r}_t^k = \frac{\underline{A} - \underline{\iota}_t}{q_t}\, dt + \Big(\phi(\underline{\iota}_t) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\Big)\,dt + (\sigma + \sigma_t^q)\, dZ_t.$$

B.2 Expert Net Worth Evolution

Over $(t, t + dt)$ the expert earns output net of investment $(A - \iota_t)k_t\, dt$, realizes capital gains $d(q_t k_t)$, pays interest $\nu_t d_t r_t\, dt$, and retains $(1 - \nu_t)d_t\, dt$ of the principal it owes through default. Collecting terms,

$$dn_t = (A - \iota_t)k_t\, dt + d(q_t k_t) - \nu_t d_t r_t\, dt + (1 - \nu_t)d_t\, dt.$$

Dividing through by n_t and using $d_t = (x_t - 1)n_t$ together with (7),

$$\frac{dn_t}{n_t} = x_t\, dr_t^k - \nu_t(x_t - 1)(1 + r_t)\, dt + (x_t - 1)\, dt = x_t\, dr_t^k - (x_t - 1)\Big[\nu_t(1 + r_t) - 1\Big]dt,$$

which is equation (11).

B.3 Household Net Worth Evolution

The household receives output $(\underline{A} - \underline{\iota}_t)\underline{k}_t\,dt$, realizes capital gains $d(q_t\underline{k}_t)$, receives interest $\nu_t\underline{d}_t r_t\,dt$ from experts, suffers the default loss $(1-\nu_t)\underline{d}_t\,dt$ on the face value of bonds, and pays for consumption $\underline{c}_t\,dt$. The dynamic budget constraint is therefore

$$d\underline{n}_t = (\underline{A} - \underline{\iota}_t)\underline{k}_t\,dt + d(q_t\underline{k}_t) - \underline{c}_t\,dt + \nu_t\underline{d}_t r_t\,dt - (1-\nu_t)\underline{d}_t\,dt. \tag{44}$$

Dividing by $\underline{n}_t$ and using $\underline{x}_t \equiv q_t\underline{k}_t/\underline{n}_t$ and $1 - \underline{x}_t = \underline{d}_t/\underline{n}_t$,

$$\frac{d\underline{n}_t}{\underline{n}_t} = \underline{x}_t\,d\underline{r}_t^k + (1-\underline{x}_t)\Big[\nu_t(1+r_t) - 1\Big]dt - \frac{\underline{c}_t}{\underline{n}_t}\,dt, \tag{45}$$

which is equation (18).

B.4 Market Clearing

Capital market. The total capital held by experts and households equals the aggregate supply:

$$\int_0^1 k_t^i\,di + \int_1^2 \underline{k}_t^j\,dj = K_t, \tag{46}$$

where aggregate capital evolves as

$$dK_t = \left(\int_0^1 \Big(\phi(\iota_t^i) - \delta\Big)k_t^i\,di + \int_1^2 \Big(\phi(\underline{\iota}_t^j) - \underline{\delta}\Big)\underline{k}_t^j\,dj\right)dt + \sigma K_t\,dZ_t. \tag{47}$$

Debt market. The face value of expert debt equals the face value of household bond holdings:

$$\int_0^1 d_t^i\,di = \int_1^2 \underline{d}_t^j\,dj. \tag{48}$$

Using the balance sheet identities $d_t^i = q_t k_t^i - n_t^i$ and $\underline{d}_t^j = \underline{n}_t^j - q_t\underline{k}_t^j$ and integrating, debt market clearing implies

$$N_t + \underline{N}_t = q_t K_t, \tag{49}$$

which is equation (20) in the main text.

Goods market. Output produced by experts and households equals household consumption:

$$\int_0^1 k_t^i(A - \iota_t^i)\,di + \int_1^2 \underline{k}_t^j(\underline{A} - \underline{\iota}_t^j)\,dj = \int_1^2 \underline{c}_t^j\,dj. \tag{50}$$

Using $\underline{c}_t^j = \gamma \underline{n}_t^j$ and the leverage definitions, this simplifies to the condition that net output equals aggregate household consumption, pinning down the investment rates jointly with (9).

Aggregate repayment rate. The macro repayment rate is the debt-weighted average of individual repayment choices:

$$\nu_t = \begin{cases} \dfrac{\int_0^1 \nu_t^i \, \mu_t^i \, di}{\int_0^1 \mu_t^i \, di}, & \text{if } \int_0^1 \mu_t^i \, di \neq 0, \\ \text{arbitrary}, & \text{otherwise}, \end{cases}$$

where ν_t^i is the repayment rate of expert i and μ_t^i is the amount owed by expert i to households. In a symmetric equilibrium, $\nu_t^i = \nu_t$ for all i.

B.5 Formal Equilibrium Definition

Definition. A stochastic equilibrium with leverage and default consists of stochastic processes for the capital price $\{q_t, t \geq 0\}$, the interest rate $\{r_t, t \geq 0\}$, expert decisions $\{(k_t^i, \iota_t^i, \nu_t^i), t \geq 0\}$ for $i \in (0, 1]$, and household decisions $\{(\underline{k}_t^j, \underline{\iota}_t^j, \underline{c}_t^j), t \geq 0\}$ for $j \in (1, 2]$, such that:

1. Given $\{q_t, r_t\}$, each expert i maximizes (13) subject to (11) and (12), and each household j maximizes expected lifetime utility subject to (18) and $\underline{x}_t \geq 0$.

2. The capital market (46), debt market (48), and goods market (50) all clear.

The leverage constraint limits how much capital productive experts can finance, while strategic repayment changes the returns received by households. These frictions motivate the welfare comparisons. As in the collateral-equilibrium setting of Geanakoplos and Polemarchakis (1985), a claim of constrained inefficiency requires a comparison with feasible allocations that respect the relevant financial constraints.

C Numerical Solution and Welfare Computation

This appendix records how the equilibrium functions of Section 5 are computed, which equilibrium conditions are verified on them and where, and how the welfare measures of Section 5.4 are obtained from them.

C.1 Solution of the equilibrium functions

The state space is a grid of 1,000 equally spaced wealth shares between 0.001 and 0.999. The solution proceeds from the lowest wealth share upward, one state at a time, in three branches that correspond to the regions of the equilibrium.

On the interior branch, where households hold part of the capital, the three conditions of Proposition 9 are solved jointly for the capital price q, the expert capital share ψ and the endogenous price volatility σ^q at each state by Newton's method, with the price derivative $q'(\eta)$ taken as the finite difference from the preceding state. The first state at which ψ reaches one begins the full-ownership branch. There experts hold all the capital, and goods-market clearing with $\psi = 1$ gives the price in closed form,

$$q_1(\eta) = \sqrt{\left[\theta\gamma(1-\eta)\right]^2 + 1 + 2\theta A} - \theta\gamma(1-\eta), \tag{51}$$

with its derivative taken analytically; the volatility of the price then follows from the volatility identity of Proposition 6. On both branches the promised rate follows from Proposition 5 and the repayment fraction from Proposition 2. The repayment fraction falls as leverage falls, and at the state where it reaches zero the solution switches to the complete-default branch, on which repayment is zero and the rate solves the expert's indifference condition with no debt relief. At the calibrated parameters that state is $\eta = 0.56$, far above the comparison range, and nothing reported in the paper depends on it. The margin enters only where leverage would exceed $1/m$, which does not happen at the calibrated parameters.

C.2 Verification of the equilibrium conditions

The computed functions are checked against the conditions of Section 4 at every state: the expert's repayment condition, that the amount withheld per unit of equity equals $1/(2\lambda)$ where repayment is partial; the household's portfolio condition, that the return on the capital households hold exceeds their effective return on debt by their risk compensation $\underline{x}(\sigma + \sigma^q)^2$ where they hold capital and falls short of it where they do not; the expert's leverage condition, that the promised rate equals the return on capital in expert hands where leverage is interior; and non-negativity of the effective return on debt.

At the calibrated margin the conditions hold for penalties of 0.60 and above, and for wealth shares up to a bound that rises with the penalty: 0.03 at the calibrated penalty, 0.09 at a penalty of 2, 0.33 at 10 and 0.59 at 30. The lower bound on the penalty is the one derived in Section 5.4: below it the leverage lenders are willing to fund exceeds the margin at every

state, so no state satisfies the household and expert conditions together. The upper bound on the wealth share comes from the full-ownership branch. With experts holding all the capital, debt per unit of equity is $1/\eta - 1$ and the loss per unit lent, $1/[2\lambda(1/\eta - 1)]$, rises with the wealth share; once it exceeds the productivity gap, the promised rate that compensates lenders exceeds the return experts earn on capital, and the lenders' participation condition fails. The bound is $\bar{\eta}(\lambda) = \Delta/[\Delta + 1/(2\lambda)]$ with Δ the productivity gap per unit of capital value. The comparison range of Section 5, which ends at a wealth share of 0.12, therefore contains verified equilibrium states on the interior branch and, at the calibrated penalty, extends beyond the verified bound on the full-ownership branch. On states where a margin binds, the household's portfolio condition does not hold, because households would lend more than the capped experts can borrow at the going rate; the margin comparisons of Section 5.4 are comparisons of the computed functions on those states.

Two further properties of the computed functions are recorded for completeness. The drift of the wealth share that the solution produces and the drift implied by the budget identity of Proposition 3 differ on the interior branch near the allocation switch, where the price derivative changes quickly; the welfare computations use the computed drift. And the stationary density of the computed law of motion places its mass at the upper boundary of the grid, which is why the regional means of Section 5 are means over states and the welfare comparisons are made state by state rather than under a stationary weighting.

C.3 Welfare measures

Fix a policy, held constant forever, and an initial capital stock normalized to one. Households consume all net output, so their consumption is $\underline{C}_t = K_t\, c(\eta_t)$ with

$$c(\eta) = A(\eta) - \iota(\eta), \qquad A(\eta) = A\psi(\eta) + \underline{A}\Big[1 - \psi(\eta)\Big], \tag{52}$$

and in equilibrium $c(\eta) = \gamma(1-\eta)q(\eta)$. Taking logarithms separates household utility into the size of the capital stock, the productivity of its allocation, and the share of output left after investment,

$$\log \underline{C}_t = \log K_t + \log A(\eta_t) + \log\Big(1 - \frac{\iota(\eta_t)}{A(\eta_t)}\Big). \tag{53}$$

Writing the wealth share as $d\eta_t = b(\eta_t)\,dt + s(\eta_t)\,dZ_t$, its generator is $\mathcal{L}f = bf' + \frac{1}{2}s^2 f''$, and the drift of log capital is

$$\mu^K(\eta) = \phi(\iota(\eta)) - \delta\psi(\eta) - \underline{\delta}\Big[1 - \psi(\eta)\Big] - \tfrac{1}{2}\sigma^2, \tag{54}$$

which under zero depreciation is $(q-1)/\theta - \sigma^2/2$. Household welfare then separates as $W_H(K,\eta) = \log K/\gamma + w(\eta)$ with

$$(\gamma - \mathcal{L})\, w = \log c + \frac{\mu^K}{\gamma}, \tag{55}$$

and solving the same equation separately for the capital, allocation and investment terms of (53) gives the three additive components of welfare. The experts' objective is obtained from the same generator,

$$(\gamma - \mathcal{L})\, U_E = \mu^N - \lambda \Upsilon, \tag{56}$$

with μ^N the expected growth of net worth of Section 5.4. Both equations are solved on the grid with reflecting boundaries, after which the component equations and the consumption equivalents $100[\exp(\gamma \Delta W_H) - 1]$ follow directly.

