

First Year Macroeconomics

Charles Christopher Hyland

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1 Dynamic Programming

1.1 Introduction

We are interested in infinite discrete time dynamic programming. Suppose that we have an agent who wishes to maximise their lifetime discounted utility. We let our discount factor be δ . We let x_t be our **control variable** (e.g. consumption) and s_t be our **state variable** (e.g. savings). We also say that our state in the next time period s_{t+1} depends on the current state and the control we choose $\phi(x_t, s_t)$. From this, we can define our sequence problem:

Definition 1 (Sequence Problem)

$$\begin{aligned} \max_{\{x_t\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \delta^t u(x_t, s_t) \\ \text{s.t.} \quad & s_{t+1} = \phi(x_t, s_t) \\ & s_0 \text{ is given} \end{aligned} \tag{1}$$

To summarise, we are choosing an infinite sequence of controls $\{x_t\}_{t=0}^{\infty}$ that maximises our discounted lifetime utility.

However, this is actually quite a tricky problem to solve. **What we can instead do is to take our sequence problem and turn it into a different problem. We can then solve this new problem and we can mathematically prove that the solution to our new problem is actually a solution to our original sequence problem.**

1.2 Bellman Equation

We first give a short motivation for our *new problem*. Suppose we are now in a finite-horizon version of our sequence problem with periods indexed by $t = 0, \dots, T$. Our problem is

$$\begin{aligned} \max_{\{x_t\}_{t=0}^T} \quad & \sum_{t=0}^T \delta^t u(x_t, s_t) \\ \text{s.t.} \quad & s_{t+1} = \phi(x_t, s_t) \\ & s_0 \text{ is given} \end{aligned} \tag{2}$$

We can now solve this problem using a form of **backward** induction. Suppose that we are currently at the final period $t = T$. Our problem is now to pick an x_T (that is feasible) that maximises

$$\max_{x_T} u(x_T, s_T)$$

We call the maximum utility value that we achieved the **value function** (similar to the indirect utility function)

$$V_T(s_T) = \max_{x_T} u(x_T, s_T)$$

whereby it is important to notice that our value function is a function of the final **state variable** s_T .

Now let us step back to prior period $t = T - 1$. Our problem is to now pick

$$\begin{aligned} \max_{x_{T-1}, x_T} \quad & u(x_{T-1}, s_{T-1}) + \delta u(x_T, s_T) \\ \text{s.t.} \quad & s_T = \phi(x_{T-1}, s_{T-1}) \end{aligned} \tag{3}$$

However, we have already *solved* the problem at the final time period $t = T$! Furthermore, we denoted the solution to final period as our value function V_T . Therefore, we can in fact replace the $u(x_T, s_T)$ term and have our problem now be:

$$\max_{x_{T-1}} u(x_{T-1}, s_{T-1}) + \delta V_T(s_T)$$

but we know that $s_T = \phi(x_{T-1}, s_{T-1})$ so we therefore have

$$\max_{x_{T-1}} u(x_{T-1}, s_{T-1}) + \delta V_T(\phi(x_{T-1}, s_{T-1})) \quad (4)$$

Let us interpret this equation. Our problem at $t = T - 1$ is to pick a control variable x_{T-1} that maximises our immediate utility payoff $u(x_{T-1}, s_{T-1})$. We then know that our state variable in the final period will then be determined by $s_T = \phi(x_{T-1}, s_{T-1})$. Finally, we know that the highest obtainable value of the total reward we can get is $V_T(\phi(x_{T-1}, s_{T-1}))$ in the final period T .

We can now generalise this. Suppose that we are at time period $t = p$ and we are currently at state s_p . Our problem is to choose a sequence $x_p, x_{p+1}, \dots, x_T$ that maximises our discounted utility (where discounting is with reference to the current period p).

$$\begin{aligned} \max_{x_p, x_{p+1}, \dots, x_T} \sum_{t=p}^T \delta^{t-p} u(x_t, s_t) \\ \text{s.t. } s_{t+1} = \phi(x_t, s_t) \end{aligned} \quad (5)$$

We can then break up our objective function into current utility and the sum of future discounted utility:

$$\sum_{t=p}^T \delta^{t-p} u(x_t, s_t) = u(x_p, s_p) + \sum_{t=p+1}^T \delta^{t-p} u(x_t, s_t)$$

However, through induction, we already know the optimal controls from time $t = p + 1$ onwards. Therefore, we can rewrite our problem as

$$\max_{x_p} u(x_p, s_p) + V_{p+1}(s_{p+1})$$

Substituting in $s_{p+1} = \phi(x_p, s_p)$, we have that our problem is now:

$$\max_{x_p} u(x_p, s_p) + V_{p+1}(\phi(x_p, s_p))$$

We also need to discount our V_{p+1} term back by one period as well:

$$\max_{x_p} u(x_p, s_p) + \delta V_{p+1}(\phi(x_p, s_p)) \quad (6)$$

To interpret this, at time $t = p$, we look for the optimal control x_p that will maximise our instant utility payoff $u(x_p, s_p)$. Then, by the state evolution equation, we have $s_{p+1} = \phi(x_p, s_p)$. At $t = p + 1$, we already know our optimal solution is given by $V_{p+1}(\phi(x_p, s_p))$. Therefore, we are done!

We can now introduce our new problem:

Definition 2 (Bellman Equation Problem)

Our Bellman equation problem is:

$$V(s) = \max_x \left\{ u(x, s) + \delta V(\phi(x, s)) \right\} \quad (7)$$

whereby $V(\cdot)$ is the **value function** and $\phi(\cdot, \cdot)$ is the equation for the evolution of the state variable.

Remark 3. Notice that, in contrast to Eq. (6), we no longer have time subscripts in our Bellman equation problem. There are a few mathematical reasons for this which I won't go into, but it is due to the principle of optimality and the fact that we can show that this Bellman equation gives rise to a time-invariant policy function.

When we talk about solving the Bellman equation, we need to find a function $V(\cdot)$ such that Eq. (7) holds. Once we find such a function, we can also find the control sequence $\{x_t\}_{t=0}^{\infty}$. One common point is that the Bellman equation is a **functional equation**: the unknown object is the function $V(\cdot)$, which appears on both sides of the equation. As we change the state s , the maximum value we can achieve will also change.

Proposition 4 (Equivalency of solutions)

The solution to the Bellman equation problem is a solution to the sequence problem.

Remark 5. This result in fact holds both ways! That is, a solution to the sequence problem is also a solution to the Bellman equation problem.

Using the Bellman Equation to find Euler Equations

Suppose you have a sequence problem:

$$\begin{aligned} \max_{\{x_t\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \delta^t u(x_t, s_t) \\ \text{s.t.} \quad & s_{t+1} = \phi(x_t, s_t) \\ & s_0 \text{ is given} \end{aligned} \quad (8)$$

We give a cookbook on using the Bellman equation to get our Euler equations from it:

1. Write out the Bellman equation

$$V(s) = \max_x \left\{ u(x, s) + \delta V(\phi(x, s)) \right\}$$

2. Take the first-order condition with respect to the control x

$$0 = \nabla_x u(x, s) + \delta V'(\phi(x, s)) \nabla_x \phi(x, s)$$

whereby ∇_x denotes differentiation with respect to the control variable x .

3. Use the envelope theorem and differentiate with respect to the state s

$$\nabla_s V(s) = \nabla_s u(x, s) + \delta V'(\phi(x, s)) \nabla_s \phi(x, s),$$

evaluated at the optimal control.

4. You then iterate your envelope theorem condition and plug it into your first order condition (see example).

Example: Finding Euler Equations

Suppose our problem is

$$\begin{aligned} \max_{\{c_{t+s}, a_{t+s+1}\}} \quad & \sum_{s=0}^{\infty} \beta^s \frac{(c_{t+s})^{1-\sigma}}{1-\sigma} \\ \text{s.t.} \quad & a_{t+s+1} = y_{t+s} - c_{t+s} + R_{t+s} a_{t+s} \\ & \lim_{T \rightarrow \infty} \frac{a_{T+1}}{\prod_{s=t}^T R_s} = 0 \end{aligned} \tag{9}$$

Here, c_t is the control variable and a_t is our state variable. The first step is to write out the Bellman equation.

$$V(a_t) = \max_{c_t} \left\{ u(c_t, a_t) + \beta V(\phi(c_t, a_t)) \right\}$$

Remark 6. We have included time subscripts here. Strictly speaking, there should not be any subscripts (but I left them there to make it easier to read).

We therefore have that the Bellman equation is

$$V(a_t) = \max_{a_{t+1}} \left\{ u(R_t a_t + y_t - a_{t+1}) + \beta V(a_{t+1}) \right\} \tag{10}$$

whereby we have substituted $c_t = R_t a_t + y_t - a_{t+1}$. As a result, we now have that a_t is our state variable and a_{t+1} is still our control variable.

Now, we take the first-order condition for the Bellman equation in Eq. (10):

$$0 = \frac{\partial}{\partial a_{t+1}} u(R_t a_t + y_t - a_{t+1}) + \beta \frac{\partial}{\partial a_{t+1}} V(a_{t+1}) = -U_{c,t} + \beta V'(a_{t+1}) \tag{11}$$

One thing you will notice is that we now have $U_{c,t}$, which denotes the derivative of the utility function with respect to consumption c at time t . This may seem strange, as we substituted out the consumption control c_t . The budget constraint gives a one-to-one mapping between consumption c_t and next period's savings a_{t+1} for a given state a_t . I find it easier to use $U_{c,t}$ notation, as we will also differentiate with respect to a_{t+1} and the notation may otherwise get messy.

We can then take envelope theorem condition with respect to a_t to get:

$$\frac{\partial V}{\partial a_t} = \frac{\partial u(R_t a_t + y_t - a_{t+1})}{\partial a_t} = R_t U_{c,t} \tag{12}$$

We then iterate forward one step the envelope theorem Eq. (12). This means adding 1 to the indices.

$$\frac{\partial V}{\partial a_{t+1}} = V'(a_{t+1}) = R_{t+1} U_{c,t+1} \tag{13}$$

We now plug the iterated envelope condition in Eq. (13) into the first-order condition in Eq. (11):

$$-U_{c,t} + \beta [R_{t+1} U_{c,t+1}] = 0$$

We arrive at the **Euler equation**

$$U_{c,t} = \beta R_{t+1} U_{c,t+1} \tag{14}$$

We then recall that the derivative of the CRRA utility function is

$$U_{c,t} = (c_t^i)^{-\sigma}$$

Substituting the CRRA marginal utility into the Euler equation gives us:

$$(c_t)^{-\sigma} = \beta R_{t+1} (c_{t+1})^{-\sigma}$$

We can re-arrange to get our final answer for the Euler equation:

$$\beta(1 + r_{t+1}) = \left(\frac{c_{t+1}}{c_t} \right)^\sigma \quad (15)$$

Example 2: Finding Euler Equations

Suppose our problem is:

$$\begin{aligned} \max_{\{c_{t+s}^1, a_{t+s+1}^1\}} \quad & \sum_{s=0}^{\infty} \beta^s \ln(c_{t+s}^1) \\ \text{s.t.} \quad & a_{t+s+1}^1 = \bar{e} - c_{t+s}^1 + R_{t+s} a_{t+s}^1 \\ & \lim_{T \rightarrow \infty} \frac{a_{T+1}^1}{\prod_{s=t}^T R_s} = 0 \end{aligned} \quad (16)$$

We will use dynamic programming to solve this. The Bellman equation is given by:

$$V(a_t^1) = \max_{a_{t+1}^1} \left\{ \ln(R_t a_t^1 + \bar{e} - a_{t+1}^1) + \beta V(a_{t+1}^1) \right\} \quad (17)$$

Take the first order condition for Eq. (17) and we get:

$$0 = -\frac{1}{R_t a_t^1 + \bar{e} - a_{t+1}^1} + \beta V'(a_{t+1}^1) \quad (18)$$

Now apply the envelope theorem to our Bellman equation Eq. (17):

$$\frac{\partial V}{\partial a_t^1} = \frac{R_t}{R_t a_t^1 + \bar{e} - a_{t+1}^1}$$

Iterate the envelope theorem condition forward one step:

$$\frac{\partial V}{\partial a_{t+1}^1} = V'(a_{t+1}^1) = \frac{R_{t+1}}{R_{t+1} a_{t+1}^1 + \bar{e} - a_{t+2}^1} \quad (19)$$

Plug our iterated envelope condition in Eq. (19) back into the first-order equation in Eq. (18):

$$\frac{1}{R_t a_t^1 + \bar{e} - a_{t+1}^1} = \beta V'(a_{t+1}^1) = \beta \frac{R_{t+1}}{R_{t+1} a_{t+1}^1 + \bar{e} - a_{t+2}^1}$$

Recall that $c_t^1 = R_t a_t^1 + \bar{e} - a_{t+1}^1$ so we then have

$$\frac{1}{c_t^1} = \frac{R_{t+1} \beta}{c_{t+1}^1}$$

We arrive at the **Euler equation**:

$$R_{t+1} \beta = \frac{c_{t+1}^1}{c_t^1} \quad (20)$$

Example: Finding the Value and Consumption Function

We now show how we can solve the Bellman equation for the value function $V(\cdot)$.

Suppose that we had

$$\begin{aligned} \max_{\{c_t\}} \quad & \sum_{t=0}^{\infty} \delta^t \ln(c_t) \\ \text{s.t.} \quad & s_{t+1} = (1+r)(s_t - c_t) \end{aligned} \quad (21)$$

Here, c_t is the control variable and s_t is the state variable.

First, we write the Bellman equation

$$V(s_t) = \max_{c_t} \left\{ \ln(c_t) + \delta V((1+r)(s_t - c_t)) \right\} \quad (22)$$

To find out what the form is for the value function $V(\cdot)$, our strategy is to **guess** what form our value function takes and solve for the coefficients of our guess.

Remark 7. *Guessing may seem strange, but we also do this a lot when solving differential equations. This is quite a common technique.*

Generally, our guess for the value function will be based on our utility function. As our utility function was $u(c_t) = \ln(c_t)$, our guess for the value function will be

$$V(s_t) = C_1 + C_2 \ln(s_t) \quad (23)$$

whereby C_1, C_2 are coefficients that we need to solve for. We plug in our guess for the value function into our Bellman equation Eq. (22)

$$V(s_t) = \ln(c_t) + \delta C_1 + \delta C_2 \ln((1+r)(s_t - c_t)) \quad (24)$$

We then take first order conditions (with respect to c_t)

$$\frac{1}{c_t} - \delta C_2 \frac{1}{s_t - c_t} = 0 \quad (25)$$

We then re-arrange to solve for our control variable c_t

$$c_t = \frac{s_t}{1 + \delta C_2} \quad (26)$$

We then equate our value function guess 23, and our Bellman function with our guess Eq. (24) to each other

$$C_1 + C_2 \ln(s_t) = \ln(c_t) + \delta C_1 + \delta C_2 \ln((1+r)(s_t - c_t)) \quad (27)$$

We can then substitute in our expression for consumption $c_t = \frac{s_t}{1 + \delta C_2}$ and re-arrange to get

$$(1 - \delta)C_1 + (1 - \delta)C_2 \ln(s_t) = \ln\left(\frac{1}{1 + \delta C_2}\right) + \delta C_2 \ln\left(\frac{\delta C_2}{1 + \delta C_2}(1 + r)\right) + \ln(s_t)$$

Now, notice that we have the term $\ln(s_t)$ on both sides. By matching coefficients, we have that

$$\begin{aligned} (1 - \delta)C_2 &= 1 \\ \Rightarrow C_2 &= \frac{1}{(1 - \delta)} \end{aligned}$$

We then have that the remaining terms must equal each other

$$(1 - \delta)C_1 = \ln\left(\frac{1}{1 + \delta C_2}\right) + \delta C_2 \ln\left(\frac{\delta C_2}{1 + \delta C_2}(1 + r)\right)$$

Plugging in the expression C_2 we just derived, and solving for C_1 , we can conclude that

$$C_1 = \frac{\ln(1-\delta)}{(1-\delta)} + \frac{\delta}{(1-\delta)^2} \ln(\delta(1+r))$$

Therefore, we can conclude that our value function is

$$V(s_t) = C_1 + C_2 \ln(s_t) = \frac{\ln(1-\delta)}{(1-\delta)} + \frac{\delta}{(1-\delta)^2} \ln(\delta(1+r)) + \frac{1}{(1-\delta)} \ln(s_t) \quad (28)$$

We can also solve for our consumption function in Eq. (26) by plugging in our known value for C_2

$$c_t = \frac{s_t}{1 + \delta \frac{1}{(1-\delta)}} \quad (29)$$

2 Optimal Control Problem

2.1 Introduction

We are interested in dynamic optimisation problems in which agents wish to maximise continuously discounted utility subject to constraints. There are a lot of analogies with static optimisation problems that we have already seen. We first list some of the main differences between the problems.

The first difference is that we now discount our utility function $u(\cdot)$ **continuously**. With discrete discounting, we would multiply our utility function by a **discount factor** $\delta \in (0, 1)$ and compute discounted utility by taking a sum. For example, with three time periods indexed by $t = 0, 1, 2$, our sum of discounted utilities is

$$u(c_0) + \delta u(c_1) + \delta^2 u(c_2)$$

whereby c_i is our consumption decision at time $t = i$. In continuous time, we now have an **instantaneous discount rate** $\rho > 0$ and we multiply our utility function by our **continuous discount factor** $e^{-\rho t}$

$$e^{-\rho t} u(c_t) \quad \text{for all } t$$

Graphically, you can think of plotting the function $e^{-\rho t}$ and the utility function $u(c_t)$ from $t \in [0, T]$ and multiplying the two functions together **pointwise** for every t .

The second difference is that previously, our agents would be optimising their discounted utility by choosing a sequence of consumption choices $\{c_t\}_{t=0}^T = \{c_0, c_1, \dots, c_T\}$. Since we are now in continuous time, our agent is choosing a **consumption function** $c(t)$ for $t \in [0, T]$ to optimise her discounted utility. You can think back to our differential equations lecture where to solve a differential equation

$$\begin{cases} \dot{x} = f(t, x(t)) & \text{the law of motion} \\ x(0) = x_0 & \text{initial condition} \end{cases}$$

you are looking for a function $x(t)$ that satisfies the differential equation $\dot{x} = f(t, x(t))$ and its initial conditions $x(0) = x_0$. Our agent is doing something similar in the sense of trying to find an optimal consumption function $c(\cdot)$.

2.2 Optimal Control

We can now define the optimal control problem that we will come back to repeatedly. Suppose that we have an agent who wishes to maximise their continuously discounted future utility over a time period $[0, T]$. We can state this as an optimal control problem.

Definition 8 (Canonical Optimal Control Problem)

The agent wishes to find a consumption function $c(\cdot)$ and savings function $s(\cdot)$ that maximise their continuously discounted utility subject to constraints:

$$\begin{aligned} \max_{c(\cdot), s(\cdot)} \quad & \int_{t=0}^T e^{-\rho t} u(c(t), s(t)) dt \\ \text{s.t.} \quad & \dot{s}(t) = \phi(c(t), s(t)) \\ & s(0) = A \\ & s(T) = B \end{aligned} \tag{30}$$

where $\dot{s}(t) = \frac{ds}{dt}$.

Remark 9. In the canonical optimal control problem, our agent is trying to find both an optimal consumption function $c(\cdot)$ and optimal savings function $s(\cdot)$. We see that the (instantaneous) utility function depends both on the level of consumption and savings $u(c(t), s(t))$.

We call $\dot{s}$ the **law of motion** for savings, whereby $\phi(c(t), s(t))$ tells us how savings s evolve over time as a function of consumption $c(t)$ and savings $s(t)$.

Finally, $s(0) = A$ says that our **initial condition** for savings at time $t = 0$ is equal to A , and $s(T) = B$ says that our **terminal condition** for savings at the final period $t = T$ is equal to B .

Whenever you have a control problem, it is *very important* to identify what is your **control** and what is your **state** in the problem. Roughly speaking, the **control variable** is what the agent can choose and this is generally your level of consumption $c(t)$. The **state variable** is the variable that is affected by your control variable, which in this case, will be the level of savings $s(t)$. We can see that in our control problem, we are trying to find the optimal consumption (function) $c(\cdot)$ and this in turn will tell us what our optimal savings (function) $s(\cdot)$ is as well!

2.3 Present Value Hamiltonians

We now wish to solve the optimal control problem. Similar to how we used **lagrangians** to solve static optimisation problems, we will use **hamiltonians** to solve dynamic optimisation problems. Note that there are a few small differences between them.

We will first define (what I call anyways) the **Present Valued Hamiltonian**.

Definition 10 (Present Valued Hamiltonian)

The **Present Valued Hamiltonian** of the optimal control problem is

$$\mathcal{H}^d = e^{-\rho t} u(c(t), s(t)) + \lambda(t) [\phi(c(t), s(t))] \tag{31}$$

Let's break down our Present Value Hamiltonian in equation Eq. (31). The first term is the objective function in our optimal control problem **without** the integral. This is similar to our first term when setting up a Lagrangian in static optimisation. We then have the term $\lambda(t)$ and this **function** is the Hamiltonian equivalent to **lagrange multipliers**. You can think of the function $\lambda(\cdot)$ as Lagrange multipliers that are functions evolving over time, rather than fixed set numbers as seen in static optimisation. $\lambda(\cdot)$ goes by many different names such as the **costate variable** (as they are the shadow prices) and **adjoint variable**. Then, the terms in the brackets $[\phi(c), s(t)]$ is the law of

motion for capital $\dot{s}(t) = \phi(c(t), s(t))$ and you can think of this as a constraint that the differential equation must follow.

Remark 11. A notable difference between Hamiltonians and Lagrangians is that, for Hamiltonians, our constraint term is $\lambda(t)[\phi(c(t), s(t))]$ rather than $\lambda(t)[\phi(c(t), s(t)) - \dot{s}(t)]$. This is an important difference to remember.

With our Present Value Hamiltonian $\mathcal{H}^d$, we can now state the first order conditions.

Proposition 12 (First Order Conditions for Present Value Hamiltonian)

Let $\mathcal{H}^d = e^{-\rho t} u(c(t), s(t)) + \lambda(t)[\phi(c(t), s(t))]$ be our Present Value Hamiltonian. Then, our **four** first order conditions are

1. $\frac{\partial \mathcal{H}^d}{\partial c} = 0$ derivative with respect to **control variable** is equal to 0
2. $\frac{\partial \mathcal{H}^d}{\partial s} = -\dot{\lambda}(t)$ derivative with respect to **state variable** is equal to $-\dot{\lambda}(t)$
3. $\frac{\partial \mathcal{H}^d}{\partial \lambda} = \phi(c(t), s(t))$ derivative with respect to **costate variable** is the law of motion $\phi(c(t), s(t))$
4. $\lim_{t \rightarrow \infty} \lambda(t)s(t) = 0$ transversality condition for an infinite-horizon problem

Remark 13. The transversality or terminal condition depends on whether $T < \infty$ or $T = \infty$. For an infinite-horizon problem, the condition above requires $\lambda(t)s(t)$ to behave appropriately in the limit. For a fixed terminal state, the boundary condition is imposed directly instead.

Present Value Hamiltonian Example

We now work through an example. Suppose we have a utility maximising agent with log-utility and interest rate on savings as $r > 0$

$$\begin{aligned} \max_{c(\cdot), s(\cdot)} \quad & \int_{t=0}^T e^{-\rho t} \log(c(t)) dt \\ \text{s.t.} \quad & \dot{s}(t) = rs(t) - c(t) \\ & s(0) = s_0 \\ & s(T) = 0 \end{aligned} \tag{32}$$

We now recall our Present Value Hamiltonian equation

$$\mathcal{H}^d = e^{-\rho t} u(c(t), s(t)) + \lambda(t)[\phi(c(t), s(t))]$$

Following the expression, the Present Value Hamiltonian for our problem at hand is

$$\mathcal{H}^d = e^{-\rho t} \log(c(t)) + \lambda(t)[rs(t) - c(t)] \tag{33}$$

The first order conditions are

$$\begin{cases} \frac{\partial \mathcal{H}^d}{\partial c} = 0 & \leftrightarrow & \frac{e^{-\rho t}}{c(t)} = \lambda(t) \\ \frac{\partial \mathcal{H}^d}{\partial s} = -\dot{\lambda}(t) & \leftrightarrow & \lambda(t)r = -\dot{\lambda}(t) \\ \frac{\partial \mathcal{H}^d}{\partial \lambda} & \leftrightarrow & \dot{s}(t) = rs(t) - c(t) \end{cases} \tag{34}$$

We now want to somehow insert our first condition into our second condition. We can do this if we differentiate the first condition by t

$$\frac{d}{dt}\lambda(t) = \dot{\lambda}(t) = \frac{d}{dt}\left[\frac{e^{-\rho t}}{c(t)}\right] = -\rho\frac{e^{-\rho t}}{c(t)} - \frac{e^{-\rho t}\dot{c}(t)}{c^2(t)}$$

whereby we applied the product rule and then the chain rule (try to verify this for yourself!)

We then plug in our expressions $\lambda(t) = \frac{e^{-\rho t}}{c(t)}$ and $\dot{\lambda}(t) = -\rho\frac{e^{-\rho t}}{c(t)} - \frac{e^{-\rho t}\dot{c}(t)}{c^2(t)}$ into the second condition $\lambda(t)r = -\dot{\lambda}(t)$ to get

$$r\frac{e^{-\rho t}}{c(t)} = \rho\frac{e^{-\rho t}}{c(t)} + \frac{e^{-\rho t}\dot{c}(t)}{c^2(t)} \quad (35)$$

Get rid of $e^{-\rho t}$, multiply through by $c(t)$, and re-arrange to arrive

$$\dot{c}(t) = (r - \rho)c(t) \quad (36)$$

This is a **first-order linear differential equation!** We know how to solve this from our maths crash course. Also recall that we have a differential equation for $\dot{s}(t)$ from our **law of motion**. We therefore have

$$\begin{cases} \dot{c}(t) = (r - \rho)c(t) \\ \dot{s}(t) = rs(t) - c(t) \end{cases} \quad (37)$$

This is a **first-order linear ODE system!!!** We can solve for the functions $c(\cdot)$ and $s(\cdot)$ using the techniques from the math crash course. We can also plot the phase diagram by computing the eigenvalues and eigenvectors of this ODE system.

Therefore, to summarise, we have taken our continuous time optimal control problem, expressed it in terms of the Present Value Hamiltonian, solved for the first order conditions, and then ended up with a first-order linear ODE system which we can solve (using Wolfram alpha). Magic!

2.4 Current Value Hamiltonians

We have previously defined the present-value Hamiltonian:

$$\mathcal{H}^d = e^{-\rho t}u(c(t), s(t)) + \lambda(t)[\phi(c(t), s(t))]$$

However, we can also define the **current value Hamiltonian** $\mathcal{H}^c$.

Definition 14 (Current Value Hamiltonian)

The **current value Hamiltonian** of the optimal control problem is

$$\mathcal{H}^c = u(c(t), s(t)) + \mu(t)[\phi(c(t), s(t))] \quad (38)$$

where $\mu(t) = e^{\rho t}\lambda(t)$.

We can see that the difference between the current-value Hamiltonian and the present-value Hamiltonian is that the current-value Hamiltonian **does not** have a discount factor $e^{-\rho t}$ in front of the utility function. Its costate variable is rescaled according to $\mu(t) = e^{\rho t}\lambda(t)$. This also implies that the first-order conditions for the current-value Hamiltonian are different.

Proposition 15 (First Order Conditions for Current Value Hamiltonian)

Let $\mathcal{H}^c = u(c(t), s(t)) + \mu(t)[\phi(c(t), s(t))]$ be our current value Hamiltonian. Then, our **four** first order conditions are

1. $\frac{\partial \mathcal{H}^c}{\partial c} = 0$ derivative with respect to **control variable** is equal to 0
2. $\frac{\partial \mathcal{H}^c}{\partial s} = -\dot{\mu}(t) + \rho\mu(t)$ derivative with respect to **state variable** is equal to $-\dot{\mu}(t) + \rho\mu(t)$
3. $\frac{\partial \mathcal{H}^c}{\partial \mu} = \phi(c(t), s(t))$ derivative with respect to **costate variable** is the law of motion $\phi(c(t), s(t))$
4. $\lim_{t \rightarrow \infty} e^{-\rho t} \mu(t) s(t) = 0$ transversality condition holds

Remark 16. *The important thing to notice is that the second condition has changed, as we now have an extra $+\rho\mu$ term.*

Let us go back to our utility maximisation problem example from the previous section. We will now write out the current value Hamiltonian and first order conditions for Eq. (32).