The penalty is varied from 0.3 to 100 at the calibrated margin, and the leverage ceiling from 1 to 100 at the calibrated penalty; each policy is solved as a separate economy and compared at every initial state. Two margins are treated as tied at an initial state when their welfare differs by less than one twentieth of a utility unit, which is how the shaded range of Figure 7 is defined.

D Proofs and Extensions

D.1 Derivatives of the price of capital

The first and second derivatives of the capital price with respect to η_t determine its drift and volatility through Itô's lemma.

Corollary 1. *The first derivative of the price of capital, q_t, with respect to η_t, is given by*

$$q'(\eta_t) \triangleq \frac{\partial q_t(\eta_t)}{\partial \eta_t} \;=\; \theta\,\gamma \;+\; \frac{-\,\theta^2\,\gamma^2\,(1-\eta_t) \;+\; \theta\,(A-\underline{A})\left[x_t'(\eta_t)\,\eta_t + x_t(\eta_t)\right]}{\sqrt{M(\eta_t)}}, \tag{57}$$

where we defined

$$M(\eta_t) \triangleq \theta^2\,\gamma^2\,(1-\eta_t)^2 \;+\; 2\,\theta\left[x_t(\eta_t)\,\eta_t\,A + \left(1 - x_t(\eta_t)\,\eta_t\right)\underline{A}\right] \;+\; 1. \tag{58}$$

The second derivative of the price of capital, q_t, with respect to η_t, is given by

$$\frac{\partial^2 q_t(\eta_t)}{\partial \eta_t^2} = \frac{\theta^2\gamma^2 + \theta(A-\underline{A})[x_t''(\eta_t)\eta_t + 2x_t'(\eta_t)]}{[M(\eta_t)]^{1/2}} - \frac{\left[-\theta^2\gamma^2(1-\eta_t) + \theta(A-\underline{A})\left[x_t'(\eta_t)\eta_t + x_t(\eta_t)\right]\right]^2}{[M(\eta_t)]^{3/2}}. \tag{59}$$

The derivative of $q_t(\eta_t)$ describes how the capital price changes with experts' share of aggregate wealth. Equation equation (57) separates the household-consumption terms in goods clearing from the effect of reallocating production.

- The term $\theta\gamma$ comes from household consumption in the goods-market price equation. A higher expert wealth share reduces household consumption at a given capital price and allocation.
- The square-root term contributes $-\theta^2\gamma^2(1-\eta_t)$, which is negative while households hold wealth. It adjusts the direct consumption effect above. Its sign alone does not determine the slope or curvature of the capital price.
- The allocation term,

  $$\theta(A-\underline{A})\Big[x_t'(\eta_t)\eta_t + x_t(\eta_t)\Big],$$

  depends on the change in experts' capital share. When expert wealth finances more capital, production shifts toward the higher productivity A. A binding constant leverage ceiling removes the change in leverage, but greater equity still raises experts' capital holdings. Once experts hold all capital, the allocation term vanishes.

The denominator $\sqrt{M(\eta_t)}$ comes from the quadratic goods-market equation. The price response therefore depends jointly on household consumption and the allocation of capital; its shape must be evaluated using the equilibrium leverage function.

D.2 Equivalence to Brunnermeier–Sannikov (2014)

The limit in which the penalty is prohibitively harsh connects the model to the intermediary mechanism in Brunnermeier and Sannikov (2014), while keeping the leverage constraint and the endogenous lending rate. The proposition concerns the expert's leverage choice in that limit.

Proposition 10. *(Equivalence theorem) Let $\lambda = +\infty$. The equilibrium value of x_t always equals its upper limit U_t.*

D.3 Proof of Proposition 8

The calculation below derives the interior household conditions. At a portfolio boundary, the corresponding equality is replaced by its Kuhn–Tucker inequality.

The value function of the household is given by the objective in (17)

$$\max_{\underline{\iota}_t, \underline{x}_t, \underline{c}_t} \mathbb{E}\left[\int_0^{+\infty} e^{-\gamma t} \log\left(\underline{c}_t\right) dt\right].$$

The net worth evolution is given by equation (18)

$$\frac{d\underline{n}_t}{\underline{n}_t} = \underline{x}_t d\underline{r}_t^k + \nu_t(1-\underline{x}_t)(1+r_t)dt - \left(1-\underline{x}_t\right)dt - \frac{\underline{c}_t}{\underline{n}_t}dt.$$

Recall that the net worth postulate is given by equation (22)

$$\frac{d\eta_t}{\eta_t} = \alpha(\eta_t)dt + \beta(\eta_t)dZ_t$$

and the consumption to net worth ratio is $\underline{\rho}_t = \frac{\underline{c}_t}{\underline{n}_t}$. Let the value function of the household be

$$V\left(t, \underline{n}_t, \eta_t\right) = \sup_{\underline{x}_t \geq 0, \underline{\iota}_t \geq 0, \underline{\rho}_t \geq 0} \mathbb{E}_t\left[\int_t^{+\infty} e^{-\gamma s}\left[\log\left(\underline{\rho}_t \underline{n}_t\right)\right] ds\right] \tag{60}$$

The HJB equation that the value function V must satisfy to be optimized is given by

$$\gamma V\left(t, \underline{n}_t, \eta_t\right) = \sup_{\underline{x}_t \geq 0, \underline{\iota}_t \geq 0, \underline{\rho}_t \geq 0} \left\{e^{-\gamma t}\log(\underline{c}_t) + \mathbb{E}_t\left[dV\left(t, \underline{n}_t, \eta_t\right)\right]\right\}.$$

Applying Itô's lemma gives the following state-derivative terms.

$$\mathbb{E}_t\left[dV\left(t, \underline{n}_t, \eta_t\right)\right] = \frac{\partial V}{\partial \underline{n}_t}\mu_t^{\underline{n}}\underline{n}_t + \frac{1}{2}\frac{\partial^2 V}{\partial \underline{n}_t^2}(\sigma_t^{\underline{n}}\underline{n}_t)^2 + \frac{\partial V}{\partial \eta_t}\mu_t^{\eta}\eta_t + \frac{1}{2}\frac{\partial^2 V}{\partial \eta_t^2}(\sigma_t^{\eta}\eta_t)^2.$$

Therefore, the HJB equation is

$$\gamma V(t, \underline{n}_t, \eta_t) = \sup_{\underline{x}_t \geq 0, \underline{\iota}_t \geq 0, \underline{\rho}_t \geq 0} \left\{e^{-\gamma t}\log\left(\underline{c}_t\right) + \frac{\partial V}{\partial \underline{n}_t}\mu_t^{\underline{n}}\underline{n}_t + \frac{1}{2}\frac{\partial^2 V}{\partial \underline{n}_t^2}(\sigma_t^{\underline{n}}\underline{n}_t)^2 + \frac{\partial V}{\partial \eta_t}\mu_t^{\eta}\eta_t + \frac{1}{2}\frac{\partial^2 V}{\partial \eta_t^2}(\sigma_t^{\eta}\eta_t)^2\right\}, \tag{61}$$

where the drifts and the volatilities of $\underline{n}_t$ and η_t are given by (18) and (22). We conjecture

that the value function takes the form

$$V \triangleq V(t, \underline{n}_t, \eta_t) = e^{-\gamma t}\left(f(\eta_t) + \frac{\log\left(\underline{n}_t\right)}{\gamma}\right),$$

for some function f. The derivatives of this value function guess are

$$\begin{aligned}
\frac{\partial V}{\partial \underline{n}_t} &= e^{-\gamma t}\frac{1}{\gamma \underline{n}_t}, \\
\frac{\partial^2 V}{\partial \underline{n}_t^2} &= -e^{-\gamma t}\frac{1}{\gamma \underline{n}_t^2}, \\
\frac{\partial V}{\partial \eta_t} &= e^{-\gamma t} f'(\eta_t), \\
\frac{\partial^2 V}{\partial \eta_t^2} &= e^{-\gamma t} f''(\eta_t).
\end{aligned}$$

Substitute this into the HJB equation

$$\gamma V = \sup_{\substack{\underline{x}_t \geq 0 \\ \underline{\iota}_t \geq 0 \\ \underline{\rho}_t \geq 0}} \left\{ e^{-\gamma t}\log\left(\underline{c}_t\right) + \frac{\partial V}{\partial \underline{n}_t}\mu_t^n \underline{n}_t + \frac{1}{2}\frac{\partial^2 V}{\partial \underline{n}_t^2}(\sigma_t^n \underline{n}_t)^2 + \frac{\partial V}{\partial \eta_t}\mu_t^\eta \eta_t + \frac{1}{2}\frac{\partial^2 V}{\partial \eta_t^2}(\sigma_t^\eta \eta_t)^2 \right\}$$

$$\leftrightarrow \quad \gamma f(\eta_t) + \log\left(\underline{n}_t\right) = \sup_{\substack{\underline{x}_t \geq 0 \\ \underline{\iota}_t \geq 0 \\ \underline{\rho}_t \geq 0}} \left\{ \log\left(\underline{c}_t\right) + \frac{1}{\gamma \underline{n}_t}\mu_t^n \underline{n}_t - \frac{1}{2}\frac{1}{\gamma \underline{n}_t^2}(\sigma_t^n \underline{n}_t)^2 + f'(\eta_t)\mu_t^\eta \eta_t + \frac{1}{2}f''(\eta_t)(\sigma_t^\eta \eta_t)^2 \right\}.$$

The right-hand side is

$$\begin{aligned}
& \log\left(\underline{c}_t\right) + \frac{1}{\gamma \underline{n}_t}\mu_t^n \underline{n}_t - \frac{1}{2}\frac{1}{\gamma \underline{n}_t^2}(\sigma_t^n \underline{n}_t)^2 + f'(\eta_t)\mu_t^\eta \eta_t + \frac{1}{2}f''(\eta_t)(\sigma_t^\eta \eta_t)^2 \\
\leftrightarrow \quad & \log\left(\underline{c}_t\right) + \frac{1}{\gamma \underline{n}_t}\mu_t^n \underline{n}_t - \frac{1}{2}\frac{1}{\gamma \underline{n}_t^2}(\sigma_t^n \underline{n}_t)^2 + f'(\eta_t)\alpha(\eta_t)\eta_t + \frac{1}{2}f''(\eta_t)(\beta(\eta_t)\eta_t)^2.
\end{aligned}$$

The net worth of household in equation (18) can be written as

$$\frac{d\underline{n}_t}{\underline{n}_t} = \underline{x}_t d\underline{r}_t^k + \left[\nu_t(1-\underline{x}_t)(1+r_t) - \left(1-\underline{x}_t\right) - \frac{\underline{c}_t}{\underline{n}_t}\right]dt,$$

and take the household's return on capital in equation (7) with $(\underline{A}, \underline{\delta})$ in place of (A, δ)

$$d\underline{r}_t^k = \left[\frac{\underline{A} - \underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right]dt + \left(\sigma + \sigma_t^q\right)dZ_t$$

to get that the drift of net worth is

$$\begin{aligned}
& \mu_t^{\underline{n}} = \Big[\nu_t(1-\underline{x}_t)(1+r_t) - \Big(1-\underline{x}_t\Big) - \frac{\underline{c}_t}{\underline{n}_t}\Big] + \underline{x}_t\Big[\frac{\underline{A}-\underline{\iota}_t}{q_t} + \phi\Big(\underline{\iota}_t\Big) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\Big], \\
\leftrightarrow \quad & \mu_t^{\underline{n}} = \Big[\nu_t(1-\underline{x}_t)(1+r_t) - \Big(1-\underline{x}_t\Big) - \frac{\underline{c}_t}{\underline{n}_t}\Big] + \underline{x}_t E\Big[\frac{d\underline{r}_t^k}{dt}\Big], \\
\Rightarrow \quad & \mu_t^{\underline{n}} = (1-\underline{x}_t)[\nu_t(1+r_t)-1] - \frac{\underline{c}_t}{\underline{n}_t} + \underline{x}_t E\Big[\frac{d\underline{r}_t^k}{dt}\Big].
\end{aligned}$$

The volatility of net worth is

$$\sigma_t^{\underline{n}} = \underline{x}_t(\sigma + \sigma_t^q).$$

Substitute these into the right-hand side of the HJB

$$\begin{aligned}
& \log\Big(\underline{c}_t\Big) + \frac{1}{\gamma \underline{n}_t}\mu_t^{\underline{n}}\underline{n}_t - \frac{1}{2}\frac{1}{\gamma \underline{n}_t^2}(\sigma_t^{\underline{n}}\underline{n}_t)^2 + f'(\eta_t)\alpha(\eta_t)\eta_t + \frac{1}{2}f''(\eta_t)(\beta(\eta_t)\eta_t)^2 \\
\leftrightarrow \quad & \log\Big(\underline{c}_t\Big) + \frac{1}{\gamma \underline{n}_t}\Big[\underline{n}_t(1-\underline{x}_t)[\nu_t(1+r_t)-1] - \underline{c}_t + \underline{n}_t\underline{x}_t E\Big[\frac{d\underline{r}_t^k}{dt}\Big]\Big] - \frac{1}{2}\frac{1}{\gamma \underline{n}_t^2}\Big(\underline{n}_t\underline{x}_t(\sigma+\sigma_t^q)\Big)^2 \\
& + f'(\eta_t)\alpha(\eta_t)\eta_t + \frac{1}{2}f''(\eta_t)\Big(\beta(\eta_t)\eta_t\Big)^2 \\
\Rightarrow \quad & \log\Big(\underline{c}_t\Big) + \frac{1}{\gamma \underline{n}_t}\Big[\underline{x}_t\underline{n}_t\Big(E\Big[\frac{d\underline{r}_t^k}{dt}\Big] - \nu_t(1+r_t) + 1\Big) + \underline{n}_t\Big[\nu_t(1+r_t)-1\Big] - \underline{c}_t\Big] - \frac{\Big(\underline{x}_t(\sigma+\sigma_t^q)\Big)^2}{2\gamma} \\
& + f'(\eta_t)\alpha(\eta_t)\eta_t + \frac{1}{2}f''(\eta_t)\Big(\beta(\eta_t)\eta_t\Big)^2
\end{aligned}$$

We therefore have that the HJB equation after substituting in the value function V, the drift and the volatility of the household's wealth $\underline{n}_t$ and state variable η_t is given by

$$\begin{aligned}
\gamma f(\eta_t) + \log\Big(\underline{n}_t\Big) = \sup_{\substack{\underline{x}_t \geq 0 \\ \underline{\iota}_t \geq 0 \\ \underline{\rho}_t \geq 0}} \Big\{ & \frac{1}{\gamma \underline{n}_t}\Big[\underline{x}_t\underline{n}_t\Big(E\Big[\frac{d\underline{r}_t^k}{dt}\Big] - \nu_t(1+r_t) + 1\Big) + \underline{n}_t\Big[\nu_t(1+r_t)-1\Big] - \underline{c}_t\Big] \\
& - \frac{\Big(\underline{x}_t(\sigma+\sigma_t^q)\Big)^2}{2\gamma} + f'(\eta_t)\alpha(\eta_t)\eta_t + \frac{1}{2}f''(\eta_t)\Big(\beta(\eta_t)\eta_t\Big)^2 + \log\Big(\underline{c}_t\Big)\Big\}. \qquad (62)
\end{aligned}$$

We take first order conditions. First, when maximizing with respect to the investment rate $\underline{\iota}_t$, we need only look at the total return on capital for household which gives us

$$\phi'\Big(\underline{\iota}_t\Big) = \frac{1}{q_t}.$$

Substituting the functional form of the adjustment cost, the investment rate of households is

$$\underline{\iota}_t = \frac{1}{2\theta}\Big(q_t^2 - 1\Big).$$

The solution of experts follows *mutatis mutandis*. Now, set

$$\underline{b}_t \triangleq E\left[\frac{d\underline{r}_t^k}{dt}\right] - \nu_t(1+r_t) + 1.$$

Re-arranging equation (62), we obtain f which satisfies

$$\begin{aligned}\gamma f(\eta_t) + \log\left(\underline{n}_t\right) =& \frac{1}{\gamma \underline{n}_t}\left[\underline{x}_t \underline{n}_t \underline{b}_t + \underline{n}_t\left[\nu_t(1+r_t) - 1\right] - \underline{c}_t\right] \\ & - \frac{\left(\underline{x}_t(\sigma + \sigma_t^q)\right)^2}{2\gamma} + f'(\eta_t)\alpha(\eta_t)\eta_t + \frac{1}{2}f''(\eta_t)\left(\beta(\eta_t)\eta_t\right)^2 + \log\left(\underline{c}_t\right).\end{aligned}$$

Re-arrange

$$\begin{aligned}\gamma f(\eta_t) - f'(\eta_t)\alpha(\eta_t)\eta_t - \frac{1}{2}f''(\eta_t)\left(\beta(\eta_t)\eta_t\right)^2 =& \frac{\underline{x}_t \underline{b}_t}{\gamma} + \frac{\nu_t(1+r_t) - 1}{\gamma} - \frac{\underline{c}_t}{\gamma \underline{n}_t} \\ & - \frac{\left(\underline{x}_t(\sigma + \sigma_t^q)\right)^2}{2\gamma} + \log\left(\underline{c}_t\right) - \log\left(\underline{n}_t\right).\end{aligned}$$

Recall that $\underline{\rho}_t \triangleq \frac{\underline{c}_t}{\underline{n}_t}$ so then

$$\begin{aligned}\gamma f(\eta_t) - f'(\eta_t)\alpha(\eta_t)\eta_t - \frac{1}{2}f''(\eta_t)\left(\beta(\eta_t)\eta_t\right)^2 - \frac{\nu_t(1+r_t) - 1}{\gamma} =& \frac{\underline{x}_t \underline{b}_t}{\gamma} - \frac{\left(\underline{x}_t(\sigma + \sigma_t^q)\right)^2}{2\gamma} \\ & + \log\left(\underline{\rho}_t\right) - \frac{\underline{\rho}_t}{\gamma}.\end{aligned}$$

Therefore, the function f satisfies

$$\Lambda - \frac{1}{\gamma}\sup_{\underline{x}_t \geq 0}\left\{\underline{x}_t \underline{b}_t - \frac{1}{2}\left(\underline{x}_t(\sigma + \sigma_t^q)\right)^2\right\} - \sup_{\underline{\rho}_t \geq 0}\left\{\log\left(\underline{\rho}_t\right) - \frac{\underline{\rho}_t}{\gamma}\right\} = 0, \tag{63}$$

where we define

$$\Lambda \triangleq \gamma f(\eta_t) - f'(\eta_t)\alpha(\eta_t)\eta_t - \frac{1}{2}f''(\eta_t)\left(\beta(\eta_t)\eta_t\right)^2 - \frac{\nu_t(1+r_t) - 1}{\gamma}$$

The first-order condition in equation (63) with respect to the consumption-to-net-worth ratio $\underline{\rho}_t$ is

$$\frac{1}{\underline{\rho}_t} - \frac{1}{\gamma} = 0.$$

Therefore, we conclude

$$\underline{c}_t = \gamma \underline{n}_t. \tag{64}$$

Next, we take first order condition with respect to leverage $\underline{x}_t$ which gives us

$$\underline{b}_t - \underline{x}_t(\sigma + \sigma_t^q)^2 = 0.$$

Simplifying, we get that the optimal leverage for the household is

$$\underline{x}_t = \frac{E\left[\frac{d\underline{r}_t^k}{dt}\right] - \nu_t(1 + r_t) + 1}{(\sigma + \sigma_t^q)^2}. \tag{65}$$

This concludes the proof.

D.4 Proof of Proposition 1

We first show that $1 \leq x_t \leq U_t$. The right-hand side of the second inequality follows directly from the leverage constraint. We only need to show the first inequality. In equilibrium, at an instant t, the expected return on bonds (accounting for default) is r_t^d if the household is the lender and $\underline{r}_t^d$ if the expert is the lender. Then, the expert who is risk neutral will lend only if $\underline{r}_t^d - \mathbb{E}\left[dr_t^k\right] > \mathbb{E}\left[dr_t^k\right] - r_t^d$ where $\mathbb{E}\left[dr_t^k\right]$ is the expected return on capital. Likewise, households prefer to borrow only if $\mathbb{E}\left[d\underline{r}_t^k\right] - \underline{r}_t^d > r_t^d - \mathbb{E}\left[d\underline{r}_t^k\right]$. The fact that the household is risk averse implies that its excess return on capital should be higher than the excess return on lending, to borrow, since default is also strategic, and thus, anticipated due to rational expectations. Hence, both of these equations need to be satisfied, which leads to a contradiction since $\mathbb{E}\left[dr_t^k\right] > \mathbb{E}\left[d\underline{r}_t^k\right]$ or to $x_t = 1 = \underline{x}_t$[4], i.e., no lending takes place.

D.5 Proof of Proposition 3

The evolution of wealth for a given expert i is from equation (11)

$$\frac{dn_t}{n_t} = x_t dr_t^k - \nu_t \left(x_t - 1\right)\left(1 + r_t\right)dt + \left(x_t - 1\right)dt.$$

The definition of leverage is $x_t \triangleq \frac{q_t k_t}{n_t}$, which becomes

$$\frac{dn_t}{n_t} = \frac{q_t k_t}{n_t} dr_t^k - \nu_t \left(\frac{q_t k_t - n_t}{n_t}\right)(1 + r_t)dt + \left(\frac{q_t k_t - n_t}{n_t}\right)dt.$$

[4]This is not an equilibrium, as shown below.