The current value Hamiltonian is

$$\mathcal{H}^c = \log(c(t)) + \mu(t)[rs(t) - c(t)] \quad (39)$$

The first order conditions are now

$$\begin{cases} \frac{\partial \mathcal{H}^c}{\partial c} = 0 & \leftrightarrow & \frac{1}{c(t)} = \mu(t) \\ \frac{\partial \mathcal{H}^c}{\partial s} = -\dot{\mu}(t) + \rho\mu(t) & \leftrightarrow & \mu(t)r = -\dot{\mu}(t) + \rho\mu(t) \\ \frac{\partial \mathcal{H}^c}{\partial \mu} & \leftrightarrow & \dot{s}(t) = rs(t) - c(t) \end{cases} \quad (40)$$

We differentiate the first condition by t (using chain rule) to get

$$\dot{\mu}(t) = \frac{-\dot{c}(t)}{c^2(t)}$$

We then substitute our expressions for $\mu(t) = 1/c(t)$ and $\dot{\mu}(t) = \frac{-\dot{c}(t)}{c^2(t)}$ into the second condition to get

$$\frac{r}{c(t)} = \frac{\dot{c}(t)}{c^2(t)} + \frac{\rho}{c(t)}$$

We can then multiply through by $c(t)$ and re-arrange to get

$$\dot{c}(t) = (r - \rho)c(t) \quad (41)$$

Combining this with our law-of-motion equation, we have

$$\begin{cases} \dot{c}(t) = (r - \rho)c(t) \\ \dot{s}(t) = rs(t) - c(t) \end{cases} \quad (42)$$

This is the exact same first-order linear ODE system we arrived at when using the Present Value Hamiltonian approach in Eq. (37)!

2.5 Equivalence between Present-Value and Current-Value Hamiltonians

This shows us that using the Present Value and current value Hamiltonian approaches are equivalent to one another. We can prove this.

Proposition 17 (Equivalence between Present-Value and Current-Value Hamiltonians)

The approaches using the present-value Hamiltonian $\mathcal{H}^d$ and the current-value Hamiltonian $\mathcal{H}^c$, alongside their respective first-order conditions, are equivalent.

Heuristic. We will show how to go from Present Value Hamiltonian to Current Value Hamiltonian. Recall that the Present Value Hamiltonian is given by

$$\mathcal{H}^d = e^{-\rho t} u(c(t), s(t)) + \lambda(t) [\phi(c(t), s(t))]$$

We can get the Current Value Hamiltonian by multiplying the Present Value Hamiltonian by $e^{\rho t}$:

$$\mathcal{H}^c = e^{\rho t} \mathcal{H}^d = u(c(t), s(t)) + e^{\rho t} \lambda(t) [\phi(c(t), s(t))] = u(c(t), s(t)) + \mu(t) [\phi(c(t), s(t))]$$

whereby, in the last equation, we defined $\mu(t) = e^{\rho t} \lambda(t)$. This expression $\mu(t)$ is known as the **current shadow price**. We therefore have that our current-value Hamiltonian is

$$\mathcal{H}^c = u(c(t), s(t)) + \mu(t) [\phi(c(t), s(t))]$$

We now derive the first order condition for the Current Value Hamiltonian $\frac{\partial \mathcal{H}^c}{\partial s} = -\dot{\mu}(t) + \rho \mu(t)$ which was different to the first order condition for the Present Value Hamiltonian.

First, recall that $\mu(t) = e^{\rho t} \lambda(t)$. If we take the derivative with respect to t and apply the product rule, we have

$$\frac{d}{dt} \mu(t) = \dot{\mu}(t) = \rho e^{\rho t} \lambda(t) + e^{\rho t} \dot{\lambda}(t) = \rho \mu(t) + e^{\rho t} \dot{\lambda}(t)$$

whereby we use the fact that $e^{\rho t} \lambda(t) = \mu(t)$. We therefore have the expression

$$\dot{\lambda}(t) = e^{-\rho t} (\dot{\mu}(t) - \rho \mu(t)) \quad (43)$$

We have previously shown that $\mathcal{H}^c = e^{\rho t} \mathcal{H}^d$ and taking derivative with respect to s ,

$$\frac{\partial \mathcal{H}^c}{\partial s} = e^{\rho t} \left(\frac{\partial \mathcal{H}^d}{\partial s} \right) = -e^{\rho t} \dot{\lambda}(t)$$

whereby we used the fact that the first-order condition for the present-value Hamiltonian is $\frac{\partial \mathcal{H}^d}{\partial s} = -\dot{\lambda}(t)$. We can also plug in the expression for $\dot{\lambda}(t)$ derived in Eq. (43) to get

$$e^{\rho t} (-e^{-\rho t} (\dot{\mu}(t) - \rho \mu(t))) = -e^{\rho t} e^{-\rho t} (\dot{\mu}(t) - \rho \mu(t)) = -\dot{\mu}(t) + \rho \mu(t)$$

We have hence shown the first-order condition

$$\frac{\partial \mathcal{H}^c}{\partial s} = -\dot{\mu}(t) + \rho \mu(t)$$

□

3 Foundations of Neoclassical Growth

This section contains some background foundations we will constantly refer to.

3.1 Firm Production

We assume that the economy admits a **representative firm** with a representative (or aggregate) production function. This aggregate production function is given by:

$$Y_t = F(K_t, L_t, A_t) \quad (44)$$

whereby Y_t is total production of the final good at time t , K_t is the capital stock, L_t is the labour employed, and A_t is technology. We recall an important property of functions that we will need.

Definition 18 (Homogeneous of degree r)

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is homogeneous of degree r if:

$$f(\lambda \mathbf{x}) = \lambda^r f(\mathbf{x}) \quad (45)$$

for all $\lambda \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbb{R}^n$.

Definition 19 (Constant Returns to Scale)

A production function $f(\cdot)$ that is homogeneous of degree 1 is said to exhibit **constant returns to scale**:

$$f(\lambda \mathbf{x}) = \lambda f(\mathbf{x}) \quad (46)$$

for all $\lambda \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbb{R}^n$.

We can impose standard assumptions on the production function.

Assumption (The Production Function Is Continuously Differentiable, Has Positive and Diminishing Marginal Products, and Has Constant Returns to Scale)

The production function $F : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ is twice continuously differentiable in K and L .

Furthermore, the production function F has positive marginal product and diminishing returns:

$$F_K(K, L, A) \equiv \frac{\partial F(K, L, A)}{\partial K} > 0, \quad F_L(K, L, A) \equiv \frac{\partial F(K, L, A)}{\partial L} > 0$$

$$F_{KK}(K, L, A) \equiv \frac{\partial^2 F(K, L, A)}{\partial K^2} < 0, \quad F_{LL}(K, L, A) \equiv \frac{\partial^2 F(K, L, A)}{\partial L^2} < 0$$

Finally, the production function F is **homogeneous of degree 1** in K and L :

$$F(\lambda K, \lambda L, A) = \lambda F(K, L, A). \quad (47)$$

We elaborate on these assumptions. First, the production function F takes non-negative arguments K, L, A and maps them to a non-negative level of output in $\mathbb{R}_+$. Positive marginal products imply that production increases with the amount of each input. Diminishing marginal products mean that adding more capital or labour, holding everything else constant, increases output by progressively smaller amounts. Finally, a function that is homogeneous of degree 1 exhibits **constant returns to scale** in K and L . This gives us a useful result for the firm's maximisation problem.

Proposition 20 (Concavity of Production Function)

Under our assumptions, the production function is concave.

Constant returns to scale are also useful because of the following theorem.

Theorem 21 (Euler's Theorem)

If the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is homogeneous of degree r , then for all $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$:

$$x_1 f_1(\mathbf{x}) + x_2 f_2(\mathbf{x}) + \dots + x_n f_n(\mathbf{x}) = r f(\mathbf{x}) \quad (48)$$

whereby $f_j \equiv \frac{\partial f}{\partial x_j}$ for $j = 1, 2, \dots, n$. In vector notation, this can be expressed as:

$$\mathbf{x} \cdot \nabla f(\mathbf{x}) = r f(\mathbf{x}) \quad (49)$$

where ∇ is the gradient operator.

Moreover, $f_j(\mathbf{x})$ is homogeneous of degree $r - 1$ in $\mathbf{x}$ for all $j = 1, \dots, n$.

Another common set of boundary conditions we impose on the production function are the Inada conditions.

Assumption (Inada Conditions)

The production function $F(\cdot)$ satisfies the Inada conditions:

$$\lim_{K \rightarrow 0} F_K(K, L, A) = \infty \quad \text{and} \quad \lim_{K \rightarrow \infty} F_K(K, L, A) = 0 \quad \text{for all } L \geq 0 \text{ and all } A \quad (50)$$

$$\lim_{L \rightarrow 0} F_L(K, L, A) = \infty \quad \text{and} \quad \lim_{L \rightarrow \infty} F_L(K, L, A) = 0 \quad \text{for all } K \geq 0 \text{ and all } A \quad (51)$$

Remark 22. The Inada conditions say that the first units of capital and labour are highly productive. When capital or labour is abundant, their marginal product goes to zero.

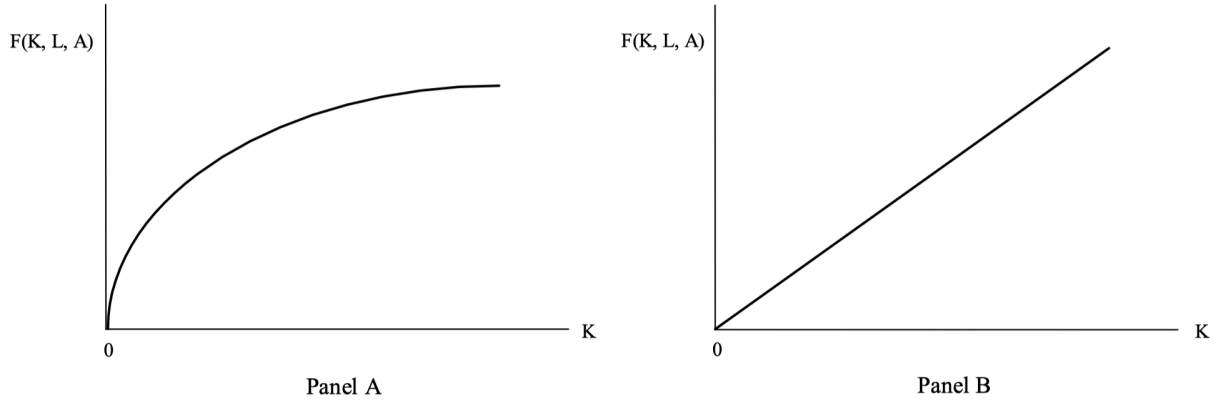


Figure 1: Two production functions. Panel A shows a production function satisfying the Inada conditions whilst panel B shows a production function **not** satisfying the Inada conditions.

3.2 Firms Make Zero Profits in Competitive Equilibrium

We constantly use the assumption that a firm with constant returns to scale in a perfectly competitive market does **not** make any economic profits. We now prove that. Furthermore, we also derive the problem of the representative firm which comes up quite a few times.

We assume that the only objective of firms is to maximize profits. We can consider the firm maximization problem for the representative firm. For notation, we drop the time subscripts and the technology term in the production function.

Definition 23 (Firm Maximisation Problem)

The firm's profit maximization problem is given by:

$$\max_{K, L} \left\{ P \cdot F(K, L) - R \cdot K - W \cdot L \right\} \quad (52)$$

where R is the rental rate of capital, W is the labour wage and P is the price of the good.

Proposition 24 (New Expression for the Firm's Profit-Maximisation Problem)

Assuming that the production function $F(\cdot, \cdot)$ exhibits constant returns to scale, we can rewrite the firm's profit-maximisation problem as:

$$\max_{k, L} \left\{ P \cdot L \left(f(k) - r \cdot k - w \right) \right\}, \quad (53)$$

whereby $f(k) \equiv F(K, 1)$ is the production technology per worker, $k \equiv \frac{K}{L}$ is the capital per worker, $r \equiv \frac{R}{P}$ is the real rental rate of capital, and $w \equiv \frac{W}{P}$ is the real wage rate.

Proof. We can use the fact that $F(\cdot, \cdot)$ exhibits constant returns to scale where we set $\lambda = \frac{1}{L}$.

$$\begin{aligned}
 P \cdot F(K, L) - R \cdot K - W \cdot L &= P \cdot \frac{L}{L} \cdot F(K, L) - R \cdot K - W \cdot L \\
 &= P \cdot L \cdot F\left(\frac{K}{L}, \frac{L}{L}\right) - R \cdot K - W \cdot L \\
 &= P \cdot L \cdot F(k, 1) - R \cdot K - W \cdot L \\
 &= P \cdot L \left[f(k) - \frac{R}{P} \cdot \frac{K}{L} - \frac{W}{P} \cdot \frac{L}{L} \right] \\
 &= P \cdot L \left[f(k) - r \cdot k - w \right]
 \end{aligned}$$

We then also have to change the choice variable from $K \rightarrow k$ in our maximisation problem. □

We now seek to find the optimal level of capital per worker k^* and labour L to maximise profits.

Proposition 25 (Optimal level of capital per worker)

The optimal level of capital per worker k^* is characterised by

$$f'(k^*) = r. \quad (54)$$

Proof. Fixing L , we can derive the FOC of our profit-maximisation problem

$$\max_k \left\{ P \cdot L \left(f(k) - r \cdot k - w \right) \right\}$$

with respect to k to get:

$$P \cdot L \cdot (f'(k)) - P \cdot L \cdot r = 0$$

Re-arrange and cancel out terms to get

$$f'(k^*) = r.$$

□

We now want to get the optimal level of labour L . Furthermore, we fix the optimal level of capital per worker k^* to be at the optimal level we previously derived.

Proposition 26 (Optimal level of labour)

Set k^* to be the optimal level of capital per worker. Then, the optimal level of labour L is characterised by:

$$f(k^*) - r k^* = w. \quad (55)$$

Proof. We can take first order conditions with respect to L of the firm's profit maximization problem

$$\max_L \left\{ P \cdot L \left(f(k^*) - r \cdot k^* - w \right) \right\}$$

to get

$$P \cdot \left(f(k^*) - r \cdot k^* - w \right) = 0$$

We can cancel out P to get our desired result

$$f(k^*) - rk^* = w.$$

□

We can now deduce our desired result that firms with constant returns to scale in perfectly competitive markets will have zero profits. To ensure a positive, finite equilibrium scale in this presentation, we make one more assumption: **labour supply is inelastic.**

Theorem 27 (Perfectly competitive firm with CRS production functions make no profits)

Suppose that we had a perfectly competitive firm with a CRS production function. Furthermore, assume that the supply of labour is inelastic. Then, the firm earns **no** economic profits in equilibrium.

Proof. We look at our characterisation for the optimal level of labour:

$$\max_L \left\{ P \cdot L \left(f(k^*) - r \cdot k^* - w \right) \right\}$$

There are three possible cases we can consider:

1. If $f(k^*) - r \cdot k^* > w$, the firm will demand labour up to ∞ , far exceeding the labour supply, which we assumed to be inelastic.
2. If $f(k^*) - r \cdot k^* < w$, the firm will employ no workers $L = 0$, which means that labour supply exceeds labour demand.
3. If $f(k^*) - r \cdot k^* = w$, the labour market clears as demand for labour will equal the (fixed) supply of labour.

We have seen earlier this third case occurs due to the optimal level of labour

$$f(k^*) - rk^* = w.$$

As a result, we have that our profits

$$\pi = P \cdot L \left(f(k^*) - r \cdot k^* - w \right) = P \cdot L(0) = 0$$

will be equal to zero in equilibrium.

□

Remark 28. In fact, given the assumptions that there are "no externalities" and all factors are priced competitively, then we can actually represent a countable number of perfectly competitive firms by a **single** representative firm.

3.3 CRRA Utility

The CRRA utility function is one that we use quite often in Macroeconomics.

Definition 29 (CRRA Utility)

The Constant Relative Risk Aversion (CRRA) utility function is given by:

$$U(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} \quad (56)$$

where $c_t > 0$ and $\sigma > 0$, with $\sigma \neq 1$. The parameter σ measures relative risk aversion; the case $\sigma = 1$ is defined by its log-utility limit below.

This utility function has some nice properties.

Proposition 30 (Homotheticity of CRRA)

The CRRA utility function is homothetic.

Proof. Recall that a utility function is homothetic if:

$$MRS(C_{t+s}, C_t) = MRS(\lambda C_{t+s}, \lambda C_t)$$

for any scalar λ . If we compute the MRS of the CRRA, we get:

$$MRS(C_{t+s}, C_t) = -\frac{u'(C_{t+s})}{u'(C_t)} = -\left(\frac{C_{t+s}}{C_t}\right)^{-\sigma}.$$

We then compare this to the scaled MRS:

$$MRS(\lambda C_{t+s}, \lambda C_t) = -\frac{u'(\lambda C_{t+s})}{u'(\lambda C_t)} = -\left(\frac{\lambda C_{t+s}}{\lambda C_t}\right)^{-\sigma} = -\left(\frac{C_{t+s}}{C_t}\right)^{-\sigma}.$$

We therefore see that the MRS are equal! □

Homotheticity implies that Marshallian demands scale proportionally with wealth, so Engel curves are straight rays through the origin. Under additional assumptions, including identical homothetic preferences and common prices, individual demands can be aggregated consistently and represented by a representative household.

Proposition 31 (CRRA Utility Properties)

The CRRA utility satisfies the following properties.

1. It is strictly concave.
2. It has a constant relative risk aversion.
3. It satisfies the Inada conditions.

Finally, we show that a common utility function that we use is a special case of CRRA utilities.

Proposition 32 (Log Utility)

The log utility is a limiting case of the CRRA utility function:

$$\lim_{\sigma \rightarrow 1} \frac{C_t^{1-\sigma} - 1}{1-\sigma} = \log(C_t). \quad (57)$$

3.4 Balanced Growth Paths

We now consider the fact that we want to be able to incorporate technological progress over time into our model, which is captured in $A(t)$. We wish for our models to satisfy a set of empirical results known as the **Kaldor facts**, some of which we lay out here.

Definition 33 (Kaldor Facts (1963))

The Kaldor facts are:

1. Output per capita increases over time;
2. Capital-output ratio is constant;
3. The interest rate is constant;
4. The distribution of income between capital and labour is constant.

An economy's path that is consistent with the Kaldor facts is known as a **balanced growth path** as these facts are temporal relationships. To try line up our models with the Kaldor facts, we can impose structure on our production function $F(K_t, L_t)$. As real wages are increasing over time, a natural structure to impose is that labour is more productive over time.

Definition 34 (Harrod-Neutral Technological Progress)

A labour-augmenting technological progress production function is given by:

$$F(K_t, L_t, A_t) = F(K_t, A_t L_t). \quad (58)$$

Remark 35. *This functional form implies that an increase in technology A_t increases output as if the economy had more labour.*

We will soon see that requiring our model to achieve a balanced growth path implies that technological progress must be labour augmenting. We shall use the fact that the capital-output ratio K_t/Y_t and the factor shares are constant to help show this. By factor shares, we mean:

$$\alpha_t^L \equiv \frac{w_t L_t}{Y_t} \quad \text{and} \quad \alpha_t^K \equiv \frac{R_t K_t}{Y_t}$$

Whilst it may seem arbitrary to use Harrod-neutral technological progress, the following lengthy proposition shows why this form arises when we want to explain balanced growth paths.

Proposition 36 (Uzawa (1961))

Consider a growth model with a constant-returns-to-scale aggregate production function $F(K_t, L_t, A_t)$ and an aggregate resource constraint specifying the evolution of capital K_t over time.

Suppose that there is a constant growth rate of the population $L_t = e^{nt} L_0$ and that there exists an asymptotic path where output, capital, and consumption grow at constant rates

$$g_Y \equiv \frac{\dot{Y}_t}{Y_t} \quad g_K \equiv \frac{\dot{K}_t}{K_t} \quad g_C \equiv \frac{\dot{C}_t}{C_t}.$$

Furthermore, suppose that $g_K + \delta > 0$ where δ is the rate of depreciation. Then,

1. $g_Y = g_K = g_C$ and
2. Asymptotically, the aggregate production function can be represented as:

$$Y_t = F(K_t, A_t L_t) \tag{59}$$

whereby

$$\frac{\dot{A}_t}{A_t} = g = g_Y - n.$$

We get really useful corollaries from this result.

Proposition 37 (Asymptotically Technological Progress)

If an economy has an asymptotic path with constant growth of output, capital, and consumption, then asymptotically technological progress can be represented as Harrod neutral. Furthermore, if factor markets are competitive, then asymptotic factor shares are constant:

$$\alpha_t^L \rightarrow \alpha_L^* \quad \alpha_t^K \rightarrow \alpha_K^*$$

as $t \rightarrow \infty$.

This implies that any asymptotic path with constant growth rates for output, capital and consumption must be a balanced growth path and can only be generated from an aggregate production function asymptotically featuring Harrod-neutral technological change.

As a result of having a Harrod-neutral production function, we generally need to analyse things in terms of **effective capital-labour ratios**:

$$k_t \equiv \frac{K_t}{A_t L_t}$$

since the **efficiency units** of labour are now given by $A_t L_t$. All of these results in this chapter will be used in the background from this point forward.

4 Simple General Equilibrium Model

4.1 Stylised Facts

In the growth-theory part of the course, the goal is to make sense of the Kaldor facts, a set of long-run empirical observations.

Theorem 38 (Kaldor Facts)

The Kaldor facts are:

1. Output per worker grows at a roughly constant rate that does not diminish over time.
2. Capital per worker grows over time.
3. The capital/output ratio is roughly constant.
4. The rate of return to capital is constant.
5. The shares of capital and labour in net income are nearly constant.
6. The real wage grows over time.
7. The ratios of consumption to GDP and investment to GDP are constant.

In the business cycle theory, we aim to try and make sense of the business cycle facts.

Theorem 39 (Business Cycle Stylised Facts)

The business cycle stylised facts are:

1. Consumption (non-durable) is smoother than output.
2. Volatility in GNP is similar in magnitude to volatility in total hours.
3. Volatility in employment is greater than volatility in average hours. Therefore most labour market adjustments operate on the extensive rather than intensive margin.
4. Productivity is slightly procyclical.
5. Wages are less variable than productivity.
6. There is no correlation between wages and output (nor for that matter with employment).

4.2 Simple General Equilibrium Model

We introduce a basic general equilibrium model. Suppose we have a consumer with a utility function $U(c_t^i)$ who wishes to maximise their discounted utility over time. Before we proceed, we need to impose an important assumption.

Assumption (Intertemporal Separability)

The utility function $U(c_t^i)$ is intertemporally separable.

Remark 40. Previous consumption choices do not influence marginal utility in the current period.

The agent has two decisions each period consisting of how much to consume c_t^i and how much to save a_{t+1}^i . They face an interest rate $R_t = (1 + r_t)$ on savings which they will take as given. Furthermore, we assume that they receive a known deterministic endowment y_t^i in every period t . With this, we can now state the consumer's problem.

Definition 41 (Optimisation Problem Facing Consumer)

The optimisation problem facing the consumer is:

$$\max_{\{a_{t+1}^i, c_t^i\}} \sum_{t=0}^{\infty} \beta^t U(c_t^i) \quad (60)$$

$$\text{s.t. } a_{t+1}^i = y_t^i - c_t^i + R_t a_t^i \quad (61)$$

$$\lim_{T \rightarrow \infty} \frac{a_{T+1}^i}{\prod_{t=0}^T R_t} \geq 0 \quad (62)$$

$$(63)$$

where this last condition is known as a **no-Ponzi condition**.

Remark 42. The no-Ponzi condition is different from a **transversality condition**. The no-Ponzi condition rules out debt that grows without bound relative to the relevant discount factor. Under the usual regularity conditions, the optimality or transversality condition holds with equality:

$$\lim_{T \rightarrow \infty} \frac{a_{T+1}^i}{\prod_{t=0}^T R_t} = 0.$$

Due to the recursive nature of this problem, we can get a nice result relating the total amount of consumption and endowment.

Proposition 43 (Intertemporal Budget Constraint)

The present discounted value of consumption must equal initial assets plus the present discounted value of the endowment stream:

$$\sum_{t=0}^{\infty} \frac{c_t^i}{Q_t} = a_0^i + \sum_{t=0}^{\infty} \frac{y_t^i}{Q_t} \quad (64)$$

where $Q_t \equiv \prod_{s=0}^t R_s$.

Proof. Look at the budget constraint:

$$a_{t+1}^i = y_t^i - c_t^i + R_t a_t^i$$

Notice that if we iterate it backward one period:

$$a_t^i = y_{t-1}^i - c_{t-1}^i + R_{t-1} a_{t-1}^i$$

We then plug the second equation into the first:

$$a_{t+1}^i = y_t^i - c_t^i + R_t \left[y_{t-1}^i - c_{t-1}^i + R_{t-1} a_{t-1}^i \right]$$

Expand to get:

$$a_{t+1}^i = y_t^i - c_t^i + R_t y_{t-1}^i - R_t c_{t-1}^i + R_{t-1} R_t a_{t-1}^i$$

Now, if we iterate the budget constraint back 2 periods, we get:

$$a_{t-1}^i = y_{t-2}^i - c_{t-2}^i + R_{t-2}a_{t-2}^i$$

which we can then plug in:

$$a_{t+1}^i = y_t^i - c_t^i + R_t y_{t-1}^i - R_t c_{t-1}^i + R_{t-1} R_t \left[y_{t-2}^i - c_{t-2}^i + R_{t-2} a_{t-2}^i \right]$$

and we end up with:

$$a_{t+1}^i = y_t^i - c_t^i + R_t y_{t-1}^i - R_t c_{t-1}^i + R_{t-1} R_t y_{t-2}^i - R_{t-1} R_t c_{t-2}^i + R_{t-2} R_{t-1} R_t a_{t-2}^i$$

or more conveniently:

$$a_{t+1}^i = \left[y_t^i + R_t y_{t-1}^i + R_{t-1} R_t y_{t-2}^i \right] - \left[c_t^i + R_t c_{t-1}^i + R_{t-1} R_t c_{t-2}^i \right] + R_{t-2} R_{t-1} R_t a_{t-2}^i$$

By induction, we obtain

$$\frac{a_{t+1}^i}{Q_t} = a_0^i + \sum_{j=0}^t \frac{y_j^i - c_j^i}{Q_j},$$

where $Q_j = \prod_{s=0}^j R_s$. Letting $t \rightarrow \infty$ and imposing the transversality condition gives

$$\sum_{t=0}^{\infty} \frac{c_t^i}{Q_t} = a_0^i + \sum_{t=0}^{\infty} \frac{y_t^i}{Q_t}.$$

□

4.3 Consumption Euler Equation

We show that it is easy to solve this optimisation problem and we get a nice result from it.

Proposition 44 (Euler Equation for Consumption)

The Euler equation for consumption is:

$$U'(c_t^i) = \beta R_{t+1} U'(c_{t+1}^i). \quad (65)$$

Proof. Construct the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[U(c_t^i) + \lambda_t \left[y_t^i - c_t^i + R_t a_t^i - a_{t+1}^i \right] \right]$$

Take first order conditions:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t^i} : \beta^t U'(c_t^i) - \beta^t \lambda_t &= 0 \\ \frac{\partial \mathcal{L}}{\partial a_{t+1}^i} : \beta^{t+1} \lambda_{t+1} R_{t+1} - \beta^t \lambda_t &= 0 \end{aligned}$$

We can simplify to get:

$$U'(c_t^i) = \lambda_t$$

$$\beta^{t+1} \lambda_{t+1} R_{t+1} = \beta^t \lambda_t.$$

We combine these two equations to get:

$$\beta^{t+1} U'(c_{t+1}^i) R_{t+1} = \beta^t U'(c_t^i)$$

and hence we have the result:

$$U'(c_t^i) = \beta R_{t+1} U'(c_{t+1}^i).$$

□

We now interpret the consumption Euler equation. It links the growth of marginal utility to the gross interest rate R_{t+1} . If consumers know that savings this period will earn a high return, there is an incentive to save more by lowering current consumption. For a given future consumption level, a lower level of current consumption corresponds to faster consumption growth.