Integrate both sides

$$\int_0^1 \frac{dn_t^i}{n_t^i}di = dr_t^k \int_0^1 \frac{q_t k_t^i}{n_t^i}di - (1+r_t)dt \int_0^1 \nu_t^i \left(\frac{q_t k_t^i - n_t^i}{n_t^i}\right)di + \int_0^1 \left(\frac{q_t k_t^i - n_t^i}{n_t^i}\right)didt. \tag{66}$$

We work through each integral. First, recall that the aggregation of wealth over all experts is

$$N_t \triangleq \int_0^1 n_t^i di$$

and the fraction of capital held by experts is

$$\psi_t \triangleq \frac{\int_0^1 k_t^i di}{K_t}.$$

Therefore, the left-hand side

$$\int_0^1 \frac{dn_t^i}{n_t^i}di = \frac{dN_t}{N_t}.$$

Next,

$$\int_0^1 \frac{q_t k_t^i}{n_t^i}di = \frac{\psi_t q_t K_t}{N_t}.$$

Also

$$\int_0^1 \nu_t^i \left(\frac{q_t k_t^i - n_t^i}{n_t^i}\right)di = \nu_t\left(\frac{\psi_t q_t K_t}{N_t} - 1\right).$$

Finally

$$\int_0^1 \left(\frac{q_t k_t^i - n_t^i}{n_t^i}\right)di = \left(\frac{\psi_t q_t K_t}{N_t} - 1\right).$$

Substitute the integrals into equation (66)

$$\frac{dN_t}{N_t} = dr_t^k \frac{\psi_t q_t K_t}{N_t} - \nu_t\left(\frac{\psi_t q_t K_t}{N_t} - 1\right)(1+r_t)dt + \left(\frac{\psi_t q_t K_t}{N_t} - 1\right)dt.$$

Therefore, we get that the aggregate wealth evolution over experts is

$$dN_t = \psi_t q_t K_t dr_t^k - \nu_t \psi_t q_t K_t (1+r_t)\,dt + \nu_t N_t (1+r_t)\,dt + \psi_t q_t K_t dt - N_t dt. \tag{67}$$

In equilibrium, the wealth share of experts from equation (21) is

$$\eta_t = \frac{N_t}{q_t K_t}.$$

By Ito's quotient rule, we have that

$$\frac{d\eta_t}{\eta_t} = \frac{dN_t}{N_t} - \frac{d\left(q_t K_t\right)}{q_t K_t} + \left(\frac{d\left(q_t K_t\right)}{q_t K_t}\right)^2 - \frac{dN_t}{N_t}\frac{d\left(q_t K_t\right)}{q_t K_t}. \tag{68}$$

Recall that the evolution of total supply of capital from equation (47) is

$$dK_t = \left(\int_0^1 \left(\phi\left(\iota_t^i\right) - \delta\right) k_t^i di + \int_1^2 \left(\phi\left(\underline{\iota}_t^j\right) - \underline{\delta}\right) \underline{k}_t^j dj\right) dt + \sigma K_t dZ_t.$$

The optimal investment by experts in (9) and the optimal investment by households from Theorem 8 are identical

$$\iota_t \triangleq \iota_t^i = \underline{\iota}_t^j = \frac{1}{2\theta}\left(q_t^2 - 1\right),$$

for $i \in [0,1]$ and $j \in [1,2]$.

$$\begin{aligned}
dK_t &= \left(\int_0^1 \left(\phi\left(\iota_t^i\right) - \delta\right) k_t^i di + \int_1^2 \left(\phi\left(\underline{\iota}_t^j\right) - \underline{\delta}\right) \underline{k}_t^j dj\right) dt + \sigma K_t dZ_t \\
&= \left(\int_0^1 \left(\phi\left(\iota_t\right) - \delta\right) k_t^i di + \int_1^2 \left(\phi\left(\iota_t\right) - \underline{\delta}\right) \underline{k}_t^j dj\right) dt + \sigma K_t dZ_t \\
&= \left[\phi\left(\iota_t\right)\left(\int_0^1 k_t^i di + \int_1^2 \underline{k}_t^j dj\right) - \left(\int_0^1 \delta k_t^i di + \int_1^2 \underline{\delta}\underline{k}_t^j dj\right)\right] dt + \sigma K_t dZ_t \\
&= \left[\phi\left(\iota_t\right) K_t - \left(\int_0^1 \delta k_t^i di + \int_1^2 \underline{\delta}\underline{k}_t^j dj\right)\right] dt + \sigma K_t dZ_t \\
&= \left[\phi\left(\iota_t\right) K_t - \left(\delta \int_0^1 k_t^i di + \underline{\delta} \int_1^2 \underline{k}_t^j dj\right)\right] dt + \sigma K_t dZ_t \\
&= \left[\phi\left(\iota_t\right) K_t - \left(\delta\psi_t K_t + \underline{\delta}(1-\psi_t) K_t\right)\right] dt + \sigma K_t dZ_t \\
&= \left[\phi\left(\iota_t\right) K_t - \left(\delta\psi_t + \underline{\delta}(1-\psi_t)\right) K_t\right] dt + \sigma K_t dZ_t \\
\Rightarrow \frac{dK_t}{K_t} &= \left[\phi\left(\iota_t\right) - \delta\psi_t - \underline{\delta}(1-\psi_t)\right] dt + \sigma dZ_t
\end{aligned}$$

The postulate for the price of capital from equation (6) is

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t.$$

Applying Ito's product rule

$$\begin{aligned}
d(q_t K_t) &= q_t dK_t + K_t dq_t + dq_t dK_t \\
&= q_t\Big[\Big[\phi(\iota_t) - \delta\psi_t - \underline{\delta}(1-\psi_t)\Big]K_t dt + \sigma K_t dZ_t\Big] + K_t\Big[\mu_t^q q_t dt + \sigma_t^q q_t dZ_t\Big] + (\sigma K_t \sigma_t^q q_t)dt \\
&= q_t K_t\Big[\Big[\phi(\iota_t) - \delta\psi_t - \underline{\delta}(1-\psi_t)\Big]dt + \sigma dZ_t\Big] + q_t K_t\Big[\mu_t^q dt + \sigma_t^q dZ_t\Big] + q_t K_t(\sigma\sigma_t^q)dt \\
\Rightarrow \frac{d(q_t K_t)}{q_t K_t} &= \Big[\phi(\iota_t) - \delta\psi_t - \underline{\delta}(1-\psi_t) + \mu_t^q + \sigma\sigma_t^q\Big]dt + (\sigma + \sigma_t^q)dZ_t.
\end{aligned}$$

Now, compare this to the return on capital for experts in equation (7)

$$dr_t^k = \frac{A-\iota_t}{q_t}dt + \Big(\phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q\Big)dt + \Big(\sigma + \sigma_t^q\Big)dZ_t.$$

Re-arrange this expression

$$\Big(\phi(\iota_t) + \mu_t^q + \sigma\sigma_t^q\Big)dt + \Big(\sigma + \sigma_t^q\Big)dZ_t = dr_t^k - \frac{A-\iota_t}{q_t}dt + \delta dt.$$

Substitute this into the capital gains on aggregate capital

$$\begin{aligned}
\frac{d(q_t K_t)}{q_t K_t} &= \Big[\phi(\iota_t) - \delta\psi_t - \underline{\delta}(1-\psi_t) + \mu_t^q + \sigma\sigma_t^q\Big]dt + (\sigma + \sigma_t^q)dZ_t \\
&= \Big[\phi(\iota_t) + \mu_t^q + \sigma\sigma_t^q\Big]dt + (\sigma + \sigma_t^q)dZ_t + \Big[-\delta\psi_t - \underline{\delta}(1-\psi_t)\Big]dt \\
&= dr_t^k - \frac{A-\iota_t}{q_t}dt + \delta dt + \Big[-\delta\psi_t - \underline{\delta}(1-\psi_t)\Big]dt \\
&= dr_t^k - \frac{A-\iota_t}{q_t}dt - (1-\psi_t)(\underline{\delta} - \delta)dt.
\end{aligned}$$

Therefore, the capital gains on aggregate capital is

$$\frac{d(q_t K_t)}{q_t K_t} = dr_t^k - \frac{A-\iota_t}{q_t}dt - (1-\psi_t)\Big(\underline{\delta} - \delta\Big)dt \tag{69}$$

where the first two terms of the right hand side are the expert capital gains and the third term is the adjustment for the household's capital.

Substitute equation (67) and equation (69) into equation (68)

$$\begin{aligned}
\frac{d\eta_t}{\eta_t} &= \frac{dN_t}{N_t} - \frac{d\left(q_t K_t\right)}{q_t K_t} + \left(\frac{d\left(q_t K_t\right)}{q_t K_t}\right)^2 - \frac{dN_t}{N_t}\frac{d\left(q_t K_t\right)}{q_t K_t} \\
&= \left[dr_t^k \frac{\psi_t q_t K_t}{N_t} - \nu_t\left(\frac{\psi_t q_t K_t}{N_t} - 1\right)(1 + r_t)dt + \left(\frac{\psi_t q_t K_t}{N_t} - 1\right)dt\right] \\
&- \left[dr_t^k - \frac{A - \iota_t}{q_t}dt - (1 - \psi_t)\left(\underline{\delta} - \delta\right)dt\right] + \left[dr_t^k - \frac{A - \iota_t}{q_t}dt - (1 - \psi_t)\left(\underline{\delta} - \delta\right)dt\right]^2 \\
&- \left[dr_t^k \frac{\psi_t q_t K_t}{N_t} - \nu_t\left(\frac{\psi_t q_t K_t}{N_t} - 1\right)(1 + r_t)dt + \left(\frac{\psi_t q_t K_t}{N_t} - 1\right)dt\right] \\
&\times \left[dr_t^k - \frac{A - \iota_t}{q_t}dt - (1 - \psi_t)\left(\underline{\delta} - \delta\right)dt\right]
\end{aligned}$$

The quadratic variation term only depends on the Brownian coefficient loading on the expert's return of capital

$$\left[dr_t^k - \frac{A - \iota_t}{q_t}dt - (1 - \psi_t)\left(\underline{\delta} - \delta\right)dt\right]^2 = \left(\sigma + \sigma_t^q\right)^2 dt.$$

The same fact applies to the cross-variation term

$$\begin{aligned}
&- \left[dr_t^k \frac{\psi_t q_t K_t}{N_t} - \nu_t\left(\frac{\psi_t q_t K_t}{N_t} - 1\right)(1 + r_t)dt + \left(\frac{\psi_t q_t K_t}{N_t} - 1\right)dt\right] \\
&\times \left[dr_t^k - \frac{A - \iota_t}{q_t}dt - (1 - \psi_t)\left(\underline{\delta} - \delta\right)dt\right] \\
&= -\left[\frac{\psi_t q_t K_t}{N_t}\right]\left(\sigma + \sigma_t^q\right)^2 dt.
\end{aligned}$$

We can group these terms

$$\left(\sigma + \sigma_t^q\right)^2 dt - \left[\frac{\psi_t q_t K_t}{N_t}\right]\left(\sigma + \sigma_t^q\right)^2 dt = \left[1 - \frac{\psi_t}{\eta_t}\right]\left(\sigma + \sigma_t^q\right)^2 dt = \left[\frac{\eta_t - \psi_t}{\eta_t}\right]\left(\sigma + \sigma_t^q\right)^2 dt,$$

where we used $\eta_t \triangleq \frac{N_t}{q_t K_t}$. Substitute this in

$$\begin{aligned}
\frac{d\eta_t}{\eta_t} &= \left[dr_t^k \frac{\psi_t q_t K_t}{N_t} - \nu_t\left(\frac{\psi_t q_t K_t}{N_t} - 1\right)(1+r_t)dt + \left(\frac{\psi_t q_t K_t}{N_t} - 1\right)dt\right] \\
&\quad - \left[dr_t^k - \frac{A-\iota_t}{q_t}dt - (1-\psi_t)\left(\underline{\delta}-\delta\right)dt\right] + \left[\frac{\eta_t - \psi_t}{\eta_t}\right]\left(\sigma + \sigma_t^q\right)^2 dt \\
&= \left[dr_t^k \frac{\psi_t}{\eta_t} - \nu_t\left(\frac{\psi_t - \eta_t}{\eta_t}\right)(1+r_t)dt + \left(\frac{\psi_t - \eta_t}{\eta_t}\right)dt\right] \\
&\quad - \left[dr_t^k - \frac{A-\iota_t}{q_t}dt - (1-\psi_t)\left(\underline{\delta}-\delta\right)dt\right] + \left[\frac{\eta_t - \psi_t}{\eta_t}\right]\left(\sigma + \sigma_t^q\right)^2 dt \\
&= \frac{\psi_t - \eta_t}{\eta_t}dr_t^k + \left[-\nu_t\left(\frac{\psi_t - \eta_t}{\eta_t}\right)(1+r_t)dt + \left(\frac{\psi_t - \eta_t}{\eta_t}\right)dt\right] \\
&\quad - \left[-\frac{A-\iota_t}{q_t}dt - (1-\psi_t)\left(\underline{\delta}-\delta\right)dt\right] + \left[\frac{\eta_t - \psi_t}{\eta_t}\right]\left(\sigma + \sigma_t^q\right)^2 dt \\
&= \frac{\psi_t - \eta_t}{\eta_t}\left(dr_t^k - \left(\sigma + \sigma_t^q\right)^2 dt\right) - \nu_t\left(\frac{\psi_t - \eta_t}{\eta_t}\right)(1+r_t)dt + \left(\frac{\psi_t - \eta_t}{\eta_t}\right)dt \\
&\quad + \frac{A-\iota_t}{q_t}dt + (1-\psi_t)\left(\underline{\delta}-\delta\right)dt.
\end{aligned}$$