4.4 General Equilibrium

We now want to see what the general equilibrium of this model is. To this end, we impose three assumptions.

Assumption 45 (Log-Utility)

The utility function $U(c_t^i)$ is a log-utility function:

$$U(c_t^i) = \log(c_t^i). \quad (66)$$

The next assumption says that the interest rate clears the borrowing and saving market. As there is no aggregate savings device (e.g. capital or bonds in positive net supply), market clearing requires individual net claims to sum to zero.

Assumption 46 (Savings Market Clears)

Net savings is equal to zero:

$$\sum_i a_t^i = 0. \quad (67)$$

Finally, as a result of the savings market clearing, we must have that all goods are consumed each period.

Assumption 47 (Goods Market Clearing)

Aggregate endowment must equal aggregate consumption:

$$\sum_i y_t^i = \sum_i c_t^i. \quad (68)$$

Under log utility and a common interest rate, we can aggregate the individual Euler equations directly.

Proposition 48 (Aggregate Consumption)

Aggregate consumption under log utility is:

$$\sum_i c_{t+1}^i = \beta R_{t+1} \sum_i c_t^i. \quad (69)$$

We can now define what the interest rate will be in this model that will clear the savings market.

Proposition 49 (Interest Rate under General Equilibrium)

The interest rate under equilibrium is:

$$R_{t+1} = \frac{1}{\beta} \frac{\sum_i y_{t+1}^i}{\sum_i y_t^i}. \quad (70)$$

Proof. Recall that aggregate consumption is:

$$\sum_i c_{t+1}^i = \beta R_{t+1} \sum_i c_t^i.$$

Due to market clearing, this aggregate consumption then becomes:

$$\sum_i y_{t+1}^i = \beta R_{t+1} \sum_i y_t^i.$$

□

This says that the equilibrium interest rate depends only on aggregate endowment. Under the assumptions made here, the distribution of endowment across households does not affect this equation; only aggregate endowment over time matters.

4.5 Two Period Consumption and Savings Model

This section will elaborate more on consumption-saving models to explain the relationship between interest rates and consumption. We are interested in two-period consumption-savings models. We will be looking at **endowment economies** where total income will be exogenous and there is no production. As a result, in equilibrium, total consumption equals total income. Since aggregate consumption must equal aggregate income, prices will adjust to make this happen.

4.5.1 One Type of Agent: Discounted Bonds

Suppose that we have many agents with identical preferences and endowments. Furthermore, we assume that agents are price-takers and live for only two periods, t and $t + 1$. They receive deterministic and exogenous income Y_t and Y_{t+1} . The household can borrow or save in period t by buying or selling bonds, which we denote by B_t . A positive B_t means that the household saves, whilst a negative B_t means that it borrows. Saving here means lending resources by buying bonds; borrowers issue those bonds and repay them later. Thus, a borrower is a debtor and a saver is a lender. These bonds cost q_t units of consumption, so q_t is a real price measured in units of the consumption good.

The agents therefore have two flow budget constraints:

$$C_t + q_t B_t \leq Y_t$$

$$C_{t+1} \leq Y_{t+1} + B_t.$$

Interest does not appear explicitly because one bond costs q_t today and pays one unit of consumption tomorrow. From this, we can form the agent's optimisation problem.

Definition 50 (Optimisation Problem Facing Consumer in Two Period Model)

The optimisation problem facing the agent is:

$$\max_{C_t, C_{t+1}, B_t} \ln(C_t) + \beta \ln(C_{t+1}) \quad (71)$$

$$\text{s.t. } C_t + q_t B_t \leq Y_t \quad (72)$$

$$C_{t+1} \leq Y_{t+1} + B_t \quad (73)$$

$$(74)$$

The Lagrangian formed is then:

$$\mathcal{L} = \ln(C_t) + \beta \ln(C_{t+1}) + \lambda_1[Y_t - C_t - q_t B_t] + \lambda_2(Y_{t+1} + B_t - C_{t+1}).$$

We can then form the Euler equation by differentiating the first order conditions.

Proposition 51 (Euler Equation)

The Euler equation is:

$$q_t \frac{1}{C_t} = \beta \frac{1}{C_{t+1}}. \quad (75)$$

The intuition for this equation is that saving one fewer unit (reducing B_t by 1) increases current consumption by q_t , giving you a marginal utility of $q_t \frac{1}{C_t}$.

We now talk about the competitive equilibrium of this model.

Proposition 52 (Competitive Equilibrium)

The competitive equilibrium is a price q_t and allocations C_t, C_{t+1}, B_t such that:

1. All agents are maximising their utility subject to their budget constraints
2. The bond market clears:

$$B_t = 0.$$

If we look at the bond market clearing condition:

$$B_t = 0$$

since we have one type of agent, if that agent wants to save, then all agents want to save (by loaning to others through issuing bonds). This means we do not have a borrowing supply since not everyone can be a lender. Therefore, $B_t = 0$. The same logic holds if the agent wants to borrow and gives us $B_t = 0$ as well since not everyone in the economy can

be borrowers. Now, if we recall the first period budget constraint:

$$C_t + q_t B_t \leq Y_t$$

this implies that:

$$C_t = Y_t$$

and then by the second constraint:

$$C_{t+1} = Y_{t+1}.$$

We can then derive the price of the bonds.

Proposition 53 (Equilibrium Price of Bonds)

The equilibrium price of bonds is:

$$q_t = \beta \frac{Y_t}{Y_{t+1}}. \quad (76)$$

Proof.

$$C_t = Y_t$$

$$C_{t+1} = Y_{t+1}$$

Then, we plug this into the Euler equation:

$$q_t \frac{1}{C_t} = \beta \frac{1}{C_{t+1}}$$

to get:

$$q_t = \beta \frac{Y_t}{Y_{t+1}}.$$

□

This leads us to an important point. **The price of the bond q_t will always adjust such that the market-clearing condition $B_t = 0$ will hold.** An implication of this is that if endowments were constant across time $Y_t = Y_{t+1}$, we would have that the equilibrium price:

$$q_t = \beta.$$

We now perform comparative statics on this setup. Suppose that endowment Y_t increases. By the Permanent Income Hypothesis, households want to smooth their consumption intertemporally so they want to increase C_t and C_{t+1} . However, by the market clearing conditions, we must have that $C_t = Y_t$ and $C_{t+1} = Y_{t+1}$. That is, the household must consume all this extra endowment Y_t . The mechanism that is causing them to do this is the price of the bond q_t . That is, the price of the bond q_t adjusts to dissuade households from buying the bonds and consuming the extra endowment at time $t + 1$. In particular, when Y_t goes up, there is an increase in demand for bonds B_t as households want to save to increase consumption in the future. However, as the supply of bonds must equal 0, the increase in demand just drives up the price of bonds q_t . We see this in Fig. 2. As a result, we see that there are no mechanisms for agents to insure themselves against risks.

4.5.2 One Type of Agent: Normalised Bonds

We now give a different treatment of bonds where we now look at interest rates instead of prices. We now normalise the price of the bond in period t to be 1 and then the bond pays out $1 + r_t$ units of consumption in period $t + 1$ where r_t is the interest rate. We then have the new setup.

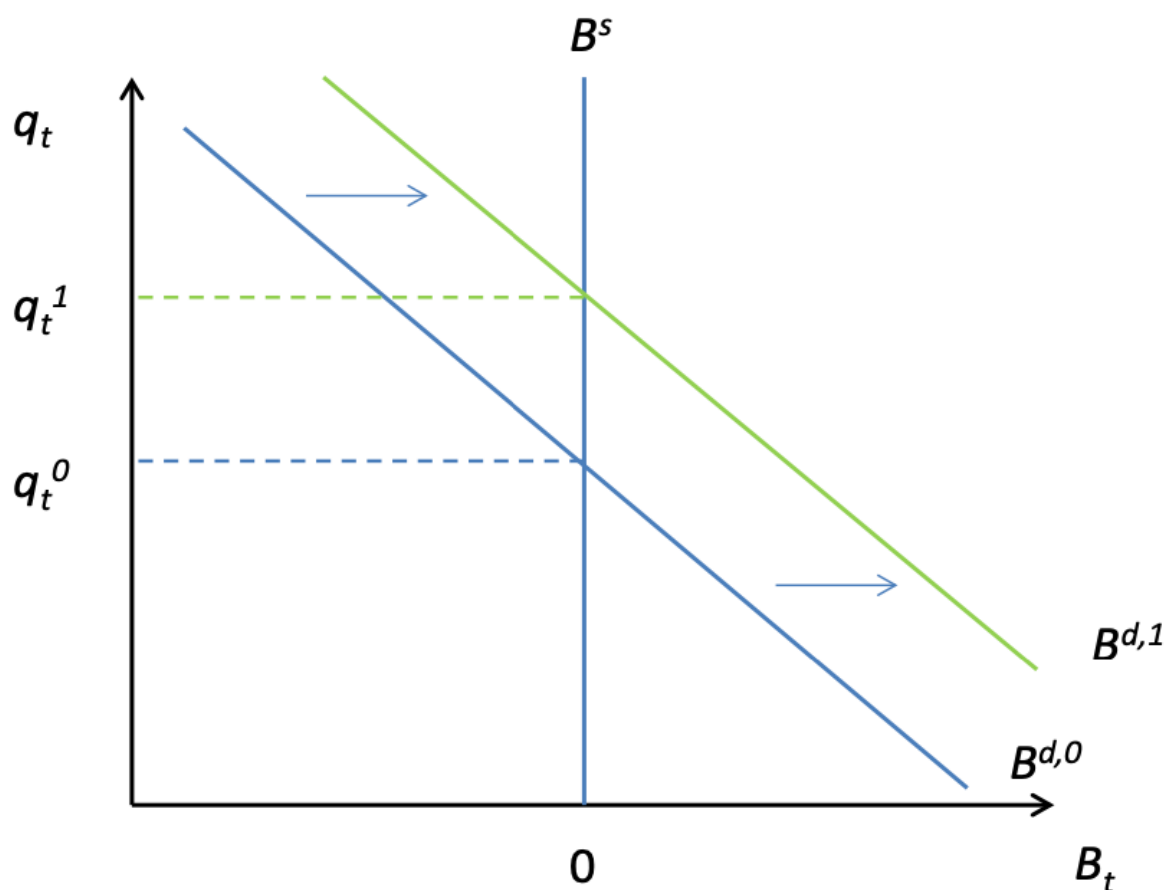


Figure 2: Current endowment Y_t increases. By the PIH, agents want to buy more bonds B in order to smooth consumption. This pushes up demand for bonds, but the fixed net supply of bonds raises the bond price q_t .

Definition 54 (Optimisation Problem Facing Consumer in Two Period Model)

The optimisation problem facing the agent is:

$$\max_{C_t, C_{t+1}, B_t} \ln(C_t) + \beta \ln(C_{t+1}) \quad (77)$$

$$\text{s.t. } C_t + B_t \leq Y_t \quad (78)$$

$$C_{t+1} \leq Y_{t+1} + (1 + r_t)B_t \quad (79)$$

$$(80)$$

Note that we have an equivalent setup to what we did previously whereby interest rates are inversely proportional

to the price of a bond:

$$1 + r_t = \frac{1}{q_t}.$$

We can derive and get the following Euler equation which is identical to what we did previously.

Proposition 55 (Euler Equation for Consumption)

The Euler equation for consumption is:

$$\frac{1}{C_t} = \beta(1 + r_t) \frac{1}{C_{t+1}}. \quad (81)$$

We have a similar mechanism in this scenario whereby if Y_t increases, agents will want to increase consumption in $t + 1$ and hence would want to buy bonds B_t . That means, in order for bond markets to clear $B_t = 0$ since we only have one type of agent, we require interest rates r_t to drop (or for prices q_t to rise). Therefore, we can see that in these models, the interest rates of the bonds, or the prices of the bonds, will always be adjusting in response to the demand and supply for bonds by consumers such that the market will clear.

So far, we have seen that no borrowing or lending occurs in these models where we only have one type of agent. However, later on, when we have heterogeneity, such as one household is more patient than another type of household, we will now have the impatient household wanting to borrow whilst the patient household wanting to lend. As a result, the interest rate of the bonds will need to adjust so that the saving markets clear (all borrowing is equal to all lending).

5 Overlapping Generations Model

5.1 Introduction

We now move on to neoclassical growth models. These are models with capital accumulation. We seek to investigate whether the accumulation of capital leads to sustained growth. On its own, it does not.

We will be looking at three markets:

1. Goods: Goods are sold and consumed by agents.
2. Labour: Agents can supply labour to earn a wage. However, in these growth models, people choosing to work more hours will not affect growth. Notice that this is a big contrast to business cycle models we will come across later.
3. Capital: We have a capital market whereby we can save through capital.

We make a big assumption throughout the next few chapters.

Assumption (Perfectly Competitive Markets)

Financial markets are redundant.

Remark 56. *The implication behind this assumption is that agents do not have any issues accessing loans or saving. This will be different when we look at macro-finance models.*

Finally, we have three agents with their own set of functions:

1. Households: They will be choosing how much to consume and work.
2. Firms: They will produce goods using capital and labour.
3. Government: They will conduct fiscal policy via taxing and spending in these models.

Bringing all this together, our main goal is to figure out the **determination of consumption and capital accumulation**.

5.2 Overlapping Generations Model

5.2.1 Household's Problem

We assume that a new generation is born every period and that each generation lives for A periods. As a result, multiple generations overlap and coexist at any time t .

We now walk through the setup of the model in great detail. For a given generation a that is born in time t , they will want to maximise their expected discounted utility $U^a(\cdot)$ where $U^a(\cdot)$ is the utility for an agent born in generation a . As they only live for A periods, their decision horizon is from $t, t + 1, \dots, t + A$. They will maximise this utility by choosing the level of capital K_{t+1}^{a+1} to hold and consumption C_t^a . To read off the notation C_{t+a}^a , the subscript represents the **period we are currently in** for an agent that is born in time t whilst the superscript represents the **age of the generation**. For example, suppose that agent was born in the year $t = 2000$. Then, if they were 12 years old, then $a = 12$. So then consumption in the year 2012 would be: $C_{2000+12}^{12}$.

Capital K_{t+a+1}^{a+1} serves as a saving vehicle for the agent. Since agents want to smooth consumption, they do not want to consume all their wealth in a given period unless it is their last period, so they require a vehicle in which to

invest. Capital therefore allows the agent to **transfer resources across periods**. To elaborate on the notation, if the agent was born in $t = 2000$ and was age $a = 12$ in 2012, the capital they choose to hold in 2013 is $K_{2000+12+1}^{12+1}$, as the agent will then be 13.

In stark contrast to other models we will later see (the real business cycle, 3-equation New Keynesian model), we shall make a big assumption on how much labour the agent will supply.

Assumption (Inelastic Labour Supply)

Labour is inelastically-supplied by the household.

The implication of this assumption is that every worker supplies a fixed amount of labour. They do not choose a level L_t to supply to the firm and instead supply $\bar{L} = 1$. This contrasts with the business-cycle models we will see later, in which agents have a disutility from working and choose how much they work.

This utility maximisation problem we have laid out is done with respect to a budget constraint whereby their purchase of capital K_{t+a+1}^{a+1} plus consumption C_{t+a}^a , must equal the income they receive, which is the wage from their labour w_t (recall they only supply 1 unit of labour regardless) and the return on capital less depreciation $(1 + r_{t+a} - \delta)K_{t+a}^a$. We can put all this together to get the optimisation problem facing generation a born at time t.

Definition 57 (Optimisation Problem Facing Generation)

The optimisation problem facing generation a born at time t is:

$$\begin{aligned} \max_{\{K_{t+a+1}^{a+1}, C_{t+a}^a\}_{a=0}^A} \quad & \sum_{a=0}^A \beta^a U^a(C_{t+a}^a) \\ \text{s.t.} \quad & K_{t+a+1}^{a+1} + C_{t+a}^a = w_{t+a} + (1 + r_{t+a} - \delta)K_{t+a}^a \end{aligned} \tag{82}$$

where δ is the depreciation, $1 + r_{t+a}$ is the rate of return on capital, and w_{t+a} is the wage.

5.2.2 Firm's Problem

We now look at the firm's problem. They will want to produce the consumption good for the household and sell it to them at a price P_t . However, we shall make an assumption to make our analysis easier.

Assumption (Numéraire Good)

The consumption good is the numéraire and hence will have a normalised price:

$$P_t = 1.$$

Therefore, the firm's revenue is simply its output y_t , since the price of the good is normalised to 1. Its costs are the amount of labour it hires, L_t , times the wage w_t , plus the amount of capital it employs, K_t , times the rental rate r_t . Its objective is to maximise revenue less costs, subject to the production function $y_t = f(L_t, K_t)$.

Definition 58 (Optimisation Problem Facing the Firm)

The firm maximisation problem is:

$$\begin{aligned} \max_{K_t, L_t} \quad & y_t - w_t L_t - r_t K_t \\ \text{s.t.} \quad & y_t = f(L_t, K_t) \end{aligned} \quad (83)$$

where $f(\cdot, \cdot)$ is a constant returns to scale production function.

Remark 59. Notice that the firm only chooses their demand for capital K_t and demand for labour L_t for a given period. This is due to the fact that markets are perfectly competitive without any rigidities and therefore they can adjust their production every period. This will be different to what we will see later on in the New Keynesian model whereby firms will face rigidities and hence need to choose a sequence.

Before, we proceed, we look at some notation again. Suppose that we are at time $t = 2000$. Then, due to the setup of our model whereby we have generations living for A periods, we will have multiple generations co-existing at a given point in time. More concretely, suppose that we are in a sad world where $A = 3$ so agents only live for 3 periods. Then, at the year $t = 2000$, we will have three generations alive where they have the ages of 0, 1, 2 (there is no one with an age of 3 as they will die at 3). Then, if we want to look at the aggregate amount of capital at time $t = 2000$, we can look at the sum of the amount of capital each of these generations are hold at $t = 2000$. However, it may be the case that a generation who has just died in this period has left behind some capital K_t^3 (however this will not be optimal for them to do so but we will include it anyways). Therefore, the total amount of capital at period t is:

$$K_t^0 + K_t^1 + K_t^2 + K_t^3 = \sum_{a=0}^A K_t^a.$$

Now that we have cleared up this notation, we state a useful result about the CRS production function in this setup.

Theorem 60 (Aggregate Output under CRS Production Function)

Suppose that $f(\cdot, \cdot)$ is a CRS production function. Then, aggregate output from all the firms is equal to:

$$\sum_{a=0}^A y_t^a = f\left(1, \sum_{a=0}^A K_t^a\right). \quad (84)$$

Proof. First, recall that the output for a firm is given by their production function:

$$y_t^a = f(L_t^a, K_t^a)$$

Using the firm's first-order conditions and the CRS production function, we can apply Euler's theorem in Eq. (48) to obtain

$$y_t^a = w_t L_t^a + r_t K_t^a.$$

Then, we can sum over all generations $a = 1, \dots, A$ at time t :

$$\sum_{a=0}^A y_t^a = w_t \sum_{a=0}^A L_t^a + r_t \sum_{a=0}^A K_t^a.$$

We assumed that labour supply is **inelastic** and hence $\sum_{a=0}^A L_t^a = 1$. Then we have that:

$$\sum_{a=0}^A y_t^a = w_t + r_t \sum_{a=0}^A K_t^a.$$

Therefore, we can apply Euler's theorem in reverse to the right-hand side to conclude that:

$$\sum_{a=0}^A y_t^a = f\left(1, \sum_{a=0}^A K_t^a\right).$$

□

5.2.3 General Equilibrium

We now want to tie everything together by looking at the equilibrium of this model. Recall that an equilibrium requires a sequence of control variables and prices such that agents are optimising and that all markets clear. We now state what is an equilibrium in this model.

Definition 61 (General Equilibrium in OLG Model)

An **equilibrium** in the OLG model is a sequence $\{r_t, w_t, \{C_t^a\}_{a=0}^A, \{K_{t+1}^a\}_{a=0}^A\}$ of interest rates, wage income, generation-specific consumption and capital holdings such that:

1. Given the sequence of interest rates and wages $\{r_t, w_t\}$, the consumption and capital allocation satisfy the household problem for each generation;
2. Given the sequence of interest rates and wages $\{r_t, w_t\}$, firms maximise profits;
3. The goods market clears for every time period t :

$$\sum_{a=0}^A C_t^a + \sum_{a=0}^A K_{t+1}^a = f\left(\sum_{a=0}^A K_t^a\right) + (1 - \delta) \sum_{a=0}^A K_t^a. \quad (85)$$

We give more interpretation to the goods market clearing condition:

$$\sum_{a=0}^A C_t^a + \sum_{a=0}^A K_{t+1}^a = f\left(\sum_{a=0}^A K_t^a\right) + (1 - \delta) \sum_{a=0}^A K_t^a.$$

The first term represents aggregate consumption (the consumption for every generation alive at period t). The second term represents aggregate savings for that period. On the right-hand side, the first term is how much production there is at time t . The final term is the capital (savings) left over from the previous periods net of depreciation.

To see the behaviour of the agents in this model, we need to use software to plot the results.

In Fig. 3, we look at a cross-sectional picture of the different generations' consumption at time t . On the x-axis, we plot the age of the generation. As a generation is older, they will have a higher wage. Therefore, we can see that when a generation is young, they consume less whilst they consume more when they are older as they have more wealth.

In Fig. 4, we look at another cross-sectional picture of the different generations' savings at time t . Generations borrow when young and save when older in order to repay their debts. The area under the entire curve is aggregate savings at time t .

In Fig. 5, we now look at a micro vs macro look of the model. On the x-axis, we now plot time and on the y-axis, we plot consumption. Then, we look at the consumption path for each generation over the course of their

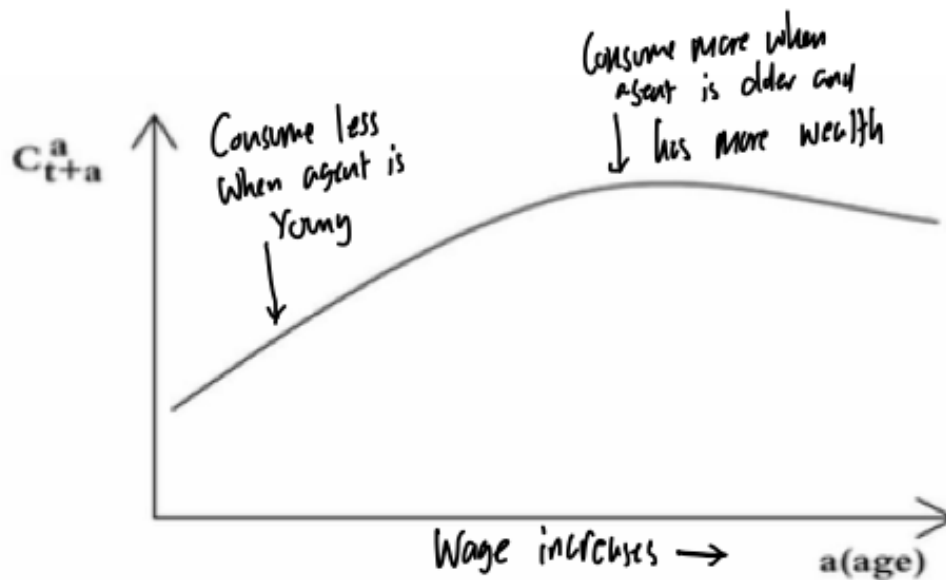


Figure 3: Cross-sectional plot of consumption at time t for all generations a .

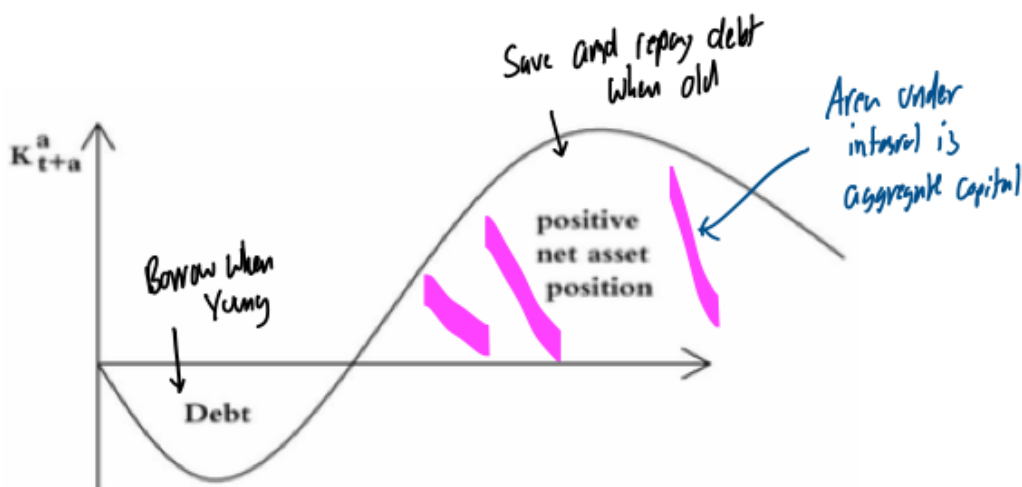


Figure 4: Cross-sectional plot of savings at time t for all generations a .

lifetime. We can see that a given generation will have an increasing consumption profile over time. This gives us a nice microeconomic result. Unfortunately, when we look at the aggregate consumption $\bar{C}$, we see that it is the same level of aggregate consumption for every period. This is not a very interesting result.

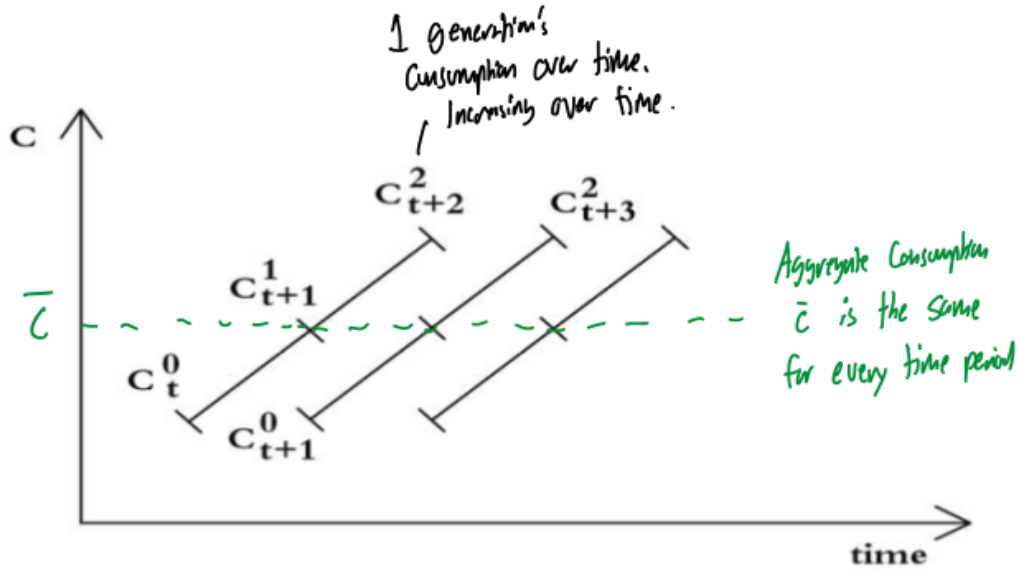


Figure 5: Time series plot of consumption for each time t for all generations a .

5.3 Two Generation Problem

So far, what we have developed is intractable for a general number of age groups A . We now restrict our analysis to two age groups, $A = 2$, and assume that each generation lives for two periods, denoted by 0 and 1. Households may work only in the first period and do not receive bequests. We also assume full depreciation, $\delta = 1$.

We also make three big assumptions.

Assumption (Household Inheritance, Labour and Borrowing)

Households are born penniless. Households only work in the first period. Retired households can only consume from savings they accumulated from when they were young.

With these assumptions, we can now state the optimisation problem facing the 2-generation household.

Definition 62 (Optimisation Problem Facing Generation)

The optimisation problem facing generation a born at time t is:

$$\begin{aligned} \max_{C_t^0, C_{t+1}^1, K_{t+1}^1} \quad & U(C_t^0) + \beta U(C_{t+1}^1) \\ \text{s.t.} \quad & K_{t+1}^1 + C_t^0 = w_t \\ & C_{t+1}^1 = r_{t+1} K_{t+1}^1 \end{aligned} \tag{86}$$

where $r_{t+1} > 0$ is the gross return on saving when depreciation is complete.