Therefore, we obtain

$$\frac{d\eta_t}{\eta_t} = \frac{\psi_t - \eta_t}{\eta_t}\left(dr_t^k + \left[1 - \nu_t(1+r_t)\right]dt - \left(\sigma + \sigma_t^q\right)^2 dt\right) + \left[\frac{A-\iota_t}{q_t} + (1-\psi_t)\left(\underline{\delta}-\delta\right)\right]dt.$$

Finally, take the optimal choice for the default rate, ν_t given by equation (15)

$$\nu_t = 1 - \frac{1}{2\lambda\left(x_t - 1\right)\left(1 + r_t\right)}.$$

We have that

$$\begin{aligned}
1 - \nu_t(1+r_t) &= 1 - \left[1 - \frac{1}{2\lambda\left(x_t - 1\right)\left(1 + r_t\right)}\right](1+r_t) \\
&= 1 - (1+r_t) + \frac{1}{2\lambda(x_t - 1)} \\
&= -r_t + \frac{1}{2\lambda(x_t - 1)}.
\end{aligned}$$

Therefore, we conclude

$$\frac{d\eta_t}{\eta_t} = \frac{\psi_t - \eta_t}{\eta_t}\left(dr_t^k - r_t dt + \frac{1}{2\lambda(x_t - 1)}dt - \left(\sigma + \sigma_t^q\right)^2 dt\right) + \left[\frac{A-\iota_t}{q_t} + (1-\psi_t)\left(\underline{\delta}-\delta\right)\right]dt.$$

D.6 Proof of Proposition 4

From the market clearing condition for consumption in equation (50),

$$\int_0^1 x_t^i n_t^i \left(A - \iota_t^i\right) di + \int_1^2 \underline{x}_t^j \underline{n}_t^j \left(\underline{A} - \underline{\iota}_t^j\right) dj = q_t \int_1^2 \underline{n}_t^j \underline{\rho}_t^j dj.$$

Substituting $\underline{c}_t = \underline{\rho}_t \underline{n}_t$ and aggregating,

$$x_t N_t \left(A - \iota_t\right) + \underline{x}_t \underline{N}_t \left(\underline{A} - \underline{\iota}_t\right) = q_t \underline{c}_t.$$

Using the definition of leverage $x_t N_t = \psi_t q_t K_t$ and $\underline{x}_t \underline{N}_t = \left(1 - \psi_t\right) q_t K_t$,

$$\psi_t q_t K_t \left(A - \iota_t\right) + \left(1 - \psi_t\right) q_t K_t \left(\underline{A} - \underline{\iota}_t\right) = q_t \underline{c}_t.$$

Now, use the fact that $\underline{c}_t = \gamma \underline{N}_t$, $\underline{N}_t = q_t K_t - N_t$ and $\eta_t = \frac{N_t}{q_t K_t}$

$$\underline{c}_t = \gamma \left[q_t K_t - N_t\right] = \gamma \left[q_t K_t - \eta_t q_t K_t\right] = \gamma q_t K_t \left[1 - \eta_t\right].$$

Substitute this into the goods market clearing

$$\psi_t q_t K_t \left(A - \iota_t\right) + \left(1 - \psi_t\right) q_t K_t \left(\underline{A} - \underline{\iota}_t\right) = q_t \gamma q_t K_t \left[1 - \eta_t\right].$$

This cancels out

$$\psi_t \left(A - \iota_t\right) + \left(1 - \psi_t\right) \left(\underline{A} - \underline{\iota}_t\right) = q_t \gamma \left[1 - \eta_t\right].$$

We further substitute for $\iota_t = \underline{\iota}_t = \frac{1}{2\theta}\left(q_t^2 - 1\right)$ into the previous equation and we get

$$\begin{aligned}
& \psi_t \left(A - \iota_t\right) + \left(1 - \psi_t\right) \left(\underline{A} - \underline{\iota}_t\right) = q_t \gamma \left[1 - \eta_t\right] \\
\leftrightarrow \quad & \psi_t \left(A - \frac{1}{2\theta}\left(q_t^2 - 1\right)\right) + \left(1 - \psi_t\right) \left(\underline{A} - \frac{1}{2\theta}\left(q_t^2 - 1\right)\right) = q_t \gamma \left[1 - \eta_t\right] \\
\leftrightarrow \quad & \psi_t A + \left(1 - \psi_t\right) \underline{A} - \left(\frac{\psi_t}{2\theta}\left(q_t^2 - 1\right) + \frac{\left(1 - \psi_t\right)}{2\theta}\left(q_t^2 - 1\right)\right) = q_t \gamma \left[1 - \eta_t\right] \\
\leftrightarrow \quad & 2\theta \psi_t A + 2\theta \left(1 - \psi_t\right) \underline{A} - \left(\psi_t \left(q_t^2 - 1\right) + \left(1 - \psi_t\right)\left(q_t^2 - 1\right)\right) = 2\theta q_t \gamma \left[1 - \eta_t\right] \\
\leftrightarrow \quad & 2\theta \psi_t A + 2\theta \left(1 - \psi_t\right) \underline{A} - \left(q_t^2 - 1\right) = 2\theta q_t \gamma \left(1 - \eta_t\right).
\end{aligned}$$

This gives us the quadratic equation

$$q_t^2 + 2\theta \gamma \left(1 - \eta_t\right) q_t - 2\theta \left[\psi_t A + \left(1 - \psi_t\right) \underline{A}\right] - 1 = 0.$$

Use the quadratic formula and keep the positive root

$$\begin{aligned} q_t &= \frac{-2\theta\gamma(1-\eta_t) \pm \sqrt{[2\theta\gamma(1-\eta_t)]^2 - 4\left(-2\theta\left[\psi_t A + (1-\psi_t)\underline{A}\right] - 1\right)}}{2} \\ &= \frac{-2\theta\gamma(1-\eta_t) \pm \sqrt{4\theta^2\gamma^2(1-\eta_t)^2 + 8\theta\left[\psi_t A + (1-\psi_t)\underline{A}\right] + 4}}{2} \\ &= \frac{-2\theta\gamma(1-\eta_t) + 2\sqrt{\theta^2\gamma^2(1-\eta_t)^2 + 2\theta\left[\psi_t A + (1-\psi_t)\underline{A}\right] + 1}}{2} \\ &= -\theta\gamma(1-\eta_t) + \sqrt{\theta^2\gamma^2(1-\eta_t)^2 + 2\theta\left[\psi_t A + (1-\psi_t)\underline{A}\right] + 1}. \end{aligned}$$

Substitute in $\psi_t = x_t\eta_t$ from Proposition 1 to conclude

$$q_t = -\theta\gamma(1-\eta_t) + \sqrt{\theta^2\gamma^2(1-\eta_t)^2 + 2\theta\left[x_t\eta_t A + (1-x_t\eta_t)\underline{A}\right] + 1}.$$

D.7 Proof of Corollary 1

We consider the price function

$$q_t(\eta_t) \;=\; -\,\theta\,\gamma\,[\,1-\eta_t\,] \;+\; \sqrt{\theta^2\,\gamma^2\,(1-\eta_t)^2 \;+\; 2\,\theta\left[x_t(\eta_t)\,\eta_t\,A \;+\; \left(1-x_t(\eta_t)\,\eta_t\right)\underline{A}\right] \;+\; 1}\,.$$

We wish to compute the first derivative $\frac{\partial q_t(\eta_t)}{\partial \eta_t}$. First rewrite q_t as a sum of a linear and non-linear part

$$q_t(\eta_t) \;=\; g(\eta_t) \;+\; \sqrt{M(\eta_t)}\,,$$

where

$$g(\eta_t) \;=\; -\,\theta\,\gamma\,[\,1-\eta_t\,], \quad M(\eta_t) \;=\; \theta^2\,\gamma^2\,(1-\eta_t)^2 \;+\; 2\,\theta\left[x_t(\eta_t)\,\eta_t\,A \;+\; \left(1-x_t(\eta_t)\,\eta_t\right)\underline{A}\right] \;+\; 1.$$

It follows that

$$\frac{\partial}{\partial \eta_t}\,q_t(\eta_t) \;=\; \frac{d}{d\eta_t}\,g(\eta_t) \;+\; \frac{d}{d\eta_t}\,\sqrt{M(\eta_t)}\,.$$

Differentiating the linear term $g(\eta_t)$ gives us

$$g'(\eta_t) \;=\; \theta\,\gamma$$

We differentiate the square-root term via the chain rule. First we write

$$\frac{d}{d\eta_t}\,\sqrt{M(\eta_t)} \;=\; \frac{1}{2\,\sqrt{M(\eta_t)}} \;\cdot\; f'(\eta_t).$$

So differentiating and using the product rule, we have

$$f'(\eta_t) = -\,2\,\theta^2\,\gamma^2\,(1-\eta_t) \;+\; 2\,\theta\left(A-\underline{A}\right)\left[x_t'(\eta_t)\,\eta_t + x_t(\eta_t)\right].$$

Substitute $f'(\eta_t)$ back into the square-root derivative

We have

$$\frac{1}{2\,\sqrt{M(\eta_t)}}\,f'(\eta_t) = \frac{1}{2\,\sqrt{M(\eta_t)}}\left[-\,2\,\theta^2\,\gamma^2\,(1-\eta_t) \;+\; 2\,\theta\,(A-\underline{A})\left(\,x_t'(\eta_t)\,\eta_t + x_t(\eta_t)\right)\right].$$

Factor out the 2 in the numerator and cancel with the 2 in the denominator:

$$\frac{d}{d\eta_t}\,\sqrt{M(\eta_t)} = \frac{-\,\theta^2\,\gamma^2\,(1-\eta_t) \;+\; \theta\,(A-\underline{A})\left[x_t'(\eta_t)\,\eta_t + x_t(\eta_t)\right]}{\sqrt{M(\eta_t)}}.$$

So now combine both terms for the final derivative:

$$\frac{\partial q_t(\eta_t)}{\partial \eta_t} \;=\; \theta\,\gamma \;+\; \frac{-\,\theta^2\,\gamma^2\,(1-\eta_t) \;+\; \theta\,(A-\underline{A})\left[x_t'(\eta_t)\,\eta_t + x_t(\eta_t)\right]}{\sqrt{\theta^2\,\gamma^2\,(1-\eta_t)^2 \;+\; 2\,\theta\left[x_t(\eta_t)\,\eta_t\,A + \left(1-x_t(\eta_t)\,\eta_t\right)\underline{A}\right] \;+\; 1}}.$$

Define

$$M(\eta_t) \triangleq \theta^2\,\gamma^2\,(1-\eta_t)^2 \;+\; 2\,\theta\left[x_t(\eta_t)\,\eta_t\,A + \left(1-x_t(\eta_t)\,\eta_t\right)\underline{A}\right] \;+\; 1.$$