We can solve for the Euler equation for consumption.

Proposition 63 (Euler Equation for Consumption)

The Euler Equation for consumption in the two-generation OLG model is:

$$U'(C_t^0) = \beta r_{t+1} U'(C_{t+1}^1). \quad (87)$$

Proof. First, directly substitute in the budget constraints into the utility function:

$$U(w_t - K_{t+1}^1) + \beta U(r_{t+1} K_{t+1}^1).$$

This takes us from 3 control variables to 1 control variable. We then differentiate with respect to K_{t+1}^1 to get:

$$-U'(w_t - K_{t+1}^1) + r_{t+1} \beta U'(r_{t+1} K_{t+1}^1) = 0$$

We plug our budget constraints back in and we are done:

$$U'(C_t^0) = \beta r_{t+1} U'(C_{t+1}^1).$$

Alternatively, we could have done this using a Lagrangian where we take first order conditions with respect to $\{C_t^0, C_{t+1}^1, K_{t+1}^1\}$. □

This Euler equation describes the trade-off each generation makes when solving its optimisation problem.

We now move onto the firm sector in this setup. It is in fact an identical setup to what we saw in the general overlapping generations model.

Definition 64 (Optimisation Problem Facing the Firm)

The firm maximisation problem is:

$$\begin{aligned} \max_{K_t, L_t} \quad & y_t - w_t L_t - r_t K_t \\ \text{s.t.} \quad & y_t = f(L_t, K_t) \end{aligned} \quad (88)$$

where $f(\cdot, \cdot)$ is a constant returns to scale production function.

We can solve this problem to get the firm's FOC.

Proposition 65 (Firm's FOC)

The firm's first order equations are:

$$\begin{aligned} w_t &= f_L(L_t, K_t) \\ r_t &= f_K(L_t, K_t) \end{aligned}$$

Proof. Substitute in the production function into the profit function:

$$f(L_t, K_t) - w_t L_t - r_t K_t.$$

Now take partial derivatives with respect to L_t and K_t respectively:

$$f_L(L_t, K_t) - w_t = 0$$

$$f_K(L_t, K_t) - r_t = 0$$

□

We can now state the equilibrium in this model.

Definition 66 (Equilibrium in OLG Model)

An **equilibrium** in the OLG model is a sequence $\{r_t, w_t, C_t^0, C_t^1, K_{t+1}^1\}$ of interest rates, wage income, consumption and capital holdings such that:

1. Given the sequence of interest rates and wages $\{r_t, w_t\}$, the consumption and capital allocation $\{C_t^0, C_t^1, K_{t+1}^1\}$ satisfy the household problem for each generation;
2. Given the sequence of interest rates and wages $\{r_t, w_t\}$, firms maximise profits with respect to capital and labour demand;
3. The goods market clears:

$$C_t^0 + C_t^1 + K_{t+1}^1 = f(K_t^1, 1). \quad (89)$$

Remark 67. As labour is inelastically supplied $L_t^1 = 1$, we have that $f(K_t^1, L_t^1) = f(K_t^1, 1)$.

The goods market clearing condition:

$$C_t^0 + C_t^1 + K_{t+1}^1 = f(K_t^1, 1)$$

says that at time t , consumption by the young plus old, and the amount saved by young people must equal the total amount of production in the economy.

We now want to characterize the equilibrium in the economy. To do so, we need to impose some more structure on the model.

Assumption (Log Utility and Cobb-Douglas Production)

We assume that agents have a log utility function:

$$U(C_t^i) = \log(C_t^i) \quad \text{for } i = 0, 1$$

and firms have a Cobb-Douglas production function:

$$f(L_t, K_t) = L_t^{(1-\alpha)} K_t^\alpha.$$

We are interested in the evolution of capital in the economy as we hypothesised in the beginning that the source of growth will come from the accumulation of capital. We now state what is the recursive law of dynamics for capital in this model.

Proposition 68 (Evolution of Capital)

Under log-utility and Cobb-Douglas production function, the evolution of capital is given by:

$$K_{t+1}^1 = \frac{\beta}{1+\beta} (1-\alpha)(K_t^1)^\alpha. \quad (90)$$

Proof. First, recall the Euler equation for consumption Eq. (87):

$$U'(C_t^0) = \beta r_{t+1} U'(C_{t+1}^1).$$

Now, plug in the households' budget constraints on both sides:

$$U'(w_t - K_{t+1}^1) = \beta r_{t+1} U'(r_{t+1} K_{t+1}^1).$$

Now, we know that the marginal product of labour and capital from the firm's optimisation problem is:

$$w_t = f_L(L_t, K_t)$$

$$r_t = f_K(L_t, K_t)$$

so we can plug those in to get:

$$U'(f_L(L_t, K_t) - K_{t+1}^1) = \beta r_{t+1} U'(f_K(L_t, K_{t+1}) K_{t+1}^1).$$

We assumed that labour is inelastically supplied at 1 so this becomes:

$$U'(f_L(1, K_t) - K_{t+1}^1) = \beta r_{t+1} U'(f_K(1, K_{t+1}) K_{t+1}^1).$$

Now, we will use the functional form assumptions on the utility function. The first derivative of the utility function is:

$$U(C_t^i) = \log(C_t^i) \Rightarrow U'(C_t^i) = \frac{1}{C_t^i}.$$

Also recall from the firm's FOC that:

$$f_K(1, K_{t+1}) = r_{t+1}$$

This means we have that:

$$\begin{aligned}
U'(f_L(1, K_t) - K_{t+1}^1) &= \beta r_{t+1} U'(f_K(1, K_{t+1}) K_{t+1}^1) \\
\leftrightarrow \frac{1}{f_L(1, K_t) - K_{t+1}^1} &= \beta r_{t+1} \frac{1}{f_K(1, K_{t+1}) K_{t+1}^1} \\
\leftrightarrow \frac{1}{f_L(1, K_t) - K_{t+1}^1} &= \beta f_K(1, K_{t+1}) \frac{1}{f_K(1, K_{t+1}) K_{t+1}^1} \\
\leftrightarrow \frac{1}{f_L(1, K_t) - K_{t+1}^1} &= \beta \frac{1}{K_{t+1}^1} \\
\leftrightarrow f_L(1, K_t) - K_{t+1}^1 &= \frac{1}{\beta} K_{t+1}^1 \\
\leftrightarrow f_L(1, K_t) &= \left(\frac{1}{\beta} + 1\right) K_{t+1}^1 \\
\leftrightarrow f_L(1, K_t) &= \left(\frac{1+\beta}{\beta}\right) K_{t+1}^1 \\
\leftrightarrow \left(\frac{1+\beta}{\beta}\right) K_{t+1}^1 &= f_L(1, K_t) \\
\leftrightarrow K_{t+1}^1 &= \left(\frac{\beta}{\beta+1}\right) f_L(1, K_t) \\
\leftrightarrow K_{t+1}^1 &= \left(\frac{\beta}{\beta+1}\right) [(1-\alpha) L_t^{-\alpha} (K_t^1)^\alpha] \\
\leftrightarrow K_{t+1}^1 &= \left(\frac{\beta}{\beta+1}\right) [(1-\alpha)(1)(K_t^1)^\alpha]
\end{aligned}$$

We therefore have that:

$$K_{t+1}^1 = \frac{\beta}{1+\beta} (1-\alpha) (K_t^1)^\alpha.$$

□

We now interpret the evolution of capital. First, we note that the amount of capital in next period is always a fixed proportion of output:

$$K_{t+1}^1 = \underbrace{\frac{\beta}{1+\beta} (1-\alpha)}_{\text{Fixed Proportion}} \underbrace{(K_t^1)^\alpha}_{\text{Output}}.$$

This is quite similar to the Solow model, where we have a fixed saving rate in every period. Under the Cobb–Douglas assumption, $0 < \alpha < 1$, which implies that this sequence converges. Therefore,

$$K_{t+1}^1 \rightarrow K^*$$

where K^* is the steady-state in our model. We show this graphically in Fig. 6. The intuition for what is happening is that after a certain amount of capital that is accumulated, due to diminishing returns of the marginal product of capital, there is no point in accumulating even more capital. Hence, we don't have any further growth in this model which is not so good.

Finally, convergence to a steady state does not by itself imply dynamic inefficiency. An OLG equilibrium is dynamically inefficient when it overaccumulates capital relative to the golden-rule level. Under that condition, a suitably designed transfer from each young generation to the old can generate a Pareto improvement. Whether this condition holds must be checked rather than assumed.

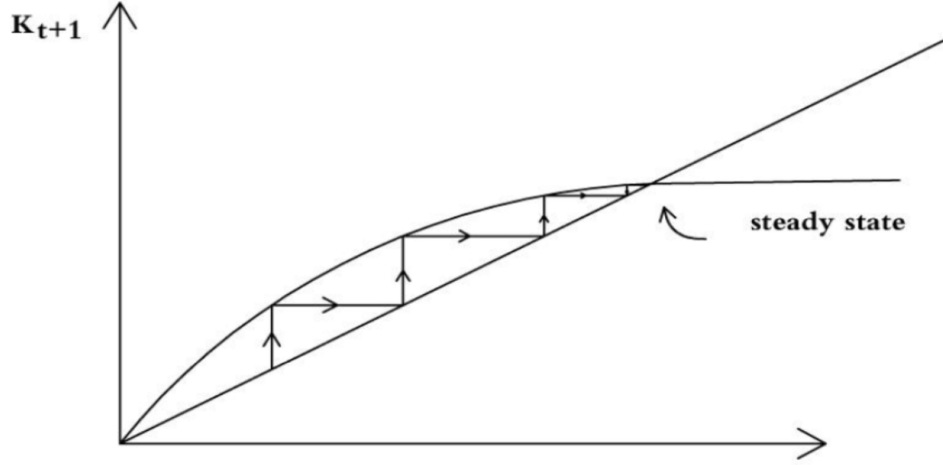


Figure 6: Evolution of capital over time. Converges to a steady-state.

5.4 Fiscal Policy and non-Ricardian Equivalence

We are now interested in fiscal policy and how it affects the model.

Definition 69 (Fiscal Policy)

Let the government's fiscal policy consist of $\{G_t, T_t^0, T_t^1, b_t\}$ where:

1. G_t is government spending on the **young**
2. T_t^0 is taxation levied on the young
3. T_t^1 is taxation levied on the old
4. b_t is government debt.

The government therefore has the decision of deciding how much to spend G_t and how to finance that spending through a mix of $\{T_t^0, T_t^1, b_t\}$. We can therefore write the government's budget constraint.

Definition 70 (Government Budget Constraint)

The government budget constraint is:

$$b_{t+1} = G_t - T_t^0 - T_t^1 + (1 + r_t^b)b_t \quad (91)$$

where $r_t^b = (r_t - 1)$ where r_t is the return on capital in order to prevent arbitrage.

Now, as we mentioned before, the government's spending G_t will directly enter the utility function of **only** the young generation. Furthermore, the young and old generation will have to pay taxes to the government. From this, we can now formulate the generation's optimisation problem.

Definition 71 (Optimisation Problem Facing Generation)

The optimisation problem facing the generation born at time t is:

$$\begin{aligned} \max_{C_t^0, C_{t+1}^1, b_{t+1}^1, K_{t+1}} \quad & U(C_t^0 + G_t) + \beta U(C_{t+1}^1) \\ \text{s.t.} \quad & K_{t+1}^1 + b_{t+1}^1 + C_t^0 = w_t - T_t^0 \\ & C_{t+1}^1 = r_{t+1}(b_{t+1}^1 + K_{t+1}^1) - T_{t+1}^1 \end{aligned} \quad (92)$$

We now want to examine the government's impact via fiscal policy on the generation's equilibrium behaviour. Before proceeding, we recall an important concept.

Definition 72 (Ricardian Equivalence)

Under the relevant assumptions, changing the timing of lump-sum taxes and debt finance does not change households' consumption choices or equilibrium allocations.

Remark 73. *The intuition is that agents can borrow to pay the tax and when they receive the transfer back, it washes out.*

We will soon see that based on the government's financing decision, Ricardian equivalence **does not** hold in the OLG model. To proceed, we make two more assumptions for our analysis.

Assumption (Log Utility and No Retirement Taxes)

We assume that households have log utility:

$$U(C_t^0 + G_t) + \beta U(C_{t+1}^1) = \log(C_t^0 + G_t) + \beta \log(C_{t+1}^1).$$

We assume no taxes on retirement:

$$T_t^1 = 0 \quad \forall t.$$

We can now derive the generation's Euler equation for consumption.

Proposition 74 (Euler Equation for Consumption)

The Euler equation for consumption under fiscal policy is:

$$C_{t+1}^1 = \beta f_K(1, K_{t+1})(C_t^0 + G_t). \quad (93)$$

Proof. First, we form the Lagrangian:

$$\mathcal{L} = U(C_t^0 + G_t) + \beta U(C_{t+1}^1) + \lambda_t[w_t - T_t^0 - K_{t+1}^1 - b_{t+1}^1 - C_t^0] + \mu_t[r_{t+1}(b_{t+1}^1 + K_{t+1}^1) - T_{t+1}^1 - C_{t+1}^1]$$

Now take first order conditions:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C_t^0} : U'(C_t^0 + G_t) - \lambda_t &= 0 \\ \frac{\partial \mathcal{L}}{\partial C_{t+1}^1} : \beta U'(C_{t+1}^1) - \mu_t &= 0 \\ \frac{\partial \mathcal{L}}{\partial b_{t+1}^1} : -\lambda_t + r_{t+1}\mu_t &= 0 \end{aligned}$$

We plug the first two order conditions for consumption into the one for bond holdings:

$$-U'(C_t^0 + G_t) + r_{t+1} \left[\beta U'(C_{t+1}^1) \right] = 0$$

Now, use the firm's first order condition $r_{t+1} = f_K(1, K_{t+1})$ to get:

$$U'(C_t^0 + G_t) = f_K(1, K_{t+1}) \beta U'(C_{t+1}^1)$$

Under the assumption of a log utility function, we have that this becomes:

$$\frac{1}{C_t^0 + G_t} = f_K(1, K_{t+1}) \beta \frac{1}{C_{t+1}^1}$$

and now re-arrange to get:

$$C_{t+1}^1 = \beta f_K(1, K_{t+1}) (C_t^0 + G_t).$$

□

If we look at the Euler equation:

$$C_{t+1}^1 = \beta f_K(1, K_{t+1}) (C_t^0 + G_t)$$

we see that the government's spending G_t now enters the Euler equation. From this, we can now derive the evolution of capital.

Proposition 75 (Evolution of Capital)

The evolution of capital is given by:

$$K_{t+1}^1 = \frac{\beta}{1 + \beta} (f_L(1, K_t) - T_t^0 + G_t) - b_{t+1}^1. \quad (94)$$

Proof. First, we look at the two budget constraints facing the generation where we solved for consumption:

$$\begin{aligned} C_t^0 &= w_t - T_t^0 - K_{t+1}^1 - b_{t+1}^1 \\ C_{t+1}^1 &= r_{t+1} (b_{t+1}^1 + K_{t+1}^1) - T_{t+1}^1 \end{aligned}$$

We now substitute these into the Euler equation and solve for the level of capital K_{t+1}^1 to get:

$$\begin{aligned}
C_{t+1}^1 &= \beta f_K(1, K_{t+1})(C_t^0 + G_t) \\
\leftrightarrow \left[r_{t+1}(b_{t+1}^1 + K_{t+1}^1) - T_{t+1}^1 \right] &= \beta f_K(1, K_{t+1}) \left[(w_t - T_t^0 - K_{t+1}^1 - b_{t+1}^1) + G_t \right] \\
\leftrightarrow r_{t+1}K_{t+1}^1 &= \beta f_K(1, K_{t+1}) \left[(w_t - T_t^0 - K_{t+1}^1 - b_{t+1}^1) + G_t \right] + \underbrace{T_{t+1}^1}_{=0 \text{ by assumption}} - r_{t+1}b_{t+1}^1 \\
\leftrightarrow r_{t+1}K_{t+1}^1 + \beta \underbrace{f_K(1, K_{t+1})}_{=r_{t+1} \text{ by firm's FOC}} K_{t+1}^1 &= \beta \underbrace{f_K(1, K_{t+1})}_{=r_{t+1}} \left[\underbrace{(w_t - T_t^0 - b_{t+1}^1)}_{=f_L(1, K_t) \text{ by FOC}} + G_t \right] - r_{t+1}b_{t+1}^1 \\
\leftrightarrow (r_{t+1} + \beta r_{t+1})K_{t+1}^1 &= \beta r_{t+1} \left[(f_L(1, K_t) - T_t^0 - b_{t+1}^1) + G_t \right] - r_{t+1}b_{t+1}^1 \\
\leftrightarrow K_{t+1}^1 &= \frac{\beta r_{t+1}}{(r_{t+1} + \beta r_{t+1})} \left[(f_L(1, K_t) - T_t^0 - b_{t+1}^1) + G_t \right] - \frac{r_{t+1}}{(r_{t+1} + \beta r_{t+1})} b_{t+1}^1 \\
\leftrightarrow K_{t+1}^1 &= \frac{\beta r_{t+1}}{(\beta + 1)r_{t+1}} \left[(f_L(1, K_t) - T_t^0 - b_{t+1}^1) + G_t \right] - \frac{r_{t+1}}{(\beta + 1)r_{t+1}} b_{t+1}^1 \\
\leftrightarrow K_{t+1}^1 &= \frac{\beta}{(\beta + 1)} \left[(f_L(1, K_t) - T_t^0 - b_{t+1}^1) + G_t \right] - \frac{1}{(\beta + 1)} b_{t+1}^1 \\
\leftrightarrow K_{t+1}^1 &= \frac{\beta}{(\beta + 1)} \left[f_L(1, K_t) - T_t^0 + G_t \right] - \frac{\beta}{\beta + 1} b_{t+1}^1 - \frac{1}{(\beta + 1)} b_{t+1}^1 \\
\leftrightarrow K_{t+1}^1 &= \frac{\beta}{(\beta + 1)} \left[f_L(1, K_t) - T_t^0 + G_t \right] - \frac{\beta + 1}{\beta + 1} b_{t+1}^1 \\
\leftrightarrow K_{t+1}^1 &= \frac{\beta}{(\beta + 1)} \left[f_L(1, K_t) - T_t^0 + G_t \right] - b_{t+1}^1
\end{aligned}$$

Therefore, we have that the law of motion of the capital stock is:

$$K_{t+1}^1 = \frac{\beta}{1 + \beta} (f_L(1, K_t) - T_t^0 + G_t) - b_{t+1}^1.$$

□

Let us analyse the law of motion of the capital stock equation:

$$K_{t+1}^1 = \frac{\beta}{1 + \beta} (f_L(1, K_t) - T_t^0 + G_t) - b_{t+1}^1.$$

We have that the young generation can save in either capital or government bonds. However, as before, the generation will still save a constant fraction of income, after taking into account any taxes they pay and the government's spending.

Non-Ricardian Equivalence

We will now look at the impact of the government's fiscal policy on the law of motion of capital stock. We make another assumption before we proceed.

Assumption (One Time Spending)

The government makes a one off spending on the young at time t :

$$G_{t+j} = \begin{cases} \bar{G} & \text{for } j = 0 \\ 0 & \text{for } j \geq 1. \end{cases}$$

From Ricardian equivalence, we would expect that financing schemes should make no difference. However, we shall in fact see that it does. We will look at two different financing schemes and use the law of motion of capital stock to help us:

$$K_{t+1}^1 = \frac{\beta}{1+\beta} (f_L(1, K_t) - T_t^0 + G_t) - b_{t+1}^1.$$

Proposition 76 (Ricardian Equivalence Under Balanced Budget)

Under a balanced budget financing scheme where the government tax the young an equivalent amount to the spending:

$$T_t^0 = G_t = \bar{G}$$

then Ricardian equivalence holds.

Proof. At time period t , the government spends the amount $\bar{G}$ on the young and will tax the young the same amount to finance this spending:

$$T_t^0 = G_t = \bar{G}.$$

Now, if we look at the law of motion of capital:

$$K_{t+1}^1 = \frac{\beta}{1+\beta} (f_L(1, K_t) - T_t^0 + G_t) - b_{t+1}^1.$$

We have that this becomes:

$$K_{t+1}^1 = \frac{\beta}{1+\beta} (f_L(1, K_t) - \bar{G} + \bar{G}) - 0$$

which then becomes:

$$K_{t+1}^1 = \frac{\beta}{1+\beta} (f_L(1, K_t)).$$

This is the exact same law of motion for capital that we had when we looked at the OLG model without any government fiscal policy. Hence, government spending here makes no difference and therefore Ricardian equivalence holds. □

Proposition 77 (Non-Ricardian Equivalence Under Deficit Financing)

Under a deficit financing scheme where the government borrows from the young the equivalent amount to the spending:

$$b_{t+1}^1 = G_t = \bar{G} \quad \text{and} \quad T_t^0 = 0$$

then Ricardian equivalence **does not** hold.

Proof. At time period t , the government spends the amount $\bar{G}$ on the young but this time, do not tax the young

$T_t^0 = 0$. However, instead they will borrow this amount from the young:

$$b_{t+1}^1 = G_t = \bar{G}.$$

Then, in the next period $t+1$, the government need to raise taxes to repay these bonds and hence will set taxes on the next generation of young people:

$$T_{t+1}^0 = (1 + r_{t+1}^b)b_{t+1}^1 = (1 + r_{t+1}^b)\bar{G}.$$

Now, we look at the law of motion for capital. At time period t , we have that this becomes:

$$\begin{aligned} K_{t+1}^1 &= \frac{\beta}{1+\beta}(f_L(1, K_t) - T_t^0 + G_t) - b_{t+1}^1 \\ &= \frac{\beta}{1+\beta}(f_L(1, K_t) - 0 + \bar{G}) - \bar{G} \\ &= \frac{\beta}{1+\beta}(f_L(1, K_t)) + \frac{\beta}{1+\beta}\bar{G} - \bar{G} \\ &= \frac{\beta}{1+\beta}(f_L(1, K_t)) - \frac{1}{1+\beta}\bar{G} \end{aligned}$$

We can see that the capital stock has fallen overall by $\frac{1}{1+\beta}\bar{G}$ compared to the scenario where there was no government spending. The intuition for this is that the government borrows from you by making you buy bonds and you cut your savings accordingly, this causes capital to fall.

Now, in the period $t + 1$, we have that the government will now need to raise taxes $T_{t+1}^0 = (1 + r_{t+1}^b)\bar{G}$ to pay off the bonds:

$$\begin{aligned} K_{t+2}^1 &= \frac{\beta}{1+\beta}(f_L(1, K_{t+1}) - T_{t+1}^0 + G_{t+1}) - b_{t+2}^1 \\ &= \frac{\beta}{1+\beta}(f_L(1, K_{t+1}) - (1 + r_{t+1}^b)\bar{G} + 0) - 0 \\ &= \frac{\beta}{1+\beta}(f_L(1, K_{t+1}) - (1 + r_{t+1}^b)\bar{G}) \end{aligned}$$

Here, we see that capital K_{t+2} also falls due to two factors, first is the taxes that the government raises and second is that since the capital stock fell in the previous period, the marginal product of capital rose and therefore the marginal product of labour $f_L(\cdot, \cdot)$ fell so that the household's wages fell and hence they have less capital.

Now, in the final period $t + 2$, we have no fiscal policy so we end up back in the original formulation of the model without fiscal policy:

$$K_{t+3}^1 = \frac{\beta}{1+\beta}f_L(1, K_{t+2}).$$

Hence, Ricardian equivalence no longer holds: the financing choice changes the transition path of capital. Under the usual stability condition, the effect is persistent, but the economy eventually converges back to the same no-policy steady state. $\square$

We end this section by drawing out the implications of the OLG model under fiscal policy. It says that if the government borrows, it is not productive. This is because the household is forced to reduce their savings in capital K_{t+1} which is productive and contributes to the capital stock and is instead forced to hold bonds b_{t+1}^1 which does not contribute to the capital stock. We graphically show this in Fig. 7.

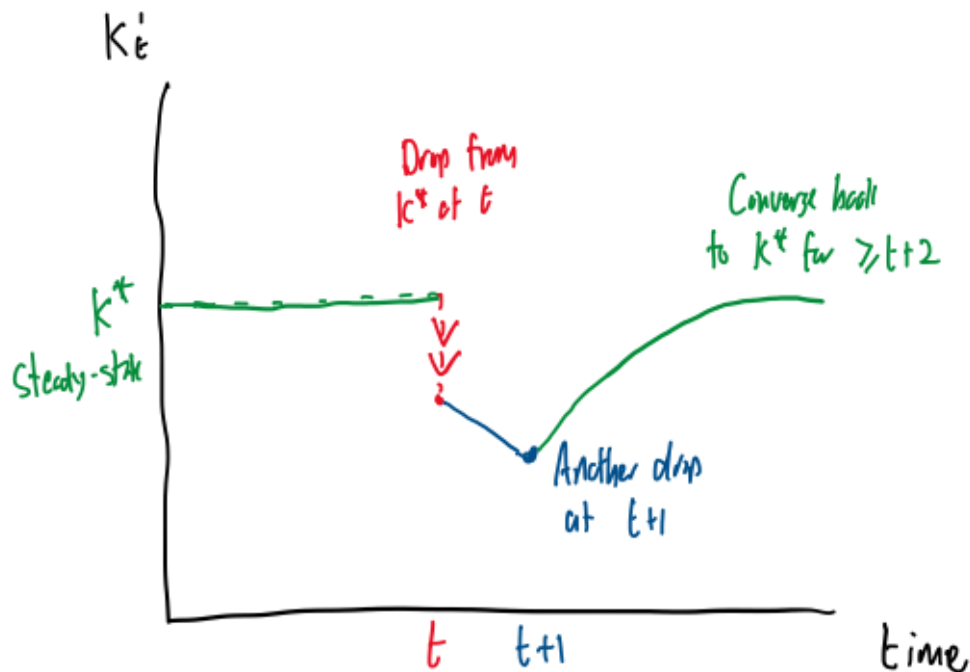


Figure 7: Non-Ricardian equivalence in the OLG model under a deficit financing scheme.

5.5 Technological Progress

Let us see how well the OLG model performs against the Kaldor facts. First, recall that the Kaldor facts are:

1. Output per worker increases over time;
2. Capital per worker grows over time;
3. The capital to output ratio is constant;
4. The rate of return to capital is constant;
5. The share of capital and labour in net income are constant;
6. Real wages grow over time;
7. The ratio of consumption to GDP and investment to GDP are constant.

We now assess how well the OLG model addresses these facts.

1. If capital has converged but labour is inelastically supplied, then it cannot be that output per worker and capital per worker growing at a constant rate over time.
2. As capital and output both constant, then we get a constant capital to output ratio.
3. As capital is constant, we get that the marginal product of capital is constant and hence the rate of return to capital is constant.
4. We also have that the share of capital and labour in net income is constant too.
5. We do not see real wages growing over time as the marginal product of labour is not changing.

6. Finally, consumption to output and investment to output are constant.

We have 4 out of 7 of the facts satisfied in the OLG model. However, we have the issue that we have a growth model that is not growing. We can try to remedy this by adding technological change.

Definition 78 (Technology)

Technology will be represented by θ_t which is growing over time at rate g :

$$\theta_{t+1} = (1 + g)\theta_t. \quad (95)$$

We can also embed technological change through the production function.

Definition 79 (Production Function with Growth)

There are three different ways to embed technology into the production function:

1. Hicks Neutral: $y_t = \theta_t f(K_t, L_t)$
2. Capital-Augmenting: $y_t = f(\theta_t K_t, L_t)$
3. Labour-Augmenting: $y_t = f(K_t, \theta_t L_t)$

Under labour-augmenting technological change, it says that over time, labour will get more productive whilst the return on capital will not change. As this technological change only enters the production function, we have that the Euler equation won't change:

$$\frac{u'(C_t^0)}{\beta u'(C_{t+1}^1)} = r_{t+1}.$$

The budget constraints won't change either:

$$\begin{aligned} C_t^0 + K_{t+1}^1 &= w_t \\ C_{t+1}^1 &= K_{t+1}^1 r_{t+1}. \end{aligned}$$

However, the firm's FOC's will now change:

$$\begin{aligned} r_t &= f_K(K_t, \theta_t L_t) \\ w_t &= \theta_t f_L(K_t, \theta_t L_t). \end{aligned}$$

We can follow a similar procedure to derive the evolution of capital.

Theorem 80 (Evolution of Capital)

The evolution of capital is given by:

$$\frac{K_{t+1}^1}{\theta_{t+1}} = \frac{\beta}{(1 + \beta)(1 + g)} (1 - \alpha) \left(\frac{K_t^1}{\theta_t} \right)^\alpha$$

Notice that we now have capital divided by the technology. Now, notice that again we have that we will be saving a constant fraction of capital per technology. The growth in technology $(1 + g)$ will have an impact on what we converge too. Since the form of this evolution of capital looks similar to what we have seen previously, we know that we will have convergence to a steady state denoted by K/θ . However, if K/θ is constant, but we assumed that θ_t is growing at a rate $(1 + g)$, then it must be the case that capital K_{t+1}^1 is growing too! Intuitively, this is happening because

as time goes by, labour is becoming more and more productive due to the labour-augmenting technology assumption we made. We now have that capital per worker and output per worker is growing over time. Furthermore, as labour productivity is growing, then the real wage will also be growing over time. Hence, this exogenous technological change has helped! Whilst this may seem like a good thing, the issue with this trick is that we are using an exogenous process θ to explain growth and hence we have not really explained growth in the end. We will see later on when looking at endogenous growth models, a way to account for this seemingly exogenous growth.

6 Neoclassical Growth Model

6.1 Introduction

We now move onto another famous neoclassical growth model known as the Ramsey model. The Ramsey model explicitly models the household's optimisation problem and as a result, endogenizes savings. We shall assume a **representative agent** for both the household and the firm.

6.2 Ramsey Model

6.2.1 Household Problem

In the Ramsey model, we move into continuous rather than discrete time. First, suppose the household has an **instantaneous** utility function $u(c(t))$. We impose the following assumptions.

Assumption 81 (Neoclassical Preferences on Utility Functions)

The instantaneous utility function $u(c)$ is defined on $\mathbb{R}_+$. It is

1. Strictly increasing
2. Strictly concave
3. Twice continuously differentiable, with $u'(c) > 0$ and $u''(c) < 0$ for all c in the interior of its domain.

The household wants to maximise discounted instantaneous utility. In continuous time, let $\rho > 0$ denote the discount rate, so the discount factor over t units of time is $e^{-\rho t}$. If we start from a one-period discrete discount factor β , the corresponding continuously compounded rate is $\rho = -\log \beta$. Discounted utility is therefore

$$\int_0^\infty e^{-\rho t} u(c_t) dt.$$

The household chooses a consumption path to maximise this present discounted value of utility, subject to the **capital accumulation equation**. The change in capital $\dot{k}_t$ depends on wage income $w_t l$, capital income $r_t k_t$, profits Π_t (the household owns the firm), consumption c_t , and depreciation δk_t . For convenience, we assume that the household supplies labour inelastically, with $l_t = 1$.

Definition 82 (Household Optimisation Problem)

The household optimisation problem is:

$$\begin{aligned} \max_{c(\cdot)} \quad & \int_0^\infty e^{-\rho t} u(c_t) dt \\ \text{s.t.} \quad & \dot{k}_t = w_t l + r_t k_t + \Pi_t - c_t - \delta k_t \end{aligned} \tag{96}$$

where l is inelastic supply of labour, Π_t is the firm's profits, δ is the depreciation rate and $\dot{k}_t$ is the change in capital stock k_t .

To solve the household problem, we need to use **Hamiltonians**. Refer to Section 2 for more details on Hamiltonians.

The **Present Value Hamiltonian** is given by:

$$\mathcal{H}^d = e^{-\rho t} u(c_t) + \mu_t \left[w_t l + (r_t - \delta) k_t + \Pi_t - c_t \right]. \quad (97)$$

Proposition 83 (First Order Conditions for Present Value Hamiltonian)

Let $\mathcal{H}^d = e^{-\rho t} u(c_t) + \mu_t \left[w_t l + (r_t - \delta) k_t + \Pi_t - c_t \right]$ be our Present Value Hamiltonian. Then, our **four** first order conditions are:

1. $\frac{\partial \mathcal{H}^d}{\partial c_t} = 0$ derivative with respect to **control variable** is equal to 0
2. $\frac{\partial \mathcal{H}^d}{\partial k_t} = -\dot{\mu}(t)$ derivative with respect to **state variable** is equal to $-\dot{\mu}(t)$
3. $\frac{\partial \mathcal{H}^d}{\partial \mu_t} = w_t l + (r_t - \delta) k_t + \Pi_t - c_t$ derivative with respect to **costate variable** is the law of motion.
4. $\lim_{t \rightarrow \infty} \mu_t k_t = 0$ transversality condition.

Actually writing out the four first order conditions, we have:

$$\begin{aligned} e^{-\rho t} u'(c_t) &= \mu_t \\ \mu_t (r_t - \delta) &= -\dot{\mu}_t \\ w_t l + (r_t - \delta) k_t + \Pi_t - c_t &= \dot{k}_t \\ \lim_{t \rightarrow \infty} \mu_t k_t &= 0. \end{aligned}$$

We can then derive the Euler equation for consumption.

Proposition 84 (Euler Equation for Consumption)

The Euler equation for consumption is:

$$\dot{c}_t = \frac{u'(c_t)(r_t - \delta - \rho)}{-u''(c_t)}. \quad (98)$$

Proof. If we look at the first FOC:

$$\mu_t = e^{-\rho t} u'(c_t)$$

taking derivatives with respect to t and using the product rule gives us:

$$\dot{\mu}_t = -u'(c_t) \rho e^{-\rho t} + u''(c_t) \dot{c}_t e^{-\rho t}.$$

We then plug this into the second FOC:

$$\mu_t (r_t - \delta) = -\dot{\mu}_t \quad (99)$$

$$\Leftrightarrow \mu_t (r_t - \delta) = -[-u'(c_t) \rho e^{-\rho t} + u''(c_t) \dot{c}_t e^{-\rho t}] \quad (100)$$

$$(101)$$

We can then solve this to get:

$$\begin{aligned}
\mu_t(r_t - \delta) &= -[-u'(c_t)\rho e^{-\rho t} + u''(c_t)\dot{c}_t e^{-\rho t}] \\
\leftrightarrow \mu_t r_t - \mu_t \delta &= u'(c_t)\rho e^{-\rho t} - u''(c_t)\dot{c}_t e^{-\rho t} \\
\leftrightarrow e^{-\rho t} u'(c_t) r_t - e^{-\rho t} u'(c_t) \delta &= u'(c_t)\rho e^{-\rho t} - u''(c_t)\dot{c}_t e^{-\rho t} \\
\leftrightarrow u'(c_t) r_t - u'(c_t) \delta &= u'(c_t)\rho - u''(c_t)\dot{c}_t \\
\leftrightarrow u''(c_t)\dot{c}_t &= -u'(c_t) r_t + u'(c_t)\delta + u'(c_t)\rho \\
\leftrightarrow u''(c_t)\dot{c}_t &= u'(c_t)(-\delta + \rho) \\
\leftrightarrow u''(c_t)\dot{c}_t &= -u'(c_t)(r_t - \delta - \rho) \\
\leftrightarrow \dot{c}_t &= -\frac{u'(c_t)}{u''(c_t)}(r_t - \delta - \rho)
\end{aligned}$$

□

This Euler equation has a nice interpretation: if the rental rate of capital satisfies $r_t > \delta + \rho$, consumption is increasing over time. Consumption growth is higher when the return is high, depreciation is low, or the household is more patient (so the discount rate ρ is low).

6.2.2 Firm Problem

The firm's optimisation problem is quite standard.

Definition 85 (Optimisation Problem Facing the Firm)

The firm maximisation problem is:

$$\begin{aligned}
\max_{k_t, l_t} \quad & y_t - w_t l_t - r_t k_t \\
\text{s.t.} \quad & y_t = f(l_t, k_t)
\end{aligned} \tag{102}$$

where $f(\cdot, \cdot)$ is a constant returns to scale production function and satisfies the Inada conditions.

We then have the usual FOC's for the firm.

Proposition 86 (Firm's First Order Conditions)

The firm's first order equations are:

$$\begin{aligned}
w_t &= f_l(l_t, k_t) \\
r_t &= f_k(l_t, k_t)
\end{aligned}$$

Proof. Substitute in output into the profit function:

$$f(l_t, k_t) - w_t l_t - r_t k_t$$

and then take partial derivatives:

$$\begin{aligned}\frac{\partial}{\partial k_t} : f_k(l_t, k_t) - r_t &= 0 \\ \frac{\partial}{\partial l_t} : f_l(l_t, k_t) - w_t &= 0.\end{aligned}$$

□

Proposition 87 (No Profits under Constant Returns to Scale)

Under constant returns to scale, the firm makes 0 profits.

Proof. Look at the production function:

$$y_t = k_t^\alpha l_t^{1-\alpha}.$$

Then look at total income, we can apply Euler's theorem:

$$\begin{aligned}w_t l_t + r_t k_t &= f_l(l_t, k_t) l_t + f_k(l_t, k_t) k_t \\ &= (1 - \alpha) k_t^\alpha l_t^{1-\alpha} + \alpha k_t^\alpha l_t^{1-\alpha} \\ &= k_t^\alpha l_t^{1-\alpha} = y_t.\end{aligned}$$

We therefore can see that profits are zero and that factor returns exhaust all the amount of output.

□

6.2.3 General Equilibrium

We now state the general equilibrium of this model.

Definition 88 (Competitive Equilibrium)

A competitive equilibrium of the Ramsey model consists of a sequence $\{l_t, k_t^f, k_t^h, c_t, r_t, w_t, \mu_t\}_{t=0}^\infty$ such that the representative household and firm are optimising:

$$\begin{aligned}w_t &= f_l(l_t, k_t^f) \\ r_t &= f_k(l_t, k_t^f) \\ u'(c_t) e^{-\rho t} - \mu_t &= 0 \\ \mu_t(r_t - \delta) &= -\dot{\mu}_t \\ k_t^h &= w_t l + (r_t - \delta) k_t^h + \Pi_t - c_t\end{aligned}$$

and all markets clear:

$$\begin{aligned}k_t^h &= k_t^f \\ l_t &= l\end{aligned}$$

where k_t^f is the demand for capital by the firm, k_t^h is the supply of capital by the household, l_t is the labour demanded by the firm, and l is the inelastic supply of labor.

We are now interested in looking at the dynamics and steady state of this model. First recall the Euler equation for consumption:

$$\dot{c}_t = \frac{u'(c_t)(r_t - \delta - \rho)}{-u''(c_t)}.$$

Before proceeding, we shall impose some more assumptions.

Assumption (CRRA Utility Function)

We assume CRRA preferences:

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$$

with $\sigma > 0$, $\sigma \neq 1$, as the coefficient of relative risk aversion. The case $\sigma = 1$ is log utility.

We then state a useful characterisation of such preferences.

Proposition 89 (Coefficient of Relative Risk Aversion)

Let $u(c_t)$ be CRRA preferences:

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$$

with σ as the coefficient of relative risk aversion. Then, the coefficient of relative risk aversion is given by:

$$\sigma = -\frac{u''(c_t)}{u'(c_t)} c_t.$$

Proof. If we take the first and second derivative of:

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$$

we get:

$$\begin{aligned} u'(c_t) &= c_t^{-\sigma} \\ u''(c_t) &= -\sigma c_t^{-\sigma-1}. \end{aligned}$$

Then dividing these two derivatives and taking the negative gives us:

$$-\frac{u''(c_t)}{u'(c_t)} c_t = -\frac{(-\sigma c_t^{-\sigma-1})}{c_t^{-\sigma}} c_t = \sigma.$$

□

Proposition 90 (Euler Equation for Consumption With RRA)

The Euler equation for consumption in terms of relative risk aversion is given by:

$$\dot{c}_t = \frac{r_t - \delta - \rho}{\sigma} c_t \tag{103}$$

where σ is the coefficient of relative risk aversion.

Proof. If we go back to our Euler equation for consumption:

$$\dot{c}_t = \frac{u'(c_t)(r_t - \delta - \rho)}{-u''(c_t)}$$

we can then rewrite this using the coefficient of relative risk aversion since

$$-\frac{u''(c_t)}{u'(c_t)} = \frac{\sigma}{c_t}.$$

□

6.2.4 Steady State

We look at the long-run behaviour of the Ramsey model by analysing the steady state of the model. In particular, we are interested in how does the control variable consumption c_t evolve over time and how does the state variable capital k_t evolve over time. From the previous section, we showed that the evolution of consumption can be written as:

$$\dot{c}_t = \frac{r_t - \delta - \rho}{\sigma} c_t$$

where σ is the coefficient of relative risk aversion. Additionally, we know from the competitive equilibrium conditions that the evolution of capital is given by:

$$\dot{k}_t^h = w_t l + (r_t - \delta) k_t^h + \Pi_t - c_t$$

but by market clearing, this can be written as:

$$\dot{k}_t = w_t l + k_t r_t - \delta k_t + \Pi_t - c_t.$$

This equation looks at the evolution of capital in the model. We can take both evolution equations and analyse the steady state results, whereby neither variable is evolving. We now show how to derive the steady state equations.

Proposition 91 (Steady State of Ramsey Model)

The steady state of the Ramsey model is:

$$\dot{c}_t = \frac{f_k(l, k^s) - \delta - \rho}{\sigma} c^s = 0 \quad (104)$$

$$\dot{k}_t = f(l, k^s) - c^s - \delta k^s = 0 \quad (105)$$

where c^s and k^s are the steady state levels of consumption and capital respectively.

Proof. First let us derive the steady state for consumption. We have the Euler equation:

$$\dot{c}_t = \frac{r_t - \delta - \rho}{\sigma} c_t.$$

By definition of a steady state, consumption should **not** be evolving in steady state and hence we have that $\dot{c}_t = 0$. We then plug in the steady state level of consumption as well $c_t = c^s$ and we get:

$$\dot{c}_t = \frac{r_t - \delta - \rho}{\sigma} c^s = 0.$$

Now, we notice that by the firm's FOC, the rental rate of capital r_t (at the steady state) is equal to the firm's marginal product of capital at the steady state level of capital k^s , which we denote by $f_k(l, k^s)$. Note that labour is inelastically supplied so it is just l . Therefore, we plug in $r_t = f_k(l, k^s)$ and we get our first result:

$$\dot{c}_t = \frac{f_k(l, k^s) - \delta - \rho}{\sigma} c^s = 0.$$

We now derive the steady state level of capital. First, we start off from the evolution of capital:

$$\dot{k}_t = w_t l + r_t k_t - \delta k_t + \Pi_t - c_t.$$

and by the same argument, in steady state, the evolution of capital should be 0 so $\dot{k}_t = 0$. Furthermore, we plug in the steady state levels of consumption and capital to get:

$$\dot{k}_t = w_t l + r_t k^s - \delta k^s + \Pi_t - c^s = 0.$$

Now, we use **Euler's theorem**, whereby factor payments exhaust the firm's output, so

$$\dot{k}_t = f(l, k^s) - \delta k^s + \Pi_t - c^s = 0.$$

Finally, in equilibrium, due to perfect competition, firm's make zero profits so we get the result:

$$\dot{k}_t = f(l, k^s) - c^s - \delta k^s = 0.$$

□

We now have our two equations characterising steady state of this model. Unintuitively, to derive the steady state of capital, we need to use the evolution of consumption equation and vice versa. We now go on to find these steady state levels.

Capital Stock Steady State

At steady state, we must have that the evolution of consumption is zero:

$$\dot{c}_t = \frac{r_t - \delta - \rho}{\sigma} c^s = 0$$

or else consumption would either be rising or falling. Then it **must be** the case that in the numerator:

$$r^s = \delta + \rho.$$

But the return to capital is the marginal product of capital at steady state:

$$r^s = \delta + \rho = f_k(l, k^s)$$

but since labour is inelastically supplied, then the marginal product of capital is **completely determined by the capital stock**. Therefore, we want to know what is the value of k^s such that:

$$f_k(l, k^s) = \delta + \rho$$

as this will then imply that we are at the steady state for **consumption**: $\dot{c}_t = 0$.

We plot both the production function $f(l, k_t)$ and marginal product of capital $f_k(l, k_t)$ as a function of capital in Fig. 8.

Now, in the left-hand image, we have that the production function is concave by assumption. On the right-hand image, we plot the marginal product of capital which follows this shape due to diminishing marginal returns of capital and the Inada conditions. Due to this shape, there exists a unique single point k^s whereby $f_k(k^s) = \delta + \rho$ and hence we have that $\dot{c}_t = 0$. Going back to the figure on the left, this level of capital k^s corresponds to the point on the production function $f(l, k^s)$ whereby the slope of the production function at this point is equal to $\rho + \delta$.

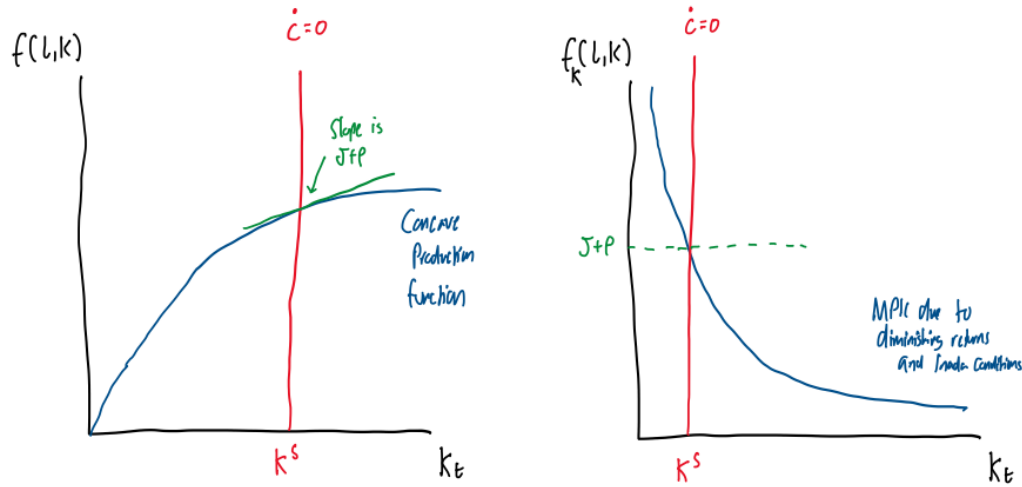


Figure 8: Production as a function of capital and marginal product of capital as a function of capital.

To summarise this argument, we have therefore shown that **there is only level of capital stock k^s that is consistent with the steady state level of consumption.**

Consumption Steady State

We now do something similar to derive the steady state level of consumption. First, we look at the evolution of capital in steady state:

$$\dot{k}_t = f(l, k^s) - c^s - \delta k^s = 0.$$

We can then re-arrange this to get the amount of consumption at steady state c^s such that the evolution of capital is 0 $\dot{k}_t = 0$, which is given by:

$$c^s = f(l, k^s) - \delta k^s.$$

We plot this in Fig. 9. If we consume this amount, the capital stock neither grows nor shrinks. With a low capital stock, output and sustainable consumption are low. As capital rises, output initially rises faster than depreciation, so sustainable consumption increases. Eventually, diminishing returns make additional output smaller than the additional depreciation, and sustainable consumption falls. Hence, every pair (k_t, c_t) on the $\dot{k}_t = 0$ locus keeps capital constant, but only its intersection with the $\dot{c}_t = 0$ locus is a steady state of the full system.

System Steady State

We now combine the two previous figures to get the steady state of the overall system. We show this in Fig. 10.

First, notice that the two loci $\dot{c}_t = 0$ and $\dot{k}_t = 0$ intersect at one point, and it is exactly at this point, which determines the steady state level of capital k^s and the steady state level of consumption c^s . Now, we notice that the $\dot{c}_t = 0$ vertical line locus intersects the $\dot{k}_t = 0$ locus to the left of its maximum. We now explain why that is the case. First, recall that the equation governing the consumption locus is:

$$c^s = f(l, k^s) - \delta k^s.$$

Now, if we want to find the **maximum sustainable level of consumption**, we can take the derivative of this with

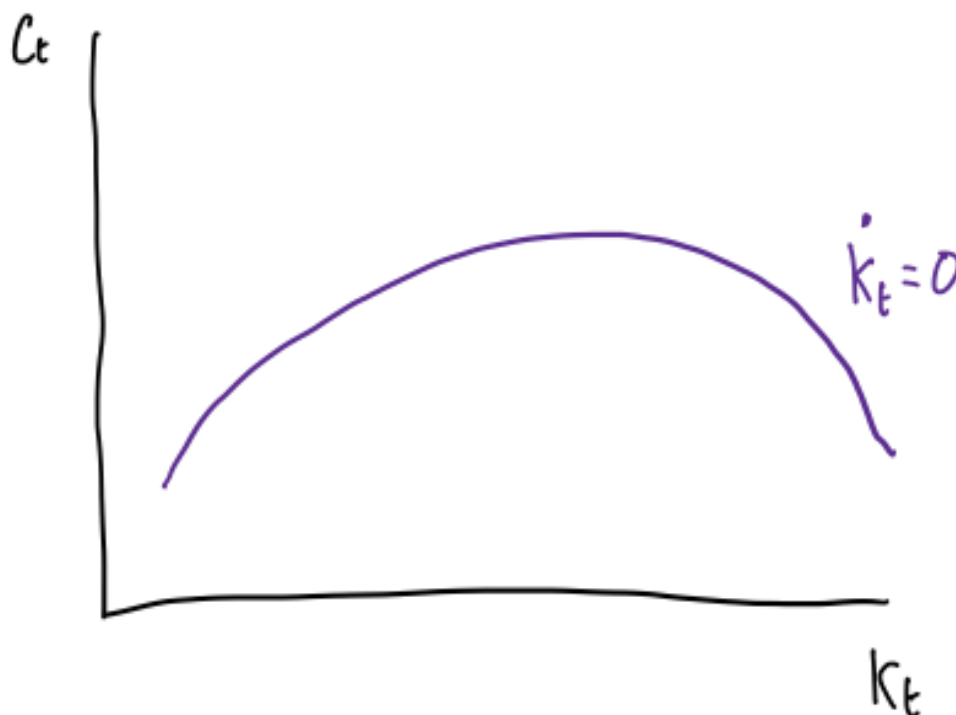


Figure 9: Consumption as a function of capital.

respect to steady state level of capital and what we get is:

$$\begin{aligned}\frac{dc^s}{dk^s} &= f_k(l, k^s) - \delta = 0 \\ \Rightarrow f_k(l, k^s) &= \delta.\end{aligned}$$

Now notice this maximum sustainable level of consumption is at the level of capital k^s such that:

$$f_k(l, k^s) = \delta$$

whilst the locus for $\dot{c}_t = 0$ we derived earlier had the condition:

$$f_k(l, k^s) = \delta + \rho.$$

That is, $\dot{c}_t = 0$ occurs at a steeper marginal product of capital, $\delta + \rho$, rather than at δ . Because households discount the future, they settle at a lower capital stock than the golden-rule level and consume sooner. **The takeaway is that discounting prevents the economy from saving enough to reach the capital stock that maximises sustainable consumption.**

6.2.5 Consumption Dynamics

We now want to look at dynamics to look at the short run behaviour of the system. We will first look at the dynamics of consumption. Previously, we have shown that there exists a single steady state of capital k^s for which **consumption** is not changing:

$$\dot{c}_t = 0.$$

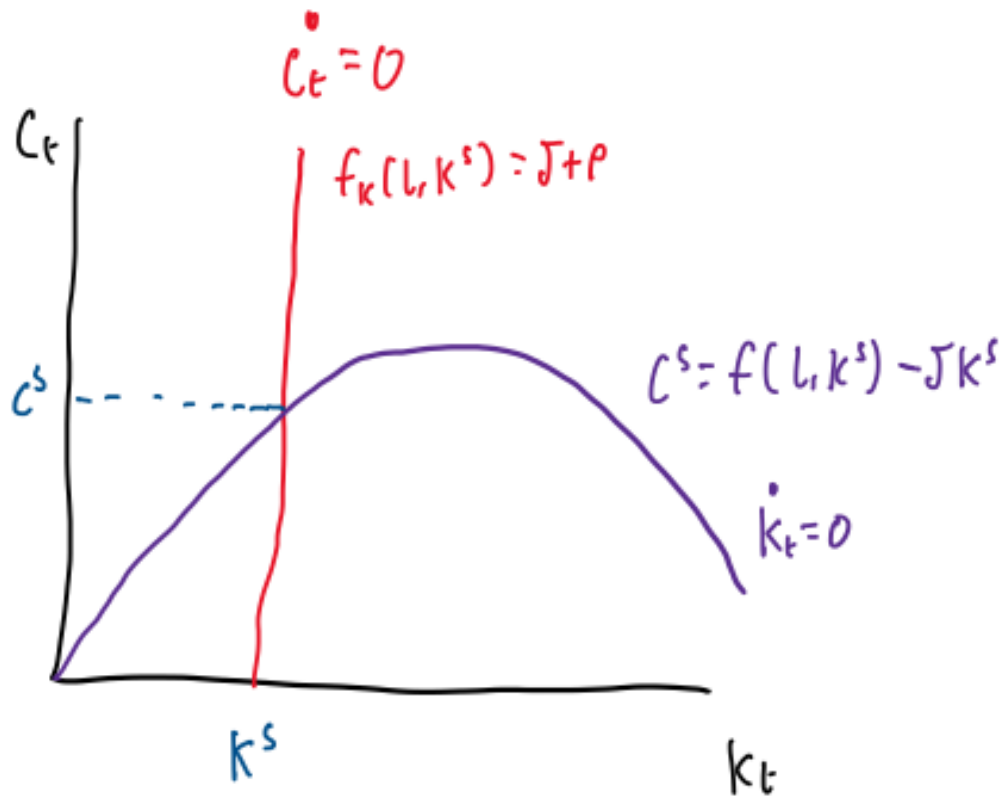


Figure 10: Consumption and capital.

Now suppose that capital is above its steady-state value, $k_t > k^s$. Then the marginal product of capital will be below its steady-state value:

$$f_k(l, k_t) < f_k(l, k^s)$$

due to diminishing returns of capital. Now recall the equation characterising the steady state level of consumption:

$$\dot{c}_t = \frac{f_k(l, k^s) - \delta - \rho}{\sigma} c^s = 0$$

Since $f_k(l, k_t) < f_k(l, k^s)$, consumption growth will be negative:

$$\dot{c}_t = \frac{f_k(l, k_t) - \delta - \rho}{\sigma} c_t < 0.$$

Therefore, any level of capital greater than the steady state level of capital $k_t > k^s$ must lead to a fall in consumption regardless the level of the consumption. To summarise the intuition of what is happening: **Capital is above its steady state value, so then marginal product of capital is low, therefore the interest rate is low, so then you better consume now and plan a decreasing path in consumption.** We show this in Fig. 11.

If capital is below its steady-state value, $k_t < k^s$, then the marginal product of capital and the interest rate are high, so agents plan an increasing consumption path. Therefore, $\dot{c}_t > 0$ when $k_t < k^s$.