Therefore we have

$$\frac{\partial q_t(\eta_t)}{\partial \eta_t} \;=\; \theta\,\gamma \;+\; \frac{-\,\theta^2\,\gamma^2\,(1-\eta_t) \;+\; \theta\,(A-\underline{A})\left[x_t'(\eta_t)\,\eta_t + x_t(\eta_t)\right]}{\sqrt{M(\eta_t)}}.$$

To get the second derivative, let

$$g(\eta_t) \triangleq -\,\theta^2\,\gamma^2\,(1-\eta_t) \;+\; \theta\,(A-\underline{A})\left[x_t'(\eta_t)\,\eta_t + x_t(\eta_t)\right],$$

so that we have

$$\frac{\partial q_t(\eta_t)}{\partial \eta_t} = \theta\gamma + \frac{g(\eta_t)}{\sqrt{M(\eta_t)}}.$$

The second derivative is

$$\frac{\partial^2 q_t(\eta_t)}{\partial \eta_t^2} = \frac{g'(\eta_t)}{\sqrt{M(\eta_t)}} - \frac{1}{2}\left[M(\eta_t)\right]^{-\frac{3}{2}} g(\eta_t) M'(\eta_t).$$

The partial derivatives are

$$\begin{aligned}
\frac{\partial M(\eta_t)}{\partial \eta_t} &= -2\theta^2\gamma^2(1-\eta_t) + 2\theta(A-\underline{A})\left[x_t'(\eta_t)\eta_t + x_t(\eta_t)\right], \\
\frac{\partial g(\eta_t)}{\partial \eta_t} &= \theta^2\gamma^2 + \theta(A-\underline{A})[x_t''(\eta_t)\eta_t + 2x_t'(\eta_t)].
\end{aligned}$$

Notice that

$$\frac{\partial M(\eta_t)}{\partial \eta_t} = 2g(\eta_t).$$

Then the second derivative is

$$\begin{aligned}
\frac{\partial^2 q_t(\eta_t)}{\partial \eta_t^2} &= \frac{g'(\eta_t)}{\sqrt{M(\eta_t)}} - \frac{1}{2}\frac{g(\eta_t)M'(\eta_t)}{\left[M(\eta_t)\right]^{3/2}} \\
&= \frac{g'(\eta_t)}{\sqrt{M(\eta_t)}} - \frac{g(\eta_t)^2}{\left[M(\eta_t)\right]^{3/2}} \\
&= \frac{g'(\eta_t)M(\eta_t) - g(\eta_t)^2}{\left[M(\eta_t)\right]^{3/2}}.
\end{aligned}$$

Therefore, we have

$$\frac{\partial^2 q_t(\eta_t)}{\partial \eta_t^2} = \frac{g'(\eta_t)M(\eta_t) - g(\eta_t)^2}{\left[M(\eta_t)\right]^{3/2}} = \frac{g'(\eta_t)}{M(\eta_t)^{1/2}} - \frac{g(\eta_t)^2}{\left[M(\eta_t)\right]^{3/2}}.$$

More explicitly, this is

$$\begin{aligned}
\frac{\partial^2 q_t(\eta_t)}{\partial \eta_t^2} = &\frac{\theta^2\gamma^2 + \theta(A-\underline{A})[x_t''(\eta_t)\eta_t + 2x_t'(\eta_t)]}{\left[\theta^2\,\gamma^2\,(1-\eta_t)^2 \;+\; 2\,\theta\left[x_t(\eta_t)\,\eta_t\,A + \left(1 - x_t(\eta_t)\,\eta_t\right)\underline{A}\right] \;+\; 1\right]^{1/2}} \\
&- \frac{\left[-\,\theta^2\,\gamma^2\,(1-\eta_t) \;+\; \theta\,(A-\underline{A})\left[x_t'(\eta_t)\,\eta_t + x_t(\eta_t)\right]\right]^2}{\left[\theta^2\,\gamma^2\,(1-\eta_t)^2 \;+\; 2\,\theta\left[x_t(\eta_t)\,\eta_t\,A + \left(1 - x_t(\eta_t)\,\eta_t\right)\underline{A}\right] \;+\; 1\right]^{3/2}}
\end{aligned}$$

D.8 Proof of Proposition 5

From Proposition 8, we have that the optimal leverage by the household is

$$\underline{x}_t = \frac{\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - \nu_t(r_t + 1) + 1}{\left(\sigma + \sigma_t^q\right)^2}. \tag{70}$$

We will solve for each term in this expression.

Using the household return on capital in equation (71), we have

$$d\underline{r}_t^k = \frac{\underline{A} - \underline{\iota}_t}{q_t} dt + \left(\phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right) dt + \left(\sigma + \sigma_t^q\right) dZ_t. \tag{71}$$

Also, from Proposition 2, the on-the-verge condition for default is given by

$$\nu_t = 1 - \frac{1}{2\lambda\left(x_t - 1\right)\left(1 + r_t\right)}. \tag{72}$$

The debt market clearing condition from equation (48) is

$$N_t + \underline{N}_t = q_t K_t.$$

Divide by $q_t K_t$ and use the definition $\eta_t \triangleq \frac{N_t}{q_t K_t}$ to rewrite this as

$$\eta_t + \underline{\eta}_t = 1. \tag{73}$$

Take the capital market clearing condition in equation (46)

$$\int_0^1 x_t^i n_t^i di + \int_1^2 \underline{x}_t^j \underline{n}_t^j dj = q_t K_t.$$

Divide by $q_t K_t$

$$\frac{x_t N_t}{q_t K_t} + \frac{\underline{x}_t \underline{N}_t}{q_t K_t} = 1.$$

Use $\eta_t \triangleq \frac{N_t}{q_t K_t}$ from equation (21)

$$x_t \eta_t + \underline{x}_t \underline{\eta}_t = 1. \tag{74}$$

Substitute equation (73) into equation (74) to get

$$\underline{x}_t = \frac{1 - x_t \eta_t}{1 - \eta_t}. \tag{75}$$

We combine the results together. Starting off with the optimal leverage equation (70):

$$
\begin{aligned}
& \underline{x}_t = \frac{\frac{\mathbb{E}\left[dr_t^k\right]}{dt} - \nu_t(r_t+1)+1}{\left(\sigma+\sigma_t^q\right)^2} \\
\leftrightarrow_{(1)} \quad & \frac{1-x_t\eta_t}{1-\eta_t} = \frac{\frac{\mathbb{E}\left[dr_t^k\right]}{dt} - \nu_t(r_t+1)+1}{\left(\sigma+\sigma_t^q\right)^2} \\
\leftrightarrow_{(2)} \quad & \frac{1-x_t\eta_t}{1-\eta_t} = \frac{\left[\frac{\underline{A}-\iota_t}{q_t} + \phi\left(\iota_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right] - \nu_t(r_t+1)+1}{\left(\sigma+\sigma_t^q\right)^2} \\
\leftrightarrow_{(3)} \quad & \frac{1-x_t\eta_t}{1-\eta_t} = \frac{\left[\frac{\underline{A}-\iota_t}{q_t} + \phi\left(\iota_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right] - \left[1-\frac{1}{2\lambda(x_t-1)(1+r_t)}\right](r_t+1)+1}{\left(\sigma+\sigma_t^q\right)^2} \\
\leftrightarrow \quad & \frac{1-x_t\eta_t}{1-\eta_t} = \frac{\left[\frac{\underline{A}-\iota_t}{q_t} + \phi\left(\iota_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right] + \frac{1}{2\lambda(x_t-1)(1+r_t)}(1+r_t) - (1+r_t)+1}{\left(\sigma+\sigma_t^q\right)^2} \\
\leftrightarrow \quad & \frac{1-x_t\eta_t}{1-\eta_t} = \frac{\left[\frac{\underline{A}-\iota_t}{q_t} + \phi\left(\iota_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right] + \frac{1}{2\lambda(x_t-1)} - r_t}{\left(\sigma+\sigma_t^q\right)^2}.
\end{aligned}
$$

$\leftrightarrow_{(1)}$ uses the capital market clearing condition in equation (75), $\leftrightarrow_{(2)}$ uses the return on capital equation (71), and $\leftrightarrow_{(3)}$ uses the repayment rate from on-the-verge default condition in equation (72). Therefore, we have

$$
\frac{1-x_t\eta_t}{1-\eta_t} = \frac{\frac{\underline{A}-\iota_t}{q_t} + \phi\left(\iota_t\right) - \underline{\delta} - r_t + \mu_t^q + \sigma\sigma_t^q + \frac{1}{2\lambda(x_t-1)}}{\left(\sigma+\sigma_t^q\right)^2}
$$

Re-arranging the above equation yields the interest rate

$$
r_t = \frac{\underline{A}-\iota_t}{q_t} + \phi\left(\iota_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q - \left(\sigma+\sigma_t^q\right)^2 \frac{1-x_t\eta_t}{1-\eta_t} + \frac{1}{2\lambda(x_t-1)}.
$$

D.9 Proof of Proposition 6

We apply Ito's lemma to the function $q_t = q\left(\eta_t\right)$ and we obtain

$$
dq_t = q'\left(\eta_t\right)d\eta_t + \frac{1}{2}q''\left(\eta_t\right)\left(d\eta_t\right)^2.
$$

Recall that

$$
\frac{d\eta_t}{\eta_t} = \alpha(\eta_t)dt + \beta(\eta_t)dZ_t.
$$

Therefore, the quadratic variation of the wealth share is

$$(d\eta_t)^2 = \eta_t^2 \beta(\eta_t)^2 dt.$$

We can then write the evolution of the price of capital as

$$dq_t = q'(\eta_t)\left[\alpha(\eta_t)\eta_t dt + \beta(\eta_t)\eta_t dZ_t\right] + \frac{1}{2}q''(\eta_t)\eta_t^2\beta(\eta_t)^2 dt.$$

Divide by q_t to obtain

$$\frac{dq_t}{q_t} = \left(\frac{q'(\eta_t)}{q_t}\alpha(\eta_t)\eta_t + \frac{q''(\eta_t)}{q_t}\frac{\beta(\eta_t)^2}{2}\eta_t^2\right)dt + \frac{q'(\eta_t)}{q_t}\beta(\eta_t)\eta_t dZ_t. \tag{76}$$

Recall the drift and volatility of the expert's relative wealth from Proposition 3 is

$$\alpha(\eta_t) = \frac{\psi_t - \eta_t}{\eta_t}\left(\frac{\mathbb{E}_t[dr_t^k]}{dt} - r_t dt + \frac{1}{2\lambda(x_t - 1)} - \left(\sigma + \sigma_t^q\right)^2\right) + \frac{A - \iota_t}{q_t} + (1 - \psi_t)\left(\underline{\delta} - \delta\right),$$
$$\beta(\eta_t) = \frac{\psi_t - \eta_t}{\eta_t}\left(\sigma + \sigma_t^q\right).$$

From Proposition 1, we have $x_t = \frac{\psi_t}{\eta_t}$, to get

$$\alpha(\eta_t) = \frac{x_t\eta_t - \eta_t}{\eta_t}\left(\frac{\mathbb{E}_t[dr_t^k]}{dt} - r_t dt + \frac{1}{2\lambda(x_t - 1)} - \left(\sigma + \sigma_t^q\right)^2\right) + \frac{A - \iota_t}{q_t} + (1 - x_t\eta_t)\left(\underline{\delta} - \delta\right),$$
$$\beta(\eta_t) = \frac{x_t\eta_t - \eta_t}{\eta_t}\left(\sigma + \sigma_t^q\right).$$