6.2.6 Capital Dynamics

Recall that the $\dot{k}_t = 0$ locus tells you the amount that you can consume such that the capital stock is neither increasing or decreasing. If you consume more than the locus, then the capital stock will fall (move to the left). If you consume

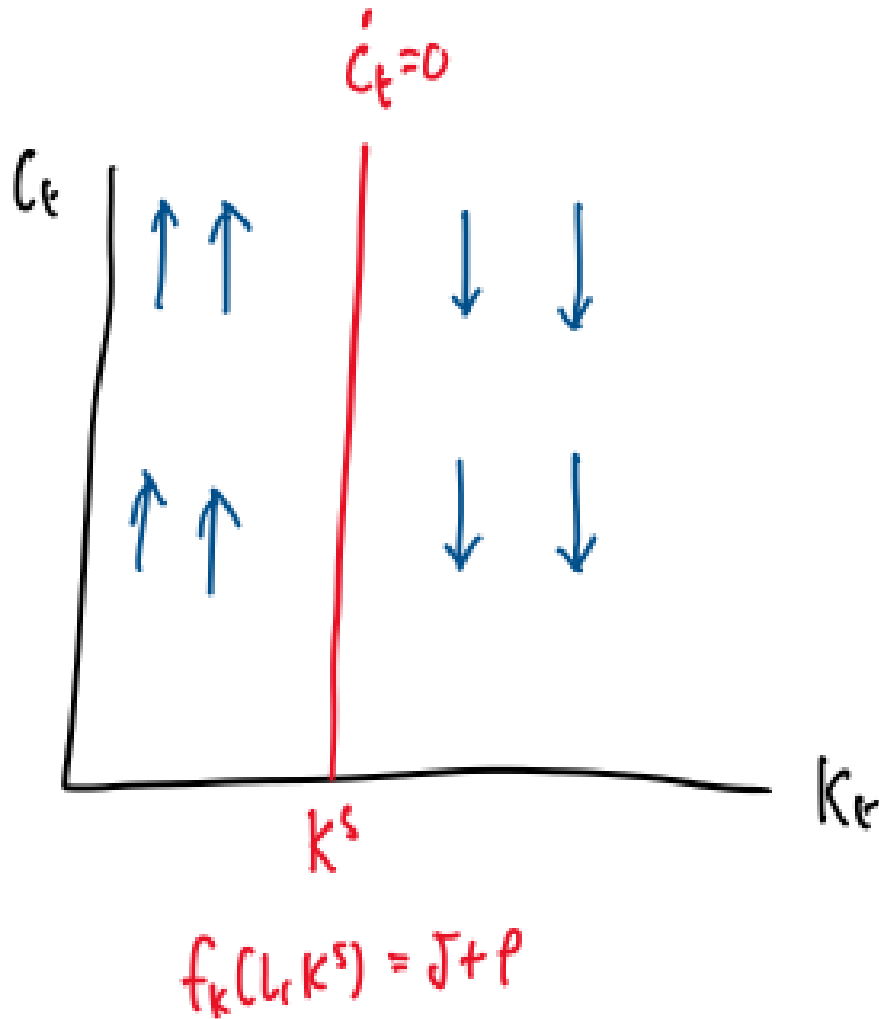


Figure 11: Dynamics of consumption as a function of capital.

less than the locus, then the capital stock will rise (move to the right).

6.2.7 System Dynamics

We now look at the joint dynamics of consumption and capital in Fig. 13, which combines Figs. 11 and 12. We can divide the system into four regions, as shown in the figure, which determine the dynamics of consumption and capital.

We now look at the different paths along which the system can evolve. In particular, there are three types of path, plotted in Fig. 14. Suppose that we start at point (1), where consumption and the capital stock are both high. The marginal product of capital and the interest rate are low, so the optimal consumption plan is decreasing. Along the black path, consumption and capital initially both fall. Eventually, the path crosses the $\dot{c}_t = 0$ locus. Once capital is below its steady-state level, its marginal product and the interest rate are high, so consumption begins to rise. Although this path satisfies the Euler equation and the law of motion locally, it eventually reaches $k_t = 0$ with $c_t > 0$. This violates the feasibility and non-negativity condition on capital, so we can rule out the path.

Now suppose we start at point (2) in Fig. 14. Capital exceeds its steady-state level, so the marginal product

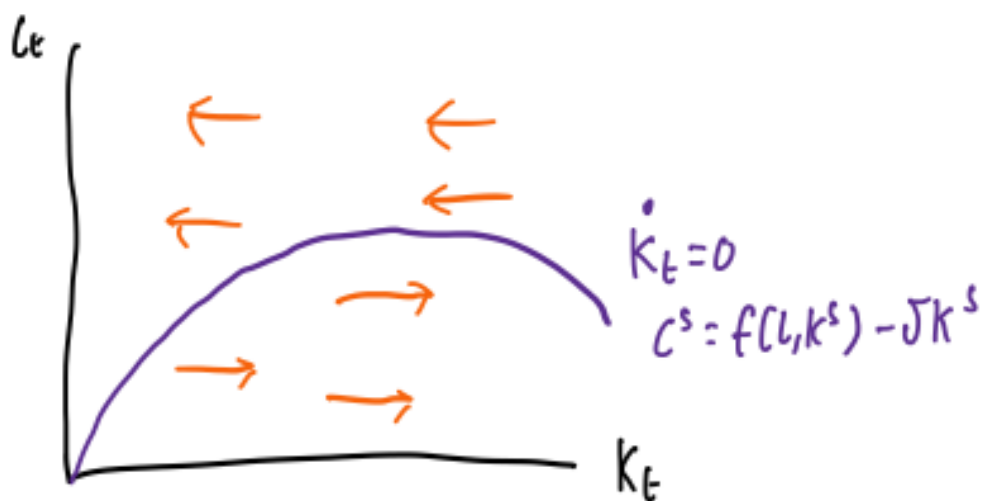


Figure 12: Dynamics of capital as a function of consumption.

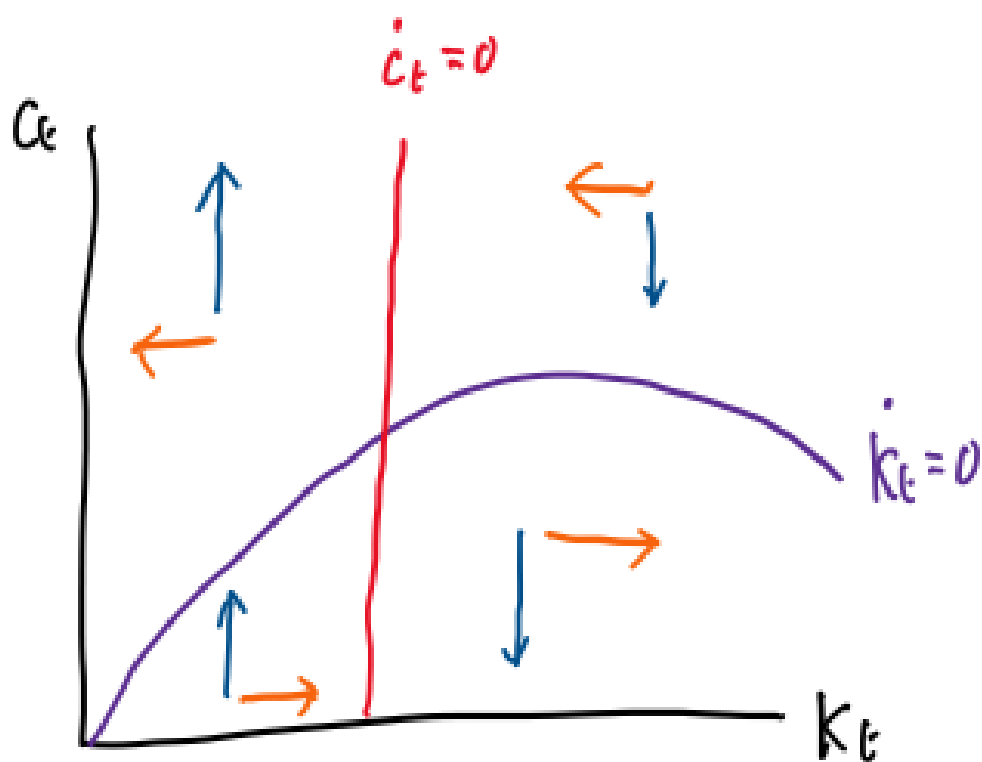


Figure 13: Dynamics of capital as a function of consumption.

of capital and the interest rate are low and consumption decreases. The path crosses the $\dot{k}_t = 0$ locus; beyond it, consumption remains below the amount consistent with constant capital, so capital begins to rise. Although this path satisfies the local dynamic equations, it approaches a point $(k^*, 0)$ with $k^* > 0$. Leaving valuable capital unconsumed

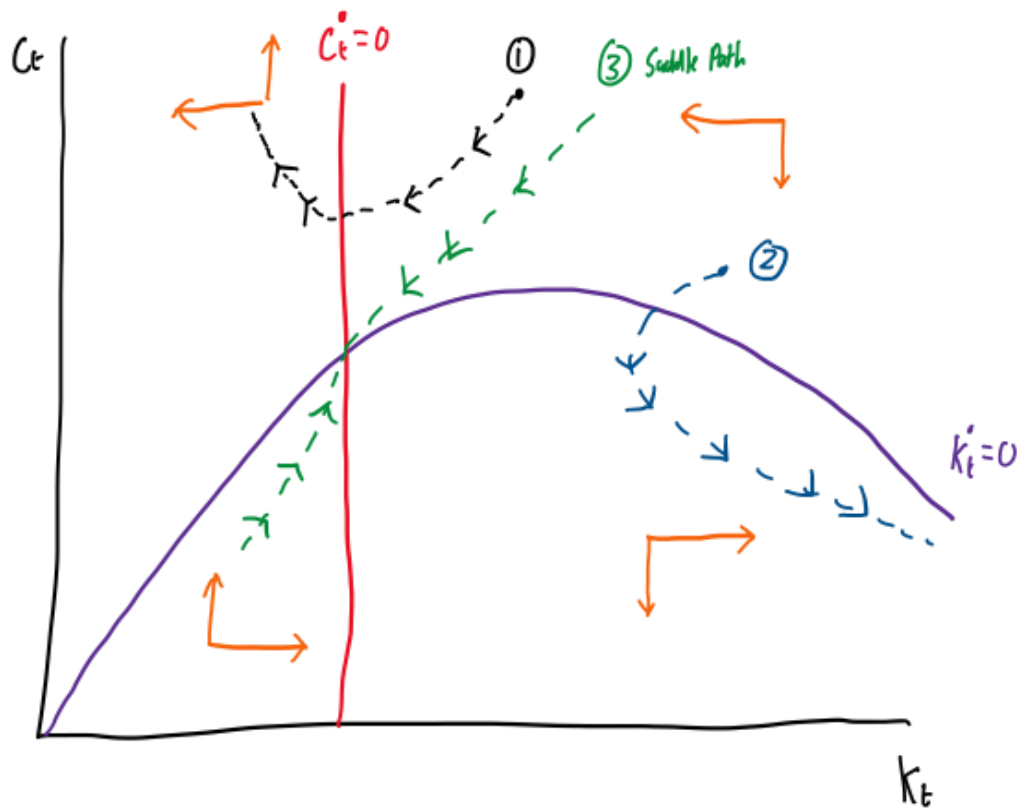


Figure 14: Three types of paths for Ramsey system. The third (green) path denotes the saddle path of the system.

violates the transversality condition $\lim_{t \rightarrow \infty} \mu_t k_t = 0$, so we rule out this path.

Now, if we start off at point (3) in Fig. 14, this is the path that is just right so that the dynamics will take us into the steady state. Hence, this is the **unique saddle path for which there is a unique consumption path**.

6.3 Incorporating Taxation

We now want to introduce taxation into the Ramsey model. We now assume that the government imposes a tax on capital income but rebates tax receipts as a lump sum to households. That is, every \$ of capital income is taxed, but the household will be rebated the average tax revenue of the economy. This means that households are aware of what they will pay for their tax on capital income, but won't be making strategic decisions on how much tax to pay to affect how much rebate they receive from the government.

We now introduce the notation for this tax.

Assumption 92 (Capital Income Taxes and Lump Sum Transfer)

Let τ_t denote the fraction of capital income $r_t k_t$ to be taxed at time t . The lump sum transfer T_t is given by:

$$T_t = \tau_t r_t k_t. \quad (106)$$

Here notice that the government transfer T_t is taken as given, where again we stress that it is not the case that if the household pays more tax, they get back a larger rebate.

We now set up a new household optimisation problem.

Definition 93 (Household Optimisation Problem with Taxation)

The modified household optimisation problem is:

$$\begin{aligned} \max_{\{c_t\}} \quad & \int_0^\infty e^{-\rho t} u(c_t) dt \\ \text{s.t.} \quad & c_t + I_t = w_t l + (1 - \tau_t) r_t k_t + \Pi_t + T_t \\ & \dot{k}_t = I_t - \delta k_t \end{aligned} \quad (107)$$

where τ_t is the capital-income tax rate, $(1 - \tau_t)$ is the after-tax share of capital income, I_t is investment, T_t is the government transfer, l is inelastic labour supply, Π_t is the firm's profits, and δ is the depreciation rate.

We simplify this model by combining the constraints:

$$\begin{aligned} \dot{k}_t &= I_t - \delta k_t \\ \Leftrightarrow \dot{k}_t &= \left[w_t l + (1 - \tau_t) r_t k_t + \Pi_t + T_t - c_t \right] - \delta k_t \\ \Leftrightarrow \dot{k}_t &= w_t l + (1 - \tau_t) r_t k_t + \Pi_t + T_t - c_t - \delta k_t \end{aligned}$$

Therefore, we now have the modified household problem.

Definition 94 (Modified Household Optimisation Problem with Taxation)

The modified household optimisation problem is:

$$\begin{aligned} \max_{\{c_t\}} \quad & \int_0^\infty e^{-\rho t} u(c_t) dt \\ \text{s.t.} \quad & \dot{k}_t = w_t l + (1 - \tau_t) r_t k_t + \Pi_t - c_t - \delta k_t + T_t \end{aligned} \quad (108)$$

where τ_t is the capital-income tax rate, $(1 - \tau_t)$ is the after-tax share of capital income, T_t is the government transfer, l is inelastic labour supply, Π_t is the firm's profits, and δ is the depreciation rate.

We now want to derive the Euler equation for consumption in this model.

Proposition 95 (Euler Equation for Consumption)

Under CRRA preferences, the Euler equation consumption for the Ramsey model with taxation is:

$$\frac{\dot{c}_t}{c_t} = \frac{f_k(l, k_t)(1 - \tau_t) - \delta - \rho}{\sigma}. \quad (109)$$

Proof. First, we form our present value Hamiltonian:

$$\mathcal{H} = e^{-\rho t} u(c_t) + \mu_t \left[w_t l + (1 - \tau_t) r_t k_t + \Pi_t - c_t - \delta k_t + T_t \right]$$

Now, recall that the two of the first order conditions for the present value Hamiltonian is:

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial c_t} &= 0 \\ \frac{\partial \mathcal{H}}{\partial k_t} &= -\dot{\mu}_t. \end{aligned}$$

Therefore, given our present value Hamiltonian, the first order conditions are:

$$\begin{aligned} e^{-\rho t} u'(c_t) - \mu_t &= 0 \\ \mu_t(1 - \tau_t)r_t - \mu_t\delta &= -\dot{\mu}_t. \end{aligned}$$

From the first FOC, we have that:

$$\mu_t = e^{-\rho t} u'(c_t)$$

and taking derivatives with respect to t , we get:

$$\dot{\mu}_t = -\rho e^{-\rho t} u'(c_t) + e^{-\rho t} u''(c_t) \dot{c}_t.$$

We plug the first FOC into the second to get:

$$\begin{aligned} \mu_t(1 - \tau_t)r_t - \mu_t\delta &= -\dot{\mu}_t \\ \Leftrightarrow \mu_t[(1 - \tau_t)r_t - \delta] &= -\dot{\mu}_t \\ \Leftrightarrow \left[e^{-\rho t} u'(c_t) \right] [(1 - \tau_t)r_t - \delta] &= -\dot{\mu}_t \\ \Leftrightarrow \left[e^{-\rho t} u'(c_t) \right] [(1 - \tau_t)r_t - \delta] &= - \left[-\rho e^{-\rho t} u'(c_t) + e^{-\rho t} u''(c_t) \dot{c}_t \right] \\ \Leftrightarrow e^{-\rho t} u'(c_t) [(1 - \tau_t)r_t - \delta] &= \rho e^{-\rho t} u'(c_t) - e^{-\rho t} u''(c_t) \dot{c}_t \\ \Leftrightarrow u'(c_t) [(1 - \tau_t)r_t - \delta] &= \rho u'(c_t) - u''(c_t) \dot{c}_t \\ \Leftrightarrow u'(c_t) [(1 - \tau_t)r_t - \delta] - \rho u'(c_t) &= -u''(c_t) \dot{c}_t \\ \Leftrightarrow -u''(c_t) \dot{c}_t &= u'(c_t) [(1 - \tau_t)r_t - \delta] - \rho u'(c_t) \\ \Leftrightarrow \dot{c}_t &= -\frac{u'(c_t)}{u''(c_t)} [(1 - \tau_t)r_t - \delta] + \rho \frac{u'(c_t)}{u''(c_t)} \end{aligned}$$

Now, recall that we have that coefficient of relative risk aversion satisfies the equation:

$$\sigma = -\frac{u''(c_t)}{u'(c_t)} c_t$$

which implies that:

$$-\frac{u'(c_t)}{u''(c_t)} = \frac{c_t}{\sigma}$$

We then plug this in to get:

$$\begin{aligned} \Leftrightarrow \dot{c}_t &= -\frac{u'(c_t)}{u''(c_t)} [(1 - \tau_t)r_t - \delta] + \rho \frac{u'(c_t)}{u''(c_t)} \\ \Leftrightarrow \dot{c}_t &= \frac{c_t}{\sigma} [(1 - \tau_t)r_t - \delta] - \rho \frac{c_t}{\sigma} \\ \Leftrightarrow \dot{c}_t &= \frac{[(1 - \tau_t)r_t - \delta - \rho]}{\sigma} c_t \end{aligned}$$

and substitute in the firm's FOC to conclude that:

$$\frac{\dot{c}_t}{c_t} = \frac{f_k(l, k_t)(1 - \tau_t) - \delta - \rho}{\sigma}.$$

□

We now derive the law of motion for capital.

Proposition 96 (Law of Motion for Capital)

The law of motion for capital is:

$$\dot{k}_t = f(l, k_t) - c_t - \delta k_t. \quad (110)$$

Proof. Recall the budget constraint:

$$\dot{k}_t = w_t l + (1 - \tau_t) r_t k_t + \Pi_t - c_t - \delta k_t + T_t.$$

Under perfect competition, profits are zero:

$$\dot{k}_t = w_t l + (1 - \tau_t) r_t k_t + 0 - c_t - \delta k_t + T_t.$$

Substitute in the expression for government transfers:

$$\dot{k}_t = w_t l + (1 - \tau_t) r_t k_t + 0 - c_t - \delta k_t + r_t \tau_t k_t$$

Expand out brackets and cancel terms:

$$\begin{aligned} \dot{k}_t &= w_t l + (1 - \tau_t) r_t k_t - c_t - \delta k_t + r_t \tau_t k_t \\ \Leftrightarrow \dot{k}_t &= w_t l + r_t k_t - \tau_t r_t k_t - c_t - \delta k_t + r_t \tau_t k_t \\ \Leftrightarrow \dot{k}_t &= w_t l + r_t k_t - c_t - \delta k_t \\ \Leftrightarrow \dot{k}_t &= f(l, k_t) - c_t - \delta k_t \end{aligned}$$

where in the last line, we used Euler's theorem. □

We can now state the equilibrium for the Ramsey model with taxation.

Proposition 97 (Equilibrium for Ramsey Model with Taxation)

Under CRRA preferences and capital taxation, the Euler equation for consumption, law of motion for capital, and transversality condition are:

$$\frac{\dot{c}_t}{c_t} = \frac{f_k(l, k_t)(1 - \tau_t) - \delta - \rho}{\sigma} \quad (111)$$

$$\dot{k}_t = f(l, k_t) - c_t - \delta k_t \quad (112)$$

$$\lim_{t \rightarrow \infty} e^{-\rho t} u'(c_t) k_t = 0. \quad (113)$$

Remark 98. Notice that the lump-sum rebate T_t does not appear in our optimality conditions!

6.3.1 Steady State with Taxation

We now look at the steady state when taxation is present. We follow a similar procedure to the one used without taxation.

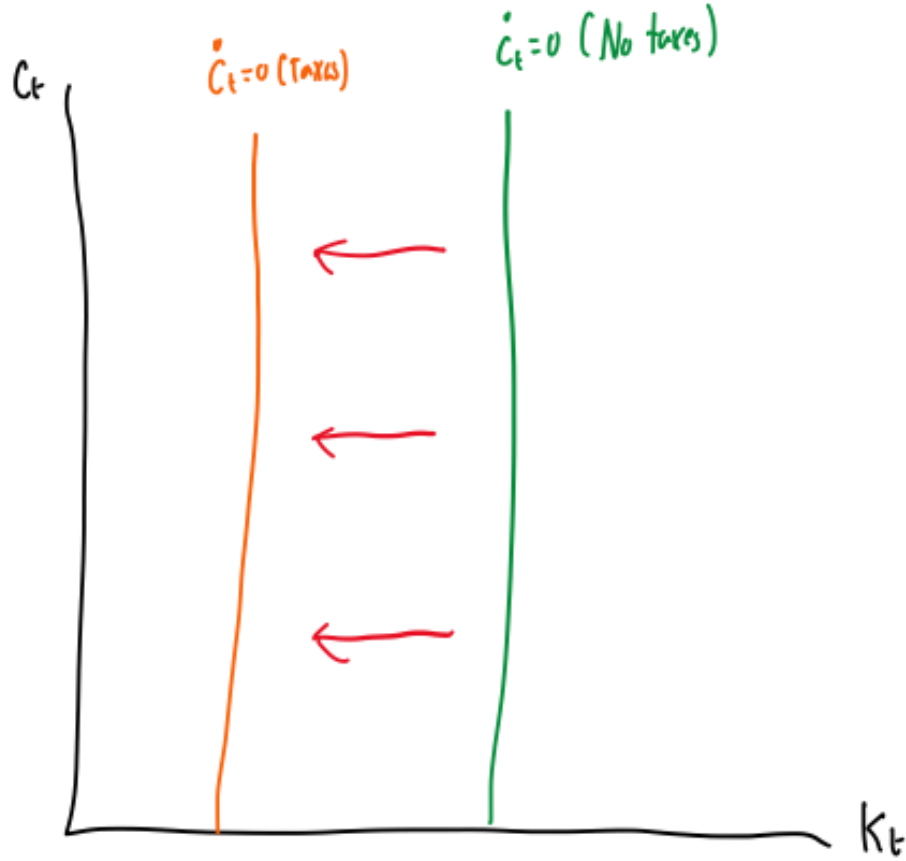


Figure 15: The locus shifts to the left due to the tax.

By introducing taxation of capital, we are reducing the return to savings and hence affecting future consumption. To see this more carefully, we examine the first order condition for consumption more closely:

$$\frac{\dot{c}_t}{c_t} = \frac{f_k(l, k_t)(1 - \tau_t) - \delta - \rho}{\sigma}.$$

This looks like the Euler equation for consumption from before. Now, suppose that we are in steady state, $\dot{c}_t = 0$ so then we have that:

$$\frac{\dot{c}_t}{c_t} = \frac{f_k(l, k_t)(1 - \tau_t) - \delta - \rho}{\sigma} = 0.$$

Then, it must be the case that:

$$f_k(l, k^s)(1 - \tau^s) = \delta + \rho$$

Now, if capital is taxed, then $(1 - \tau^s) < 1$ and since $\delta + \rho$ are fixed numbers, it must be the case that the marginal product of capital $f_k(l, k^s)$ has increased. If the marginal product of capital has increased, it must be the case that the steady state level of capital stock k_s has fallen. Therefore, if we plot this on $k_t - c_t$ space, we have that the locus has shifted to the left at a lower capital stock. We see this in Fig. 15.

Now, let us look at the evolution of capital:

$$\dot{k}_t = f(l, k_t) - c_t - \delta k_t.$$

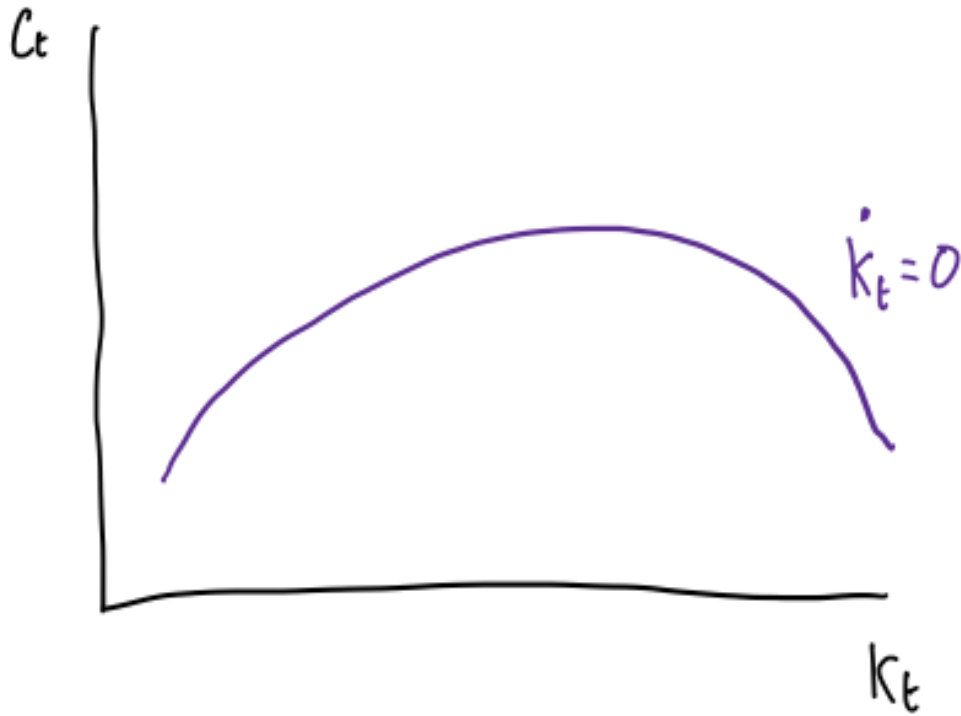


Figure 16: Same locus as before when there was no taxation.

This is exactly the same equation as we saw before which we show in Fig. 16! Nothing has changed since the agent has been rebated. As the government taxes and repays the agents, nothing has changed.

We now combine the two graphs. The $\dot{k}_t = 0$ locus does not move, whilst the $\dot{c}_t = 0$ locus shifts because of the tax. In Fig. 17, both steady-state consumption and capital fall. This is not surprising: taxing capital income reduces the incentive to save. With a lower future capital stock, the economy produces and consumes less. Therefore, a tax on capital is detrimental to the capital stock and lowers consumption in the long run.

6.3.2 Dynamics with Taxation

We now want to look at dynamics with taxation. To do so, we need to consider whether any changes to taxes are implemented immediately or with a lag. Now, suppose that we have taxed capital. If we impose transversality conditions, we can determine a unique saddle in both cases where we do and don't have taxation. We show this in Fig. 18.

We now look at two types of tax increases and see the effects of these tax increases.

The first type is an **unexpected** tax increase announced and implemented at time t^* . We show the dynamics in Fig. 19. Consumption, which is a control, jumps onto the new saddle path at t^* . Capital is predetermined and cannot jump, but it begins to fall because the tax reduces the incentive to save. The economy then follows the new saddle path to permanently lower steady-state levels of consumption and capital.

The second type is an unexpected tax increase **announced** at time t^* but implemented later at t^{**} . We show the dynamics in Fig. 20. Between announcement and implementation, the old tax regime still determines the vector field, but the economy is no longer on the old saddle path. Consumption jumps at t^* and the economy follows a transition path that places it on the new saddle path at t^{**} . Capital remains continuous throughout: it begins adjusting after

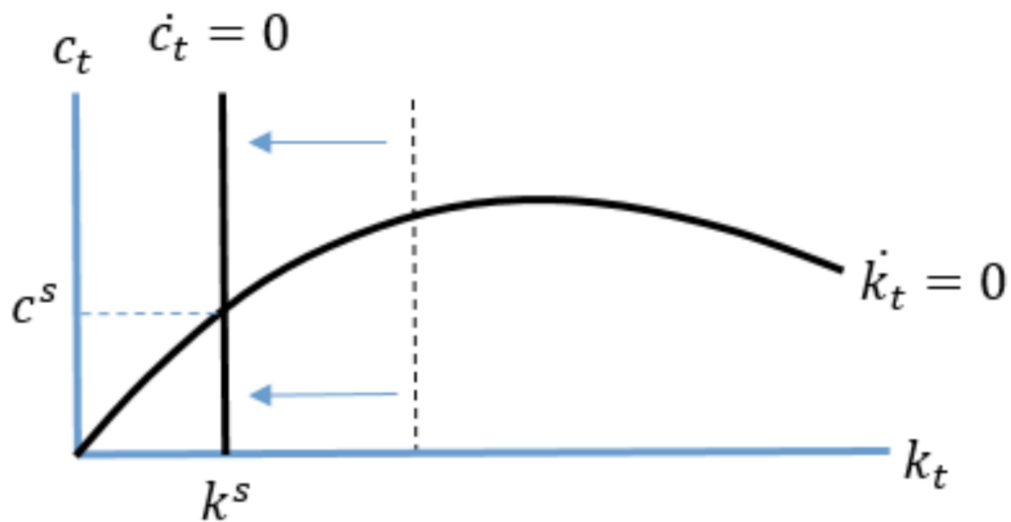


Figure 17: Steady state of consumption has dropped.