To derive the volatility of the price of capital, first recall from equation (76)

$$\frac{dq_t}{q_t} = \left(\frac{q'(\eta_t)}{q_t}\alpha(\eta_t)\eta_t + \frac{q''(\eta_t)}{q_t}\frac{\beta(\eta_t)^2}{2}\eta_t^2\right)dt + \frac{q'(\eta_t)}{q_t}\beta(\eta_t)\eta_t dZ_t.$$

We also know $\beta(\eta_t)$ from Proposition 3 is equal to

$$\beta(\eta_t) = \frac{x_t\eta_t - \eta_t}{\eta_t}\left(\sigma + \sigma_t^q\right).$$

Hence, from the price of capital postulate in equation (6),

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t,$$

we get that from matching coefficients

$$\begin{aligned}\sigma_t^q &= \frac{q'(\eta_t)}{q_t}\beta(\eta_t)\eta_t \\ &= \frac{q'(\eta_t)}{q_t}\frac{x_t\eta_t-\eta_t}{\eta_t}\left(\sigma+\sigma_t^q\right)\eta_t \\ &= \frac{q'(\eta_t)}{q_t}(x_t\eta_t-\eta_t)\left(\sigma+\sigma_t^q\right).\end{aligned}$$

Re-arrange to obtain

$$\sigma_t^q = \frac{(x_t\eta_t-\eta_t)\frac{q'(\eta_t)}{q_t}\sigma}{1-(x_t\eta_t-\eta_t)\frac{q'(\eta_t)}{q_t}}.$$

To derive the drift of the price of capital, we again match coefficients where

$$\begin{aligned}\mu_t^q &= \frac{q'(\eta_t)}{q_t}\alpha(\eta_t)\eta_t + \frac{q''(\eta_t)}{q_t}\frac{\beta(\eta_t)^2}{2}\eta_t^2 \\ &= \frac{q'(\eta_t)}{q_t}\eta_t\left[\frac{x_t\eta_t-\eta_t}{\eta_t}\left(\frac{\mathbb{E}_t[dr_t^k]}{dt}-r_t+\frac{1}{2\lambda(x_t-1)}-\left(\sigma+\sigma_t^q\right)^2\right)+\frac{A-\iota_t}{q_t}+(1-x_t\eta_t)\left(\underline{\delta}-\delta\right)\right] \\ &+ \frac{q''(\eta_t)}{q_t}\frac{\beta(\eta_t)^2}{2}\eta_t^2 \\ &= \frac{q'(\eta_t)}{q_t}(x_t\eta_t-\eta_t)\left(\frac{\mathbb{E}_t[dr_t^k]}{dt}-r_t+\frac{1}{2\lambda(x_t-1)}-\left(\sigma+\sigma_t^q\right)^2\right) \\ &+ \frac{q'(\eta_t)}{q_t}\eta_t\frac{A-\iota_t}{q_t}+\frac{q'(\eta_t)}{q_t}\eta_t(1-x_t\eta_t)\left(\underline{\delta}-\delta\right)+\frac{q''(\eta_t)}{q_t}\frac{\beta(\eta_t)^2}{2}\eta_t^2.\end{aligned}$$

So in the region where $\frac{\mathbb{E}_t[dr_t^k]}{dt} = r_t$, we have

$$\begin{aligned}\mu_t^q &= \frac{q'(\eta_t)}{q_t}(x_t\eta_t-\eta_t)\left(\frac{1}{2\lambda(x_t-1)}-\left(\sigma+\sigma_t^q\right)^2\right) \\ &+ \frac{q'(\eta_t)}{q_t}\eta_t\frac{A-\iota_t}{q_t}+\frac{q'(\eta_t)}{q_t}\eta_t(1-x_t\eta_t)\left(\underline{\delta}-\delta\right)+\frac{q''(\eta_t)}{q_t}\frac{\beta(\eta_t)^2}{2}\eta_t^2.\end{aligned}$$

For completeness, we look at the case where $\frac{\mathbb{E}_t[dr_t^k]}{dt} \neq r_t$. Recall the drift of return on capital for experts from equation (7)

$$\frac{\mathbb{E}_t[dr_t^k]}{dt} = \frac{A-\iota_t}{q_t}+\phi(\iota_t)-\delta+\mu_t^q+\sigma\sigma_t^q.$$

Substitute this in

$$\mu_t^q = \frac{q'(\eta_t)}{q_t}(x_t\eta_t - \eta_t)\left(\left[\frac{A-\iota_t}{q_t} + \phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q\right] - r_t + \frac{1}{2\lambda(x_t-1)} - \left(\sigma + \sigma_t^q\right)^2\right)$$
$$+ \frac{q'(\eta_t)}{q_t}\eta_t\frac{A-\iota_t}{q_t} + \frac{q'(\eta_t)}{q_t}\eta_t(1-x_t\eta_t)\left(\underline{\delta} - \delta\right) + \frac{q''(\eta_t)}{q_t}\frac{\beta(\eta_t)^2}{2}\eta_t^2.$$

Rewrite this as

$$\mu_t^q = \chi(\xi + \mu_t^q) + R,$$

where

$$\chi \triangleq \frac{q'(\eta_t)}{q_t}(x_t\eta_t - \eta_t),$$
$$\xi \triangleq \left[\frac{A-\iota_t}{q_t} + \phi(\iota_t) - \delta + \sigma\sigma_t^q\right] - r_t + \frac{1}{2\lambda(x_t-1)} - \left(\sigma + \sigma_t^q\right)^2,$$
$$R \triangleq \frac{q'(\eta_t)}{q_t}\eta_t\frac{A-\iota_t}{q_t} + \frac{q'(\eta_t)}{q_t}\eta_t(1-x_t\eta_t)\left(\underline{\delta} - \delta\right) + \frac{q''(\eta_t)}{q_t}\frac{\beta(\eta_t)^2}{2}\eta_t^2.$$

Solve for μ_t^q to get

$$\mu_t^q = \frac{\xi\chi + R}{1-\chi}.$$

Therefore, we have

$$\mu_t^q = \frac{\left(\frac{q'(\eta_t)}{q_t}(x_t\eta_t - \eta_t)\right)\left(\frac{A-\iota_t}{q_t} + \phi(\iota_t) - \delta + \sigma\sigma_t^q - r_t + \frac{1}{2\lambda(x_t-1)} - \left(\sigma + \sigma_t^q\right)^2\right)}{1 - \frac{q'(\eta_t)}{q_t}(x_t\eta_t - \eta_t)}$$
$$+ \frac{\frac{q'(\eta_t)}{q_t}\eta_t\frac{A-\iota_t}{q_t} + \frac{q'(\eta_t)}{q_t}\eta_t(1-x_t\eta_t)\left(\underline{\delta} - \delta\right)}{1 - \frac{q'(\eta_t)}{q_t}(x_t\eta_t - \eta_t)} + \frac{\frac{q''(\eta_t)}{q_t}\frac{\beta(\eta_t)^2}{2}\eta_t^2}{1 - \frac{q'(\eta_t)}{q_t}(x_t\eta_t - \eta_t)}.$$

This simplifies to

$$\mu_t^q = \left[\frac{q'(\eta_t)\eta_t}{q_t\left(1 - \frac{q'(\eta_t)}{q_t}(x_t\eta_t - \eta_t)\right)}\right]\left\{(x_t-1)\left(\phi(\iota_t) - \delta + \sigma\sigma_t^q - r_t + \frac{1}{2\lambda(x_t-1)} - \left(\sigma + \sigma_t^q\right)^2\right)\right.$$
$$\left. + \frac{A-\iota_t}{q_t}x_t + (1-x_t\eta_t)\left(\underline{\delta} - \delta\right)\right\} + \frac{q''(\eta_t)\beta(\eta_t)^2\eta_t^2}{2\left[q_t - q'(\eta_t)(x_t\eta_t - \eta_t)\right]}.$$

Further simplify

$$\mu_t^q = \left[\frac{q'(\eta_t)\eta_t}{q_t - q'(\eta_t)\left(x_t\eta_t - \eta_t\right)}\right]\Bigg\{\left(x_t - 1\right)\left(\phi\left(\iota_t\right) - \delta + \sigma\sigma_t^q - r_t + \frac{1}{2\lambda(x_t-1)} - \left(\sigma + \sigma_t^q\right)^2\right) + \frac{A - \iota_t}{q_t}x_t + \left(1 - x_t\eta_t\right)\left(\underline{\delta} - \delta\right)\Bigg\} + \frac{q''(\eta_t)\beta(\eta_t)^2\eta_t^2}{2\left[q_t - q'(\eta_t)\left(x_t\eta_t - \eta_t\right)\right]} \times \frac{q'(\eta_t)}{q'(\eta_t)}.$$

We arrive at

$$\mu_t^q = \left[\frac{q'(\eta_t)\eta_t}{q_t - q'(\eta_t)\eta_t\left(x_t - 1\right)}\right]\Bigg\{\left(x_t - 1\right)\left(\phi\left(\iota_t\right) - \delta + \sigma\sigma_t^q - r_t + \frac{1}{2\lambda(x_t-1)} - \left(\sigma + \sigma_t^q\right)^2\right) + \frac{A - \iota_t}{q_t}x_t + \left(1 - x_t\eta_t\right)\left(\underline{\delta} - \delta\right) + \frac{q''(\eta_t)\beta(\eta_t)^2\eta_t}{2q'(\eta_t)}\Bigg\}. \tag{77}$$

To write the expression in terms of exogenous variables only, substitute the interest rate from Proposition 5,

$$r_t = \frac{\underline{A} - \underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q - \left(\sigma + \sigma_t^q\right)^2\frac{1 - x_t\eta_t}{1 - \eta_t} + \frac{1}{2\lambda(x_t - 1)}.$$

So the term multiplying $x_t - 1$ becomes

$$\begin{aligned}
&\phi\left(\iota_t\right) - \delta + \sigma\sigma_t^q - r_t + \frac{1}{2\lambda(x_t-1)} - \left(\sigma + \sigma_t^q\right)^2 \\
&= \phi\left(\iota_t\right) - \delta + \sigma\sigma_t^q - \left[\frac{\underline{A} - \underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q - \left(\sigma + \sigma_t^q\right)^2\frac{1 - x_t\eta_t}{1 - \eta_t} + \frac{1}{2\lambda(x_t-1)}\right] \\
&+ \frac{1}{2\lambda(x_t-1)} - \left(\sigma + \sigma_t^q\right)^2 \\
&= \phi\left(\iota_t\right) - \delta + \sigma\sigma_t^q - \left[\frac{\underline{A} - \underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q - \left(\sigma + \sigma_t^q\right)^2\frac{1 - x_t\eta_t}{1 - \eta_t}\right] - \left(\sigma + \sigma_t^q\right)^2 \\
&= \left(\underline{\delta} - \delta\right) - \frac{\underline{A} - \underline{\iota}_t}{q_t} - \mu_t^q + \frac{\eta_t\left(1 - x_t\right)}{1 - \eta_t}\left(\sigma + \sigma_t^q\right)^2.
\end{aligned}$$

So we have

$$\mu_t^q = \left[\frac{q'(\eta_t)\eta_t}{q_t - q'(\eta_t)\eta_t\left(x_t - 1\right)}\right]\Bigg\{\left(x_t - 1\right)\left(\left(\underline{\delta} - \delta\right) - \frac{\underline{A} - \underline{\iota}_t}{q_t} - \mu_t^q + \frac{\eta_t\left(1 - x_t\right)}{1 - \eta_t}\left(\sigma + \sigma_t^q\right)^2\right) + \frac{A - \iota_t}{q_t}x_t + \left(1 - x_t\eta_t\right)\left(\underline{\delta} - \delta\right) + \frac{q''(\eta_t)\beta(\eta_t)^2\eta_t}{2q'(\eta_t)}\Bigg\}.$$