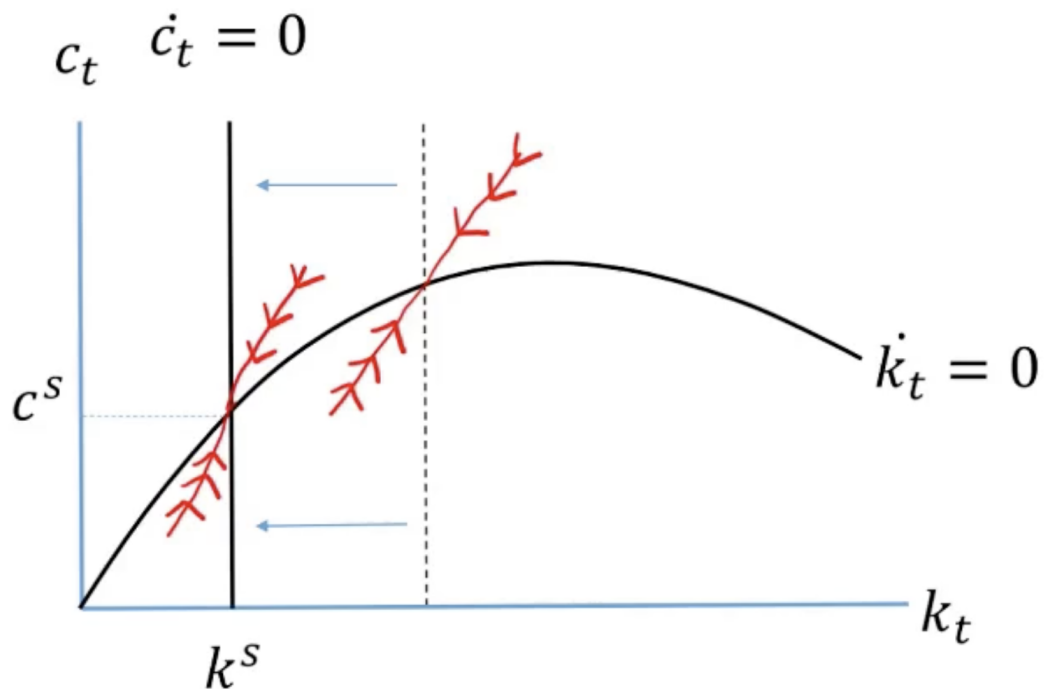


Figure 18: Saddle path with and without taxation.

the announcement, and its rate of change shifts when the tax is implemented. The economy then converges along the new saddle path to the lower steady state.

Therefore, we see that when we make tax increases with a lag in implementation, agents will want to smooth their consumption.

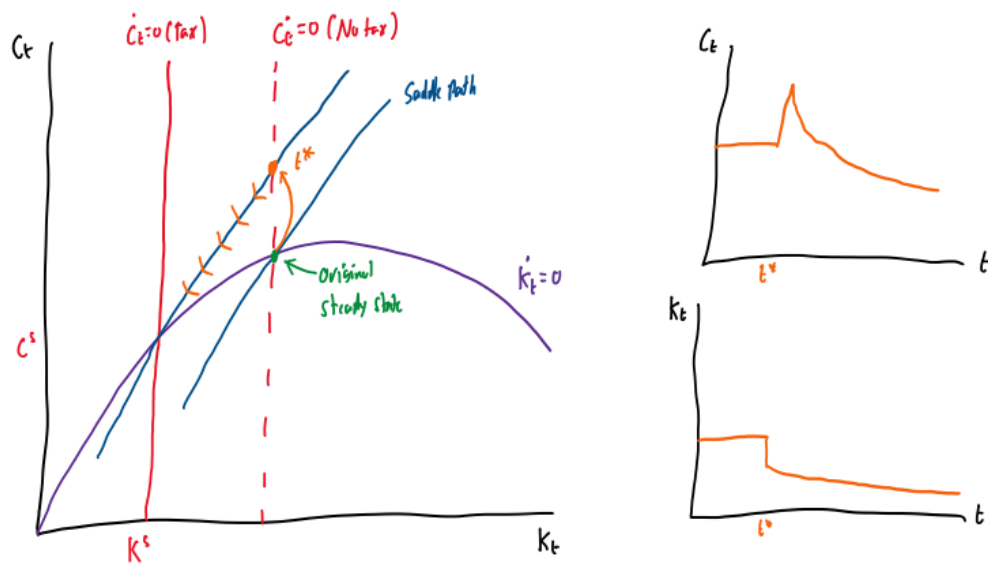


Figure 19: Consumption dynamics with unexpected tax announcement and immediate implementation.

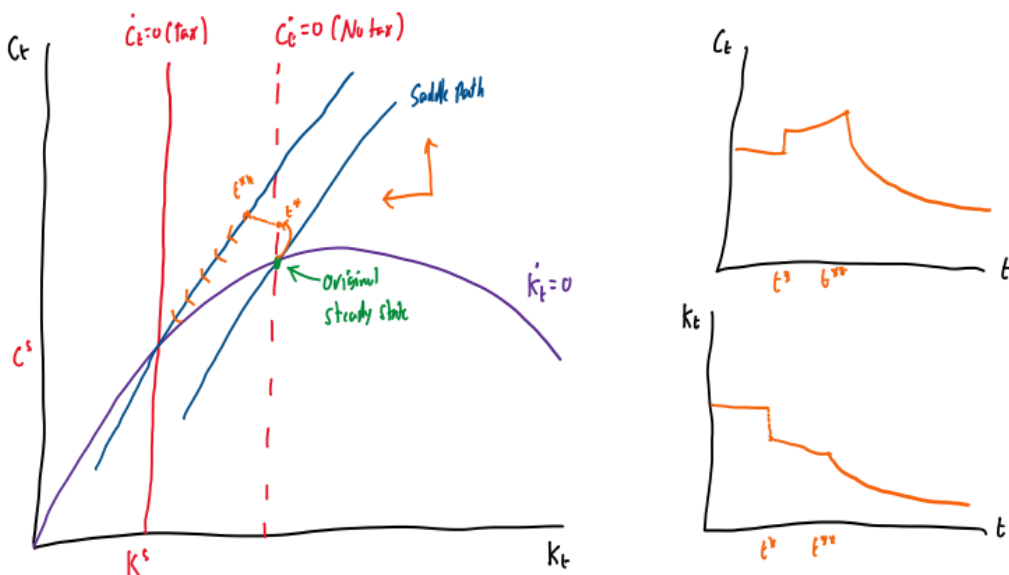


Figure 20: Consumption dynamics with unexpected tax announcement and a lag in implementation.

6.4 Incorporating Technological Progress

The issue so far is that the Kaldor facts says that economies should grow. However, in this Ramsey model, we reach a steady state. Therefore, growth cannot occur simply through the accumulation of capital. We now try to incorporate technological progress into the model via the production function. In particular, we incorporate labour-augmenting technological progress.

Definition 99 (Labour-Augmenting Technological Progress)

Technological progress is given by:

$$y_t = f(A_t l, k_t) \quad (114)$$

whereby technology is increasing exogenously at rate:

$$\frac{\dot{A}_t}{A_t} = \gamma. \quad (115)$$

Remark 100. *We want real wages to be increasing over time but not the rental rate of capital. This is why we made technology labour augmenting.*

The equilibrium under this model is the same as the Ramsey model without taxation except we now have this new labour-augmenting technological progress.

Proposition 101 (Equilibrium for Ramsey Model with Labour-Augmenting Process)

Under CRRA preferences and labour-augmenting technology, the Euler equation for consumption, law of motion for capital, and transversality condition are:

$$\frac{\dot{c}_t}{c_t} = \frac{f_k(A_t l, k_t) - \delta - \rho}{\sigma} \quad (116)$$

$$\dot{k}_t = f(A_t l, k_t) - c_t - \delta k_t \quad (117)$$

$$\lim_{t \rightarrow \infty} e^{-\rho t} u'(c_t) k_t = 0. \quad (118)$$

We see that A_t appears in the production function now. There are a few issues when we try to compute the steady state of this model. Let us look for a steady state where consumption is not growing $\dot{c}_t = 0$ for example. As before, this implies that:

$$f_k(A_t l, k_t) = \delta + \rho.$$

However, suppose that this relationship was true. As technology A_t is exogenously growing, then labour is more productive and this implies that the marginal product of **capital** is also increasing. Since $\delta + \rho$ are constants, we have that the left-hand side of this equation is increasing whilst the right hand side are fixed. Therefore, this equation is impossible to satisfy. **There is no steady state in which capital or consumption is not changing.** We therefore need to look for a **balanced growth path**.

6.4.1 Balanced Growth Path in Ramsey Model with Technology

A balanced growth path is when the key variables capital, consumption, and output (k_t, c_t, y_t) all grow at the same rate. This is fine for us as recall in the Kaldor facts, one of the facts was that consumption to output C_t/Y_t is constant over time. If these two variables have the same growth rate, then they are constant.

To proceed on this path, we define two new variables to take into account the fact that technology A_t is growing.

Definition 102 (Technology Adjusted Capital and Consumption)

Technology adjusted capital is defined as:

$$\tilde{k}_t = \frac{k_t}{A_t}. \quad (119)$$

Technology adjusted consumption is defined as:

$$\tilde{c}_t = \frac{c_t}{A_t} \quad (120)$$

Now, let us transform the marginal product of capital. Since $f(A_t l, k_t)$ is homogeneous of degree one, $f_k(A_t l, k_t)$ is homogeneous of degree zero. We can therefore multiply both arguments by $1/A_t$:

$$f_k(A_t l, k_t) = f_k\left(l, \frac{k_t}{A_t}\right) =_{(1)} f_k(l, \tilde{k}_t) \quad (121)$$

where $=_{(1)}$ follows by definition for technology adjusted capital.

We now want to look at the dynamics for both consumption and technology along the balanced growth path.

Consumption Dynamics Along Balanced Growth Path

Previously, we had the Euler equation:

$$\frac{\dot{c}_t}{c_t} = \frac{f_k(A_t l, k_t) - \delta - \rho}{\sigma}.$$

However, we have already seen why we can't use this to find the steady state and instead need to find a balanced growth path. Hence, we need to rewrite this Euler equation in terms of our new transformed variables.

Proposition 103 (Transformed Euler Equation)

The transformed Euler equation is:

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{f_k(l, \tilde{k}_t) - \delta - \rho}{\sigma} - \gamma. \quad (122)$$

Proof. By definition, we had:

$$c_t = A_t \tilde{c}_t \quad \leftrightarrow \quad \tilde{c}_t = \frac{c_t}{A_t}.$$

If we differentiate this, we have:

$$\dot{c}_t = \dot{A}_t \tilde{c}_t + A_t \dot{\tilde{c}}_t$$

Then, if we divide this equation by $c_t = A_t \tilde{c}_t$, we get:

$$\begin{aligned} \frac{\dot{c}_t}{c_t} &= \frac{\dot{c}_t}{A_t \tilde{c}_t} \\ &= \frac{\dot{A}_t \tilde{c}_t}{A_t \tilde{c}_t} + \frac{A_t \dot{\tilde{c}}_t}{A_t \tilde{c}_t} \\ &= \frac{\dot{A}_t}{A_t} + \frac{\dot{\tilde{c}}_t}{\tilde{c}_t} \end{aligned}$$

Now, by the Euler equation, we have that:

$$\frac{\dot{c}_t}{c_t} = \frac{f_k(A_t l, k_t) - \delta - \rho}{\sigma}$$

so then we have that:

$$\frac{\dot{A}_t}{A_t} + \frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{f_k(A_t l, k_t) - \delta - \rho}{\sigma}.$$

By definition, $\frac{\dot{A}_t}{A_t} = \gamma$. Hence,

$$\gamma + \frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{f_k(A_t l, k_t) - \delta - \rho}{\sigma}.$$

Using the fact that $f_k(A_t l, k_t)$ is homogeneous of degree zero, we can rewrite this as

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{f_k(l, \tilde{k}_t) - \delta - \rho}{\sigma} - \gamma.$$

□

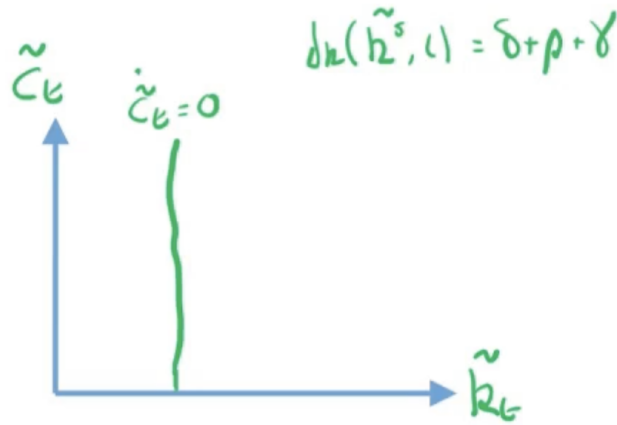


Figure 21: Balanced growth path for capital.

The solution to the balanced growth path to the transformed Euler equation:

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{f_k(l, \tilde{k}_t) - \delta - \rho}{\sigma} - \gamma = 0$$

is therefore:

$$f_k(l, \tilde{k}_t) = \delta + \rho + \sigma\gamma.$$

We show this in Fig. 21.

Capital Dynamics Along Balanced Growth Path

Theorem 104 (Transformed Law of Motion for Capital)

The Transformed Law of Motion for Capital is given by:

$$\dot{\tilde{k}}_t = f(l, \tilde{k}_t) - \tilde{c}_t - (\delta + \gamma)\tilde{k}_t. \quad (123)$$

Proof. First, look at the original law of motion for capital:

$$\dot{k}_t = f(A_t l, k_t) - c_t - \delta k_t.$$

As $f(A_t l, k_t)$ is homogeneous of degree 1, we can rewrite this as:

$$\dot{k}_t = A_t f(l, \tilde{k}_t) - c_t - \delta k_t.$$

Now, recall that by definition of the technology adjusted capital:

$$k_t = A_t \tilde{k}_t.$$

We then take derivative:

$$\dot{k}_t = \dot{A}_t \tilde{k}_t + A_t \dot{\tilde{k}}_t.$$

We then combine this with our law of motion for capital to have that:

$$A_t f(l, \tilde{k}_t) - c_t - \delta k_t = \dot{k}_t = \dot{A}_t \tilde{k}_t + A_t \dot{\tilde{k}}_t$$

Then, if we divide by A_t and use the growth-rate definition $\dot{A}_t/A_t = \gamma$, we have that:

$$\gamma \tilde{k}_t + \dot{\tilde{k}}_t = f(l, \tilde{k}_t) - \tilde{c}_t - \delta \tilde{k}_t.$$

We then re-arrange to get:

$$\dot{\tilde{k}}_t = f(l, \tilde{k}_t) - \tilde{c}_t - (\delta + \gamma) \tilde{k}_t$$

□

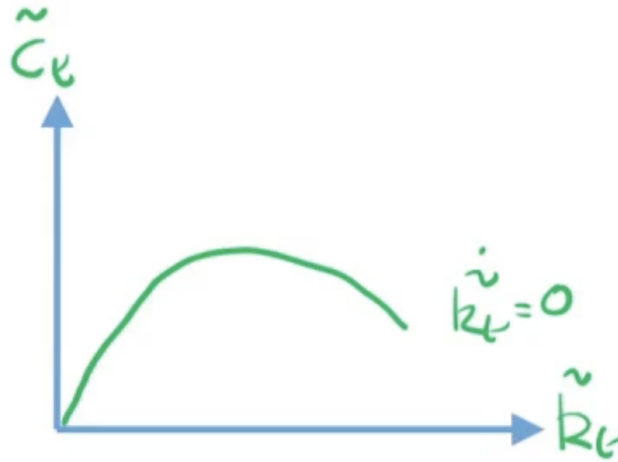


Figure 22: Balanced growth path for consumption.

Again, we can compute the balanced growth path:

$$\dot{\tilde{k}}_t = f(l, \tilde{k}_t) - \tilde{c}_t - (\delta + \gamma) \tilde{k}_t = 0$$

and therefore we have that:

$$f(l, \tilde{k}_t) = \tilde{c}_t + (\delta + \gamma) \tilde{k}_t.$$

We can plot this too in Fig. 22.

We now combine these two balanced growth rates in Fig. 23. This is the same as what we saw before! This economy converges to a steady state in **transformed variables** which implies that capital, consumption, and output

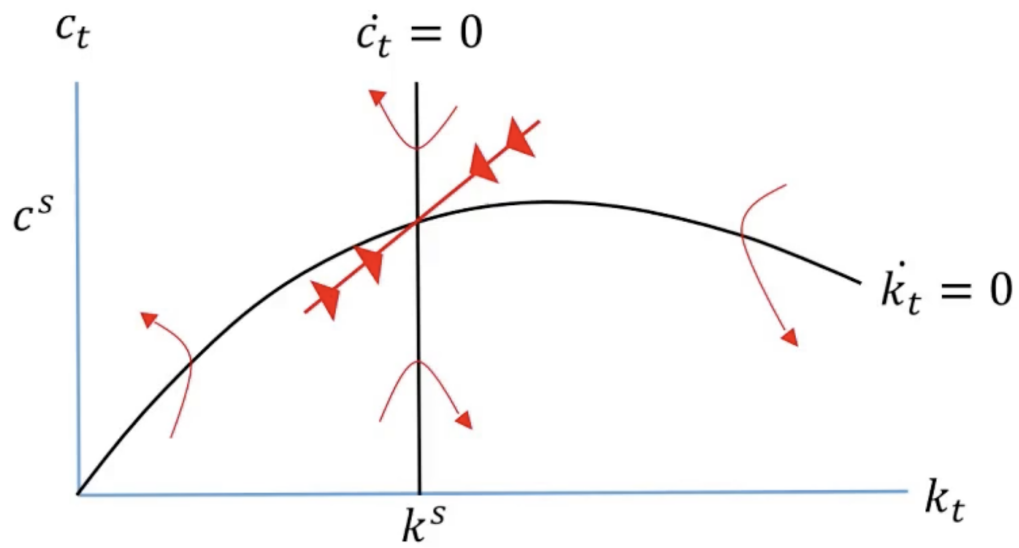


Figure 23: Balanced growth path for consumption and capital where the axis should be the transformed capital $\tilde{k}_t$ and $\tilde{c}_t$.

are growing at a steady rate γ . This is the definition of a balanced growth path. Again, the issue with this method is that γ is an exogenous growth rate and hence not endogenously explaining anything in the model.

7 Endogenous Growth Theory

In our previous models, we assumed technology was growing exogenously. We now want to explain why is technology growing over time.

7.1 AK Model

Previously, we did not have growth in our models and converged to a steady state. This is due to the decreasing returns to capital. We now seek to change this assumption. We first look at the AK model which shall assume constant returns to capital in the production.

Definition 105 (AK Production Function)

The AK production function is given by:

$$f(K_t, A) = AK_t \quad (124)$$

where A is exogenous technology and K_t is endogenous capital stock.

We have that the production function is now linear in capital. The following gives us a useful property.

Proposition 106 (Marginal Product of Capital is Constant)

The marginal product of capital of the AK production function is constant:

$$f_k(K_t, A) = A. \quad (125)$$

Proof. Just take the derivative of:

$$f(K_t, A) = AK_t$$

with respect to K_t . □

We no longer have decreasing returns to scale in capital with this production function!

Household Problem

We now look at the household problem which is unchanged from the Ramsey model which we replicate again. The difference is that now the household is not supplying labour to the firm.

Definition 107 (Household Optimisation Problem)

The household optimisation problem is:

$$\begin{aligned} \max_{c(\cdot)} \quad & \int_0^{\infty} e^{-\rho t} u(c_t) dt \\ \text{s.t.} \quad & \dot{K}_t = r_t K_t + \Pi_t - c_t - \delta K_t \end{aligned} \quad (126)$$

where Π_t is the firm's profits, δ is the depreciation rate, and $\dot{K}_t$ is the change in the capital stock K_t .

From this, we can derive the Euler equation for consumption as before.

Proposition 108 (Euler Equation for Consumption)

The Euler equation for consumption is:

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta}{\sigma}. \quad (127)$$

Proof. To solve the household problem, we need to use **Hamiltonians** again.

$$\mathcal{H}^d = e^{-\rho t} u(c_t) + \mu_t \left[(r_t - \delta)K_t + \Pi_t - c_t \right].$$

Our associated **four** first order conditions are:

1. $\frac{\partial \mathcal{H}^d}{\partial c_t} = 0$ derivative with respect to **control variable** is equal to 0
2. $\frac{\partial \mathcal{H}^d}{\partial K_t} = -\dot{\mu}_t$ derivative with respect to **state variable** is equal to $-\dot{\mu}_t$
3. $\frac{\partial \mathcal{H}^d}{\partial \mu_t} = (r_t - \delta)K_t + \Pi_t - c_t$ derivative with respect to **costate variable** is the law of motion.
4. $\lim_{t \rightarrow \infty} \mu_t K_t = 0$ transversality condition.

Actually writing out the four first order conditions, we have:

$$\begin{aligned} e^{-\rho t} u'(c_t) &= \mu_t \\ \mu_t(r_t - \delta) &= -\dot{\mu}_t \\ (r_t - \delta)K_t + \Pi_t - c_t &= \dot{K}_t \\ \lim_{t \rightarrow \infty} \mu_t K_t &= 0. \end{aligned}$$

If we look at the first FOC:

$$\mu_t = e^{-\rho t} u'(c_t)$$

taking derivatives with respect to t and using the product rule gives us:

$$\dot{\mu}_t = -u'(c_t)\rho e^{-\rho t} + u''(c_t)\dot{c}_t e^{-\rho t}.$$

We then plug this into the second FOC:

$$\begin{aligned} \mu_t(r_t - \delta) &= -\dot{\mu}_t \\ \Leftrightarrow \mu_t(r_t - \delta) &= -[-u'(c_t)\rho e^{-\rho t} + u''(c_t)\dot{c}_t e^{-\rho t}] \end{aligned}$$

We can then solve this to get:

$$\begin{aligned}
\mu_t(r_t - \delta) &= -[-u'(c_t)\rho e^{-\rho t} + u''(c_t)\dot{c}_t e^{-\rho t}] \\
\leftrightarrow \mu_t r_t - \mu_t \delta &= u'(c_t)\rho e^{-\rho t} - u''(c_t)\dot{c}_t e^{-\rho t} \\
\leftrightarrow e^{-\rho t} u'(c_t) r_t - e^{-\rho t} u'(c_t) \delta &= u'(c_t)\rho e^{-\rho t} - u''(c_t)\dot{c}_t e^{-\rho t} \\
\leftrightarrow u'(c_t) r_t - u'(c_t) \delta &= u'(c_t)\rho - u''(c_t)\dot{c}_t \\
\leftrightarrow u''(c_t)\dot{c}_t &= -u'(c_t) r_t + u'(c_t)\delta + u'(c_t)\rho \\
\leftrightarrow u''(c_t)\dot{c}_t &= u'(c_t)(-\rho + \delta + r_t) \\
\leftrightarrow u''(c_t)\dot{c}_t &= -u'(c_t)(r_t - \delta - \rho) \\
\leftrightarrow \dot{c}_t &= -\frac{u'(c_t)}{u''(c_t)}(r_t - \delta - \rho)
\end{aligned}$$

To finish this, looking at our Euler equation for consumption:

$$\dot{c}_t = \frac{u'(c_t)(r_t - \delta - \rho)}{-u''(c_t)}$$

we can then rewrite this using the coefficient of relative risk aversion since

$$-\frac{u''(c_t)}{u'(c_t)} = \frac{\sigma}{c_t}.$$

We therefore have that:

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta}{\sigma}.$$

□

Firm Problem

The firm problem is the same problem as in the Ramsey model except we don't consider labour.

Definition 109 (Optimisation Problem Facing the Firm)

The firm maximisation problem is:

$$\begin{aligned}
\max_{K_t} \quad & y_t - r_t K_t \\
\text{s.t.} \quad & y_t = f(K_t, A) = AK_t
\end{aligned} \tag{128}$$

where $f(\cdot, \cdot)$ is the AK production function.

We then have the usual FOC for the firm.

Proposition 110 (Firm's First Order Conditions)

The firm's first order equation is:

$$r_t = f_K(K_t, A) = A.$$

Proof. Substitute in output into the profit function:

$$AK_t - r_t K_t$$

and then take derivative:

$$\frac{\partial}{\partial K_t} : A - r_t = 0$$

□

The implication of this AK production function is that the return to capital is now constant:

$$r_t = r = A.$$

Balanced Growth Path

Now, if we look at the system of equations consisting of the Euler equation for consumption and the evolution of capital, we have:

$$\begin{aligned} \frac{\dot{c}_t}{c_t} &= \frac{A - \rho - \delta}{\sigma} \\ \frac{\dot{K}_t}{K_t} &= A - \frac{c_t}{K_t} - \delta \end{aligned}$$

where we used the firm's FOC $r_t = A$ in the Euler equation for consumption.

The issue with this, as seen previously, we no longer have steady state in levels since it is not necessarily the case that $A - \rho - \delta = 0$ as they are all constant. However, we now hope that we can have a steady state in growth rates. Recall that this was called a **balanced growth path**. We now derive this.

Theorem 111 (Balanced Growth Path)

The balanced growth path of the AK model is:

$$\frac{\dot{c}_t}{c_t} = \frac{A - \rho - \delta}{\sigma} \equiv g \tag{129}$$

$$\frac{\dot{K}_t}{K_t} = A - \delta - g. \tag{130}$$

Proof. If we **assume** that there is a balanced growth path, so that c_t and K_t are growing at the same rate, then we define:

$$\frac{\dot{c}_t}{c_t} = \frac{A - \rho - \delta}{\sigma} \equiv g.$$

Now, recall that the capital equation is:

$$\dot{K}_t = AK_t - c_t - \delta K_t$$

where we used Euler's equation $f(K_t, A) = AK_t = r_t K_t$. We then have that this becomes:

$$\frac{\dot{K}_t}{K_t} = A - \frac{c_t}{K_t} - \delta.$$

As we are in a balanced growth path, we must have that that K_t is growing at the same rate g as c_t so then:

$$\frac{\dot{K}_t}{K_t} = g.$$

Now, we plug this into the evolution of capital equation to get:

$$\frac{c_t}{K_t} = A - \delta - g.$$

□

To interpret this equation for $\frac{c_t}{K_t}$, it says that the ratio of consumption to capital is a constant. This makes sense since we are saying that we are in a balanced growth path where both of these variables grow at the same rate.

Consumption-to-capital ratio is constant along the balanced growth path.

Transition Dynamics

We now want to look at the short term dynamics of the system.

Theorem 112 (Transition Dynamics)

Under log-utility, the transition dynamics of the AK model are:

$$\frac{\dot{\hat{c}}_t}{\hat{c}_t} = \hat{c}_t - \rho \quad (131)$$

where $\hat{c}_t = \frac{c_t}{K_t}$.

Proof. By assumption of log-utility, the Euler equation becomes:

$$\frac{\dot{c}_t}{c_t} = \frac{A - \rho - \delta}{\sigma} = A - \rho - \delta.$$

Now, we define the new variable:

$$\hat{c}_t = \frac{c_t}{K_t}.$$

The derivative of this by quotient rule is:

$$\dot{\hat{c}}_t = \frac{K_t \dot{c}_t - c_t \dot{K}_t}{K_t^2}.$$

Now, we want to look at the evolution of this new variable, which is:

$$\begin{aligned} \frac{\dot{\hat{c}}_t}{\hat{c}_t} &= \frac{K_t \dot{c}_t - c_t \dot{K}_t}{K_t^2} \left(\frac{1}{\hat{c}_t} \right) \\ &= \frac{K_t \dot{c}_t - c_t \dot{K}_t}{K_t^2} \left(\frac{K_t}{c_t} \right) \\ &= \frac{\dot{c}_t}{c_t} - \frac{\dot{K}_t}{K_t} \\ &=_{(1)} \left(A - \rho - \delta \right) - \left(A - \frac{c_t}{K_t} - \delta \right) \\ &= \hat{c}_t - \rho \end{aligned}$$

where $=_{(1)}$ uses the Euler equation for consumption and the law of motion for capital.