We now solve for μ_t^q. Let

$$F \triangleq \frac{q'(\eta_t)\,\eta_t}{q_t - q'(\eta_t)\,\eta_t\left(x_t - 1\right)}.$$

Denote by

$$S \triangleq (x_t - 1)\Big[(\underline{\delta} - \delta) \; - \; \frac{\underline{A} - \underline{\iota}_t}{q_t} \; + \; \frac{\eta_t\,(1 - x_t)}{1 - \eta_t}\left(\sigma + \sigma_t^q\right)^2\Big] \\ + \frac{A - \iota_t}{q_t}\, x_t \; + \; \left(1 - x_t\,\eta_t\right)\left(\underline{\delta} - \delta\right) \; + \; \frac{q''(\eta_t)\,\beta(\eta_t)^2\,\eta_t}{2\,q'(\eta_t)}.$$

Observe that μ_t^q appears with a minus sign inside the bracket multiplied by $(x_t - 1)$. Distributing F, we have

$$\mu_t^q = FS - F(x_t - 1)\mu_t^q.$$

Solve for μ_t^q

$$\mu_t^q = \frac{FS}{1 + F(x_t - 1)}.$$

Therefore the explicit form we need in implementing code

$$\mu_t^q = \frac{\dfrac{q'(\eta_t)\,\eta_t}{q_t - q'(\eta_t)\,\eta_t\,(x_t - 1)} \times S}{1 \; + \; \dfrac{q'(\eta_t)\,\eta_t}{q_t - q'(\eta_t)\,\eta_t\,(x_t - 1)}\,(x_t - 1)}.$$

D.10 Proof of Proposition 7

The expert's expected capital return from equation (7) is

$$\frac{\mathbb{E}_t[dr_t^k]}{dt} = \frac{A - \iota_t}{q_t} + \phi\left(\iota_t\right) - \delta + \mu_t^q + \sigma\sigma_t^q.$$

The interest rate is given by Proposition 5

$$r_t = \frac{\underline{A} - \underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q - \left(\sigma + \sigma_t^q\right)^2 \frac{1 - x_t\eta_t}{1 - \eta_t} + \frac{1}{2\lambda(x_t - 1)}.$$

Using the interior leverage condition (16), the excess return on capital satisfies

$$\begin{aligned} & \frac{\mathbb{E}\left[dr_t^k\right]}{dt} - r_t = 0 \\ \leftrightarrow \quad & \frac{A - \iota_t}{q_t} + \phi\left(\iota_t\right) - \delta + \mu_t^q + \sigma\sigma_t^q \\ & - \left[\frac{\underline{A} - \underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q - \left(\sigma + \sigma_t^q\right)^2 \frac{1 - x_t\eta_t}{1 - \eta_t} + \frac{1}{2\lambda(x_t - 1)}\right] = 0. \end{aligned}$$

Since $\iota_t = \underline{\iota}_t$, we have

$$\frac{A - \underline{A}}{q_t(x_t, \eta_t)} + (\underline{\delta} - \delta) + \left(\sigma + \sigma_t^q(x_t, \eta_t)\right)^2 \frac{1 - x_t\eta_t}{1 - \eta_t} - \frac{1}{2\lambda(x_t - 1)} = 0.$$

Therefore we have

$$\frac{A - \underline{A}}{q_t(x_t, \eta_t)} + (\underline{\delta} - \delta) + \left(\sigma + \sigma_t^q(x_t, \eta_t)\right)^2 \frac{1 - x_t\eta_t}{1 - \eta_t} - \frac{1}{2\lambda(x_t - 1)} = 0.$$

The fact that the parameters of the function $g(x_t)$ are functions of the exogenous variables and the state variable η_t follows from Proposition 4, Corollary 1 and Proposition 6.

D.11 Proof of Proposition 10

The proof follows immediately from observing that once default is removed, then the objective function of the expert becomes linear in x_t and should, therefore, be either U_t or 1. It can be easily shown that irrespective of the value of x_t, the expert's excess return over the interest rate r_t, in the no default case is always positive i.e.,

$$\frac{\mathbb{E}\left[dr_t^k\right]}{dt} - r_t > 0$$

and hence, the only equilibrium can be the one that satisfies $x_t = U_t$ for all t.

D.12 Proof of Proposition 9

Proof. For (A), take goods market clearing condition from equation (50)

$$\int_0^1 k_t^i \left(A - \iota_t^i\right) di + \int_1^2 \underline{k}_t^j \left(\underline{A} - \underline{\iota}_t^j\right) dj = \int_1^2 \underline{c}_t^j dj.$$

Recall the definition of capital share

$$\psi_t \triangleq \frac{\int_0^1 k_t^i di}{K_t}.$$

Also define aggregate consumption of households as

$$\underline{C}_t \triangleq \int_1^2 \underline{c}_t^j dj.$$

Finally, recall that optimal investment for the expert and household are equal. Divide by K_t in the goods market clearing condition to get

$$\psi_t(A - \iota_t) + (1 - \psi_t)\left(\underline{A} - \iota_t\right) = \frac{\underline{C}_t}{K_t}.$$

We can simplify this to get

$$\psi_t(A - \underline{A}) + \underline{A} - \iota_t = \frac{\underline{C}_t}{K_t}.$$

Substitute optimal investment from equation (9)

$$\iota_t = \frac{1}{2\theta}\left(q_t^2 - 1\right),$$

to get

$$\psi_t(A - \underline{A}) + \underline{A} - \frac{1}{2\theta}\left(q_t^2 - 1\right) = \frac{\underline{C}_t}{K_t}.$$

Re-arrange

$$q_t^2 - 2\theta\left[\psi_t(A - \underline{A}) + \underline{A}\right] - 1 = -2\theta\left(\frac{\underline{C}_t}{K_t}\right).$$

Optimal aggregate consumption is a fraction of household aggregate net worth

$$\underline{C}_t = \gamma \underline{N}_t.$$

Hence

$$\begin{aligned}\frac{\underline{C}_t}{K_t} &= \frac{\gamma \underline{N}_t}{K_t} \\ &= \frac{\gamma(1 - \eta_t) q_t K_t}{K_t} \\ &= \gamma(1 - \eta_t) q_t,\end{aligned}$$

where we used the fact that the expert's net worth share is defined as

$$\eta_t \triangleq \frac{N_t}{q_t K_t}.$$

Therefore, we conclude

$$q_t^2 + 2\theta\gamma(1 - \eta_t) q_t - 2\theta\left[\psi_t(A - \underline{A}) + \underline{A}\right] - 1 = 0.$$

For (B), the volatility of the price of capital, σ_t^q, is from proposition (6)

$$\sigma_t^q = \frac{q'(\eta_t)}{q_t}(x_t\eta_t - \eta_t)\left(\sigma + \sigma_t^q\right).$$

Re-arrange

$$\sigma_t^q q_t = \left(\sigma + \sigma_t^q\right) q'(\eta_t)(x_t\eta_t - \eta_t).$$

Finally, recall that optimal leverage is

$$x_t = \frac{\psi_t}{\eta_t}.$$

So plug this in to get

$$\left(\sigma + \sigma_t^q\right) q'(\eta_t)(\psi_t - \eta_t) - \sigma_t^q q_t = 0.$$

For (C), the optimal household leverage from equation (19) is

$$\underline{x}_t = \frac{\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - \nu_t(1+r_t) + 1}{\left(\sigma + \sigma_t^q\right)^2}.$$

From Proposition 1, the optimal leverage is

$$x_t = \frac{\psi_t}{\eta_t} \qquad \text{and} \qquad \underline{x}_t = \frac{1-\psi_t}{1-\eta_t}.$$

The optimal default from equation (15) is

$$\nu_t = 1 - \frac{1}{2\lambda(x_t - 1)(1 + r_t)}.$$

Substitute both of these results in

$$\frac{1-\psi_t}{1-\eta_t} = \frac{\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - \left[1 - \frac{1}{2\lambda(x_t-1)(1+r_t)}\right](1+r_t) + 1}{\left(\sigma + \sigma_t^q\right)^2}.$$

This simplifies to

$$\frac{1-\psi_t}{1-\eta_t}\left(\sigma + \sigma_t^q\right)^2 = \frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - r_t + \frac{1}{2\lambda(x_t - 1)}.$$

Multiply by $2\lambda(1-\eta_t)$

$$2\lambda\left(1-\psi_t\right)\left(\sigma+\sigma_t^q\right)^2=2\lambda(1-\eta_t)\left[\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt}-r_t\right]+\frac{1-\eta_t}{(x_t-1)}.$$

Substituting $x_t=\frac{\psi_t}{\eta_t}$,

$$2\lambda\left(1-\psi_t\right)\left(\sigma+\sigma_t^q\right)^2=2\lambda(1-\eta_t)\left[\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt}-r_t\right]+\frac{1-\eta_t}{(\frac{\psi_t}{\eta_t}-1)}.$$

Rewrite

$$2\lambda\left(1-\psi_t\right)\left(\sigma+\sigma_t^q\right)^2=2\lambda(1-\eta_t)\left[\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt}-r_t\right]+\left[\frac{1-\eta_t}{(\frac{\psi_t-\eta_t}{\eta_t})}\right].$$

Rewrite this into

$$2\lambda\left(1-\psi_t\right)\left(\sigma+\sigma_t^q\right)^2=2\lambda(1-\eta_t)\left[\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt}-r_t\right]+\eta_t\left[\frac{1-\eta_t}{(\psi_t-\eta_t)}\right].$$

Multiply through

$$2\lambda\left(1-\psi_t\right)\left(\psi_t-\eta_t\right)\left(\sigma+\sigma_t^q\right)^2=2\lambda\left(\psi_t-\eta_t\right)(1-\eta_t)\left[\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt}-r_t\right]+\eta_t\left[1-\eta_t\right].$$

By the expert's leverage condition (16), the expected return on capital for the expert equals the interest rate

$$\mathbb{E}[dr_t^k]=r_t.$$

Therefore we can plug this in

$$2\lambda\left(1-\psi_t\right)\left(\psi_t-\eta_t\right)\left(\sigma+\sigma_t^q\right)^2=2\lambda\left(\psi_t-\eta_t\right)(1-\eta_t)\left[\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt}-\frac{\mathbb{E}\left[dr_t^k\right]}{dt}\right]+\eta_t\left[1-\eta_t\right].$$

Plugging in the expected returns and canceling terms out gives us

$$2\lambda\left(1-\psi_t\right)\left(\psi_t-\eta_t\right)\left(\sigma+\sigma_t^q\right)^2=2\lambda\left(\psi_t-\eta_t\right)(1-\eta_t)\left[\frac{\underline{A}-A}{q_t}-(\underline{\delta}-\delta)\right]+\eta_t\left[1-\eta_t\right].$$

Multiply by q_t to get

$$2\lambda\left(1-\psi_t\right)\left(\psi_t-\eta_t\right)\left(\sigma+\sigma_t^q\right)^2 q_t=2\lambda\left(\psi_t-\eta_t\right)(1-\eta_t)\left[\underline{A}-A-q_t(\underline{\delta}-\delta)\right]+\eta_t\left[1-\eta_t\right]q_t.$$

Re-arrange

$$2\lambda\left(1-\psi_t\right)\left(\psi_t-\eta_t\right)\left(\sigma+\sigma_t^q\right)^2 q_t+2\lambda\left(\psi_t-\eta_t\right)\left(1-\eta_t\right)\left[A-\underline{A}-q_t(\delta-\underline{\delta})\right]-\eta_t\left[1-\eta_t\right]q_t=0.$$

□