□

The equation:

$$\frac{\dot{\hat{c}}_t}{\hat{c}_t} = \hat{c}_t - \rho$$

says that, for a given consumption-to-capital ratio, this tells us where it will go in the future.

Proposition 113 (Constant Consumption to Capital Ratio Along Balanced Growth Path)

In the balanced growth path, $\hat{c}_t$ is a constant:

$$\hat{c}_t = \rho. \quad (132)$$

Proof. Take our transition dynamics equation and set it equal to 0:

$$\frac{\dot{\hat{c}}_t}{\hat{c}_t} = \hat{c}_t - \rho = 0.$$

Then, we immediately have that:

$$\hat{c}_t = \rho.$$

□

We now want to analyse the transition dynamics:

$$\hat{c}_t = \rho$$

If $\hat{c}_t \neq \rho$, then if $\hat{c}_t > \rho$, then it will increase $\hat{c}_t$ all the way up to infinity, this violates transversality conditions. If $\hat{c}_t < \rho$, then the system will implode. Hence, the only way for the system to be stable is if $\hat{c}_t = \rho$. This means that **there are no transition dynamics** in the AK model. Intuitively, this is the case because **the balanced growth path requires the consumption to capital ratio be constant and hence no transitional dynamics**. Therefore, we always jump to the balanced growth path. This is not an interesting model since we are always on the balanced growth path.

7.2 Microfoundations of AK Model

We now want to give some microfoundations for the AK model. We look at two types of microfoundations.

7.2.1 Microfoundations 1: Learning By Doing

We now want to explain why could it be the case that the marginal product of capital is constant in the AK model. The first idea we want to look at is learning by doing. Here, we learn more as we do things. But to do this, we need a measure of how much *doing* is happening.

Definition 114 (Learning By Doing)

Knowledge accumulates through accumulating K:

$$Y_t = f(K_t, A_t L_t) \quad (133)$$

$$A_t = \psi \frac{K_{t-1}}{L_{t-1}} \quad (134)$$

where $A_t L_t$ is labour measured in efficiency units.

The term $A_t L_t$ measures the *doing* in the economy as it is based on the accumulated capital. We see that A_t is a function of the previous period capital stock which captures the idea of accumulation.

We can now show how to go from this to the AK model.

Proposition 115 (Learning By Doing is Microfoundations for AK Model)

Assuming output Y_t follows a constant returns to scale production function, under learning by doing microfoundations, we get back the AK model:

$$\frac{Y_t}{L_t} \approx \frac{K_t A}{L_t} = \frac{f(K_t, A)}{L_t} \quad (135)$$

where the right-hand side is the production function in the AK model.

Proof.

$$\begin{aligned} \frac{Y_t}{L_t} &= \frac{1}{L_t} f(K_t, A_t L_t) \\ &= \frac{K_t}{L_t} f\left(1, \frac{A_t L_t}{K_t}\right) \\ &= \frac{K_t}{L_t} f\left(1, \frac{K_{t-1}}{K_t} \frac{L_t}{L_{t-1}} \psi\right) \\ &\stackrel{(1)}{=} \frac{K_t}{L_t} f\left(1, \frac{\psi}{1+g}\right) \equiv A \frac{K_t}{L_t} \end{aligned}$$

where in $\stackrel{(1)}{=}$, along a balanced growth path with constant labour, $L_t/L_{t-1} = 1$ and $K_{t-1}/K_t = 1/(1+g)$. We define the constant $A \equiv f(1, \psi/(1+g))$. $\square$

You have a capital stock, and as this stock gets bigger, you produce more technology too, so your marginal product of capital doesn't shrink and stays constant.

7.2.2 Microfoundations 2: Human Capital Accumulation

We now look at another way to explain why could the marginal product of capital in the AK model be constant. Now, we don't just care about the quantity of labour but also the quality of labour. Hence, the quality of labour H_t can now be treated as another alternative to be invested in. So then, we have that $Y_t = C_t + I_t^K + I_t^H$ which says that output can be either consumed or invested into either capital K or labour H . With this, we now define human capital accumulation.

Definition 116 (Human Capital Accumulation)

Output is a function of capital and quality human labour H_t . Then, human capital accumulation is given by:

$$Y_t = f(K_t, H_t) \quad (136)$$

$$Y_t = C_t + I_t^K + I_t^H \quad (137)$$

$$\dot{K}_t = I_t^K - \delta K_t \quad (138)$$

$$\dot{H}_t = I_t^H - \delta H_t \quad (139)$$

where $\dot{H}_t$ is the law of motion for human capital and $\dot{K}_t$ is law of motion for capital.

From this, we can solve this using a Hamiltonian and from the Hamiltonian, we get the following result.

Theorem 117 (Equalisation of Factors)

The marginal product of physical capital and human capital are equalised:

$$f_K(K_t, H_t) = f_H(K_t, H_t). \quad (140)$$

Proof. First, construct the Hamiltonian:

$$\mathcal{H} = e^{-\rho t} u(c_t) + \lambda_{1t} \left[f(K_t, H_t) - C_t - I_t^K - I_t^H \right] + \lambda_{2t} \left[I_t^K - \delta K_t \right] + \lambda_{3t} \left[I_t^H - \delta H_t \right].$$

Then, the first order conditions are:

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial C_t} &= u'(C_t) e^{-\rho t} - \lambda_{1t} = 0 \\ \frac{\partial \mathcal{H}}{\partial I_t^K} &= -\lambda_{1t} + \lambda_{2t} = 0 \\ \frac{\partial \mathcal{H}}{\partial I_t^H} &= -\lambda_{1t} + \lambda_{3t} = 0 \\ \frac{\partial \mathcal{H}}{\partial K_t} &= \lambda_{1t} f_K(K_t, H_t) - \delta \lambda_{2t} = -\dot{\lambda}_{2t} \\ \frac{\partial \mathcal{H}}{\partial H_t} &= \lambda_{1t} f_H(K_t, H_t) - \delta \lambda_{3t} = -\dot{\lambda}_{3t}. \end{aligned}$$

From the second and third equations, we immediately have

$$\lambda_{2t} = \lambda_{3t}.$$

Because these costates are equal at every date, their derivatives are also equal. Subtracting the fourth and fifth equations therefore gives $\lambda_{1t}(f_K - f_H) = 0$, so the marginal products are equalised at an interior solution. $\square$

The intuition behind this is that we choose to invest in either physical or human capital. Therefore, we should invest in both until the returns on both investment goods are equalised or else we can just invest in the one with the higher return. So if capital K increases, then the marginal product f_K decreases. However, this means that f_H increases so then we should invest more into human capital H . This pushes down f_H . However, this then pushes up f_K so then we now have an incentive to again invest in physical capital K . This is how the model works to counteract the forces of diminishing returns to capital.

We now can show that this model is akin to the AK model.

Theorem 118 (Human Capital Accumulation and the AK Mechanism)

Along an interior balanced growth path, the human-capital accumulation model generates the same constant marginal-return mechanism as the AK model.

Proof. Since the production function $f(\cdot)$ is homogeneous of degree one, its marginal products are homogeneous of degree zero. The condition $f_K(K_t, H_t) = f_H(K_t, H_t)$ pins down a constant ratio K_t/H_t at an interior solution. This implies that the marginal product of capital, which depends only on this ratio, is constant along the balanced growth path. This is the same constant-return mechanism as in the AK model. $\square$

7.3 Romer Endogenous Growth Model

We now want to look at an endogenous growth model where we grow due to incentives. That is, technology improves because there is a profit incentive for technology to improve. That is, changes in the production possibility set are determined as an endogenous response to economic incentives.

However, the issue is that due to Euler's theorem, firms do not make any profits and hence they have no profits to invest into technology. As a result, we need imperfection in the goods market so that there is profits to be made for firms. Hence, we need to introduce a monopolistic-competitive setup. We will have two types of firms:

1. Final Goods: Final goods production is competitive market. However, they require intermediate goods.
2. Intermediate Goods: Each intermediate good is produced by a monopolist.

Now, the question is the structure on this intermediate goods market. First, we shall assume that the variety of intermediate goods are fixed. Later, we relax this by endogeneising the number of intermediate goods produced.

7.3.1 Fixed Varieties

Final Good Sector

We first look at the final goods sector. There is a consumption good Y_t which requires a mixture of N intermediate goods X_i for $i = 1, \dots, N$ to produce. In this section, we fix the number of intermediate goods to be fixed at N_i .

Definition 119 (Final Goods Production)

The Dixit-Stiglitz production function is:

$$Y_t = L_t^{1-\alpha} \sum_{i=1}^{N_i} X_{it}^\alpha \quad (141)$$

where N_i is the number of intermediate goods.

This production function has **increasing returns to specialisation**. This is because the more intermediate goods you have, the more options you have for your input and hence you are more productive. This is a really important part of the model as we will later see since as we increase the number of varieties, this will increase output and hence lead to growth.

Now, we can set up the optimisation problem that faces the firm producing the final good.

Definition 120 (Optimisation Problem Facing the Final Good Firm)

The final good firm's profit maximisation problem is:

$$\begin{aligned} \max_{Y_t, L_t, \{X_{it}\}_{i=0}^{N_t}} \quad & Y_t - w_t L_t - \sum_{i=1}^{N_i} P_{it} X_{it} \\ \text{s.t.} \quad & Y_t = L_t^{1-\alpha} \sum_{i=1}^{N_i} X_{it}^\alpha \end{aligned} \quad (142)$$

where P_{it} is the price of intermediate good i .

We can then derive the first order conditions.

Proposition 121 (First Order Conditions for Final Good Producer)

The first-order conditions for the final-good producer are:

$$(1 - \alpha)L_t^{-\alpha} \sum_{i=1}^{N_i} X_{it}^\alpha = w_t \quad (143)$$

$$\alpha L_t^{1-\alpha} X_{it}^{\alpha-1} = P_{it} \quad \text{for } i = 1, \dots, N_i. \quad (144)$$

Proof. Plug the production function into the profit function:

$$\pi_t = L_t^{1-\alpha} \sum_{i=1}^N X_{it}^\alpha - w_t L_t - \sum_{i=1}^{N_i} P_{it} X_{it}.$$

Now, take first order conditions with respect to L_t and X_{it} to get:

$$\begin{aligned} \frac{d\pi_t}{dL_t} : (1 - \alpha)L_t^{-\alpha} \sum_{i=1}^{N_i} X_{it}^\alpha - w_t &= 0 \\ \frac{d\pi_t}{dX_{it}} : \alpha L_t^{1-\alpha} X_{it}^{\alpha-1} - P_{it} &= 0 \quad \text{for } i = 1, \dots, N_i. \end{aligned}$$

□

Remark 122. The first FOC is the marginal product of labour. The second FOC is the marginal product of an intermediate good.

Intermediate Goods Sector

We now look at the intermediate goods. Each intermediate good is produced by a monopolist. First, we look at the production function for a single intermediate good.

Definition 123 (Production Function for Intermediate Good)

The production function for a single intermediate good X_{it} is given by:

$$X_{it} = y_t \quad (145)$$

where y_t is the final good.

Remark 124. We need one unit of the final good y_t to produce one unit of the intermediate good X_{it} .

We can now state the optimisation problem facing the monopolist producer for intermediate good X_i .

Definition 125 (Optimisation Problem Facing the Intermediate Good Firm)

The monopolist intermediate good X_i firm's profit maximisation problem is:

$$\begin{aligned} \max_{P_{it}, X_{it}} \quad & P_{it} X_{it} - y_{it} \\ \text{s.t.} \quad & X_{it} = y_{it} \\ & \alpha L_t^{1-\alpha} X_{it}^{\alpha-1} = P_{it} \end{aligned} \quad (146)$$

where y_{it} is the demand for the final good y_t by intermediate good producer i and the second constraint is the demand for the intermediate good X_i .

The second constraint:

$$\alpha L_t^{1-\alpha} X_{it}^{\alpha-1} = P_{it}$$

is the first order condition from the final good producer's profit maximisation problem. This is the demand for the intermediate good i by the final good producer. Hence, this is a downward sloping demand curve for the intermediate good with respect to the price of the intermediate good P_i .

Proposition 126 (First Order Conditions for Intermediate Good Producer)

The first order conditions for the intermediate good producer are:

$$\alpha^2 L_t^{1-\alpha} X_{it}^{\alpha-1} = 1 \quad (147)$$

$$P_{it} = \frac{1}{\alpha}. \quad (148)$$

Proof. Substitute both constraints into the profit function:

$$\max_{X_{it}} \alpha L_t^{1-\alpha} X_{it}^\alpha - X_{it}.$$

Taking the derivative with respect to X_{it} gives the first condition. Substituting that condition into the demand curve gives the price $P_{it} = 1/\alpha$. □

From this, we can get how much of intermediate good i is produced and its price.

Proposition 127 (Supply and Price of Intermediate Good i)

The amount of intermediate good i that is produced is:

$$X_{it} = L_t \alpha^{\frac{2}{1-\alpha}}. \quad (149)$$

The price of intermediate good i is:

$$P_{it} = \frac{1}{\alpha}. \quad (150)$$

Proof. Re-arrange the first order conditions for the intermediate good producer:

$$\alpha^2 L_t^{1-\alpha} X_{it}^{\alpha-1} = 1$$

$$P_{it} = \frac{1}{\alpha}.$$

□

The first equation says how much of intermediate good X_{it} will be produced. The second equation says the price of the intermediate good. The price depends on the importance of the monopoly intermediate firm in the production of the final good.

Now recall that in equilibrium, the supply of labour is inelastic so then $L_t = 1$. This then implies that the amount of intermediate good i produced is:

$$X_{it} = \alpha^{\frac{2}{1-\alpha}}$$

and hence the demand for intermediate good i is fixed.

Household Problem

The household problem is standard.

Definition 128 (Household Optimisation Problem)

The household optimisation problem is:

$$\begin{aligned} \max_{c(\cdot), A(\cdot)} \quad & \int_0^{\infty} e^{-\rho t} u(c_t) dt \\ \text{s.t.} \quad & C_t + \dot{A}_t = w_t L_t + r_t A_t + \Pi_t \\ & L_t = \bar{L} \end{aligned} \tag{151}$$

where $L_t = \bar{L}$ is the inelastic labour supply, Π_t is the **intermediate-goods monopolists'** profits, A_t is bond holdings, and $\dot{A}_t$ is the change in those holdings.

An important thing that is different now is that $\Pi_t \neq 0$ since we have a monopoly rather than perfectly competitive markets. We make an assumption to make things easier.

Assumption (Logarithmic Utility)

The utility function is logarithmic:

$$u(c_t) = \log(c_t).$$

We can now derive the standard Euler equation for consumption.

Proposition 129 (Euler Equation for Consumption)

The Euler equation for consumption is:

$$\frac{\dot{c}_t}{c_t} = r_t - \rho. \tag{152}$$

Proof.

$$\mathcal{H}^d = e^{-\rho t} u(c_t) + \mu_t \left[w_t \bar{L} + r_t A_t + \Pi_t - c_t \right].$$

Our associated **four** first order conditions are:

1. $\frac{\partial \mathcal{H}^d}{\partial c_t} = 0$ derivative with respect to **control variable** is equal to 0
2. $\frac{\partial \mathcal{H}^d}{\partial A_t} = -\dot{\mu}(t)$ derivative with respect to **state variable** is equal to $-\dot{\mu}(t)$
3. $\frac{\partial \mathcal{H}^d}{\partial \mu_t} = w_t \bar{L} + r_t A_t + \Pi_t - c_t$ derivative with respect to **costate variable** is the law of motion.
4. $\lim_{t \rightarrow \infty} \mu_t A_t = 0$ transversality condition.

Actually writing out the four first order conditions, we have:

$$e^{-\rho t} u'(c_t) = \mu_t$$

$$\mu_t r_t = -\dot{\mu}_t$$

$$w_t \bar{L} + r_t A_t + \Pi_t - c_t = \dot{A}_t$$

$$\lim_{t \rightarrow \infty} \mu_t A_t = 0.$$

If we look at the first FOC:

$$\mu_t = e^{-\rho t} u'(c_t)$$

taking derivatives with respect to t and using the product rule gives us:

$$\dot{\mu}_t = -u'(c_t)\rho e^{-\rho t} + u''(c_t)\dot{c}_t e^{-\rho t}.$$

We then plug this into the second FOC:

$$\begin{aligned}\mu_t r_t &= -\dot{\mu}_t \\ \Leftrightarrow \mu_t r_t &= -[-u'(c_t)\rho e^{-\rho t} + u''(c_t)\dot{c}_t e^{-\rho t}]\end{aligned}$$

We can then solve this to get:

$$\begin{aligned}\mu_t r_t &= -[-u'(c_t)\rho e^{-\rho t} + u''(c_t)\dot{c}_t e^{-\rho t}] \\ \Leftrightarrow \mu_t r_t &= u'(c_t)\rho e^{-\rho t} - u''(c_t)\dot{c}_t e^{-\rho t} \\ \Leftrightarrow e^{-\rho t} u'(c_t) r_t &= u'(c_t)\rho e^{-\rho t} - u''(c_t)\dot{c}_t e^{-\rho t} \\ \Leftrightarrow u'(c_t) r_t &= u'(c_t)\rho - u''(c_t)\dot{c}_t \\ \Leftrightarrow u''(c_t)\dot{c}_t &= -u'(c_t) r_t + u'(c_t)\rho \\ \Leftrightarrow u''(c_t)\dot{c}_t &= u'(c_t)(-\rho + r_t) \\ \Leftrightarrow u''(c_t)\dot{c}_t &= -u'(c_t)(\rho - r_t) \\ \Leftrightarrow \dot{c}_t &= -\frac{u'(c_t)}{u''(c_t)}(\rho - r_t)\end{aligned}$$

To finish this, looking at our Euler equation for consumption:

$$\dot{c}_t = \frac{u'(c_t)(r_t - \rho)}{-u''(c_t)}$$

we can then rewrite this using the coefficient of relative risk aversion since

$$-\frac{u''(c_t)}{u'(c_t)} = \frac{\sigma}{c_t}.$$

We therefore have that:

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\sigma}.$$

and since we have log utility, $\sigma = 1$ so we get:

$$\frac{\dot{c}_t}{c_t} = r_t - \rho.$$

□

Now, under steady state, we have that:

$$\frac{\dot{c}_t}{c_t} = 0$$

which then implies that

$$r_t = \rho.$$

General Equilibrium

We can now state the equilibrium of this model.

Proposition 130 (Equilibrium of Fixed Variety Romer Model)

The general equilibrium of the model is a sequence of prices $\{P_{it}, w_t, r_t\}$ and a set of quantities $\{Y_t, \{X_{it}\}_i, C_t, A_t, L_t\}$ such that the consumer's problem, intermediate goods producer's problem and the final good producer's problem are solved. Furthermore, markets clear:

$$X_{it} = \bar{L} \alpha^{\frac{2}{1-\alpha}} \quad (153)$$

$$P_{it} = \frac{1}{\alpha} \quad (154)$$

$$Y_t = \bar{L} N_i \alpha^{\frac{2\alpha}{1-\alpha}} \quad (155)$$

$$C_t = Y_t - N_i X_{it} \quad (156)$$

$$A_t = 0 \quad (157)$$

$$r_t = \rho \quad (158)$$

$$w_t = (1 - \alpha) N_i \alpha^{\frac{2\alpha}{1-\alpha}}. \quad (159)$$

Remark 131. Note that $A_t = 0$ says that the bonds are in net supply of 0 in this model as interest rates r_t adjust so that you do not want to save in between two periods. The household will decide how much bonds they want to buy A_t every period but interest rates will adjust such that they do not want to buy any bonds in the end.

Notice that the equation for consumption can be written as:

$$C_t = Y_t - N_i X_{it} = Y_t - N_i \left[\bar{L} \alpha^{\frac{2}{1-\alpha}} \right] = \bar{L} N_i \alpha^{\frac{2\alpha}{1-\alpha}} - N_i \left[\bar{L} \alpha^{\frac{2}{1-\alpha}} \right]$$

which is also a fixed value. Hence, notice that all the variables in equilibrium are fixed values. As a result, we ended up with a static model where there is no growth. This is unfortunately not what we wanted since we wanted growth in the model.

7.3.2 Endogenous Varieties

We now want to fix the static result that we got from the last section with fixed varieties. Hence, we now want to endogenise varieties so the number of intermediate goods N_i can be increasing every period. To do so, we need a new assumption.

Assumption (Creation of Intermediate Goods)

A intermediate goods monopolist can create a new intermediate good at cost η . We further assume that η is small.

Remark 132. This can be thought of as gains from specialisation. The monopolist can create a new intermediate good at a cost and hence the monopolist will need to think about the profits it gains from creating this intermediate good.

We can now calculate the profit if a monopolist brought out a new variety.

Proposition 133 (Profit From New Variety)

The equilibrium per-period profit of a new intermediate good to the monopolist is:

$$\pi_{it} = \frac{1-\alpha}{\alpha} \bar{L} \alpha^{\frac{2}{1-\alpha}}. \quad (160)$$

Proof. The definition of profit for new intermediate good i is:

$$\pi_{it} = P_{it}X_{it} - X_{it}.$$

Recall that in the equilibrium in the static number of varieties, we had that:

$$X_{it} = \bar{L} \alpha^{\frac{2}{1-\alpha}} \quad \text{and} \quad P_{it} = \frac{1}{\alpha}.$$

Plug these in to get:

$$\pi_{it} = \left(\frac{1}{\alpha}\right) \bar{L} \alpha^{\frac{2}{1-\alpha}} - \bar{L} \alpha^{\frac{2}{1-\alpha}}.$$

Add them together to get:

$$\pi_{it} = \frac{1-\alpha}{\alpha} \bar{L} \alpha^{\frac{2}{1-\alpha}}.$$

□

We have a per-period profit of the new variety good. We can get a present discounted value of introducing the new variety good by simply discounting the per-period profit by an appropriate discount rate.

Definition 134 (Present Discounted Value of New Variety)

The present discounted value of a new variety good is:

$$\int_0^{\infty} \pi e^{-rt} dt = \frac{\pi}{r} \quad (161)$$

where r is the rate of return on a bond in net zero supply.

We can now state the condition for which the monopolist will introduce a new intermediate good.

Proposition 135 (Conditions for Introduction of New Variety)

The intermediate good monopolist will introduce a new intermediate good if:

$$\frac{\pi}{r} \geq \eta \quad (162)$$

where π is the per-period profit and η was the cost of introducing the new variety.

Proof. This equation says that PDV of profits must exceed the one-off cost of introducing the new variety. □

Remark 136. *In equilibrium, this equation must bind or else the monopolist is leaving money on the table by not introducing the new variety.*

We can then compute the interest rate in equilibrium as a result of this.

Proposition 137 (Equilibrium Level of Interest Rate)

The equilibrium level of interest rate is:

$$r = \frac{\pi}{\eta}. \quad (163)$$

Proof. Re-arrange the condition for the introduction of a new variety:

$$\frac{\pi}{r} = \eta.$$

□

Remark 138. *This says that the interest rate is the flow value of profits divided by the cost of introducing a new variety.*

We now examine the Euler equation for consumption again.

Proposition 139 (Euler Equation for Consumption)

The Euler equation for consumption is:

$$\frac{\dot{C}_t}{C_t} = \frac{\pi}{\eta} - \rho \equiv g > 0 \quad (164)$$

whereby consumption grows at rate g .

Proof. Recall the Euler equation for consumption:

$$\frac{\dot{C}_t}{C_t} = r_t - \rho.$$

Then recall that in equilibrium:

$$r = \frac{\pi}{\eta}.$$

We plug this in and we are done.

□

Remark 140. *The reason that this is no longer 0 is that the new varieties introduced into the market by the intermediate good monopolist means that we will be able to produce more goods in the future. Therefore, we now have growth.*

7.3.3 Balanced Growth Path

The question we want to answer now is that does there exists a **balanced growth path equilibrium** in which C_t, Y_t and the number of intermediate goods N_t grow at the same rate g ? Answer is yes.

Theorem 141 (Existence of Balanced Growth Path in Romer Model)

There exists a **balanced growth path equilibrium** in which C_t, Y_t, N_t all grow at rate g .

Proof. Recall that we showed in the equilibrium of fixed variety that consumption is:

$$C_t = Y_t - N_t X_{it} = \bar{L} N_i \alpha^{\frac{2\alpha}{1-\alpha}} - N_i \left[\bar{L} \alpha^{\frac{2}{1-\alpha}} \right] = N_t \bar{L} \left(\alpha^{\frac{2\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}} \right)$$

Now, the resource constraint in the economy is the total amount of final goods less the amount of final goods you give up in order to produce new varieties $\dot{N}_t\eta$:

$$\begin{aligned} C_t &= N_t \bar{L} \left(\alpha^{\frac{2\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}} \right) - \dot{N}_t \eta \\ &=_{(1)} N_t \bar{L} \left(\alpha^{\frac{2\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}} \right) - g_N N_t \eta \\ &=_{(1)} N_t \left\{ \bar{L} \left(\alpha^{\frac{2\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}} \right) - g_N \eta \right\} \end{aligned}$$

where $=_{(1)}$ uses the guess that $\dot{N}_t = g_N N_t$ where g_N is the growth rate of the number of varieties N_t (assuming it does grow). Hence, we shown that C_t is a linear function of N_t , which we assumed to be growing at rate g_N . This then implies that C_t is also growing at the same rate g_N as N_t .

Now, if we look at output, which is given by:

$$Y_t = \bar{L} N_i \alpha^{\frac{2\alpha}{1-\alpha}}$$

which is also a function of N_i which is growing at g_N . Since we have Y_t as a linear function, then we have that Y_t is also growing at rate g_N . Therefore, we have shown that all the variables of interest grow at the same rate g_N , which we shall denote by $g \equiv g_N$. $\square$

We can now solve for this BGP growth rate.

Proposition 142 (Growth Rate)

The BGP growth rate is:

$$g = \frac{1-\alpha}{\alpha\eta} \bar{L} \alpha^{\frac{2}{1-\alpha}} - \rho. \quad (165)$$

Proof. First, recall that we have shown that profits in equilibrium are:

$$\pi = \frac{1-\alpha}{\alpha} \bar{L} \alpha^{\frac{2}{1-\alpha}}.$$

Now, also use the fact that the Euler equation is:

$$\frac{\dot{C}_t}{C_t} = \frac{\pi}{\eta} - \rho.$$

In the BGP, $\frac{\dot{C}_t}{C_t} = g$ so then:

$$g = \frac{\pi}{\eta} - \rho.$$

We plug in π into this and we are done:

$$g = \frac{1}{\eta} \left[\frac{1-\alpha}{\alpha} \bar{L} \alpha^{\frac{2}{1-\alpha}} \right] - \rho.$$

$\square$

We can see what parameters affect the growth of the economy. First, as the cost of introducing new varieties η decreases, the economy grows. If an economy has a higher labour supply $\bar{L}$, then the economy grows. If the economy is more patient, so ρ falls, then the economy grows.

7.3.4 Efficiency

We now end this section by looking at the efficiency in this model.

Static Efficiency

To look at static efficiency, we ask the question: given the number of varieties, is societal utility being maximised?

First, recall the decentralised quantity of each intermediate good derived earlier:

$$X_{it} = \bar{L} \alpha^{\frac{2}{1-\alpha}}$$

which we derived under a decentralised equilibrium. We now compare this to a **social planner's problem**.

Definition 143 (Social Planner's Problem)

The social planner's problem is given by:

$$\begin{aligned} \max_{Y_t, L_t, \{X_{it}\}_i} \quad & Y_t - \sum_{i=1}^{N_i} X_{it} = C_t \\ \text{s.t.} \quad & Y_t = L_t^{1-\alpha} \sum_{i=1}^{N_i} X_{it}^\alpha \\ & L_t = \bar{L} \end{aligned} \tag{166}$$

where L is inelastic supply of labour and N_i is the number of varieties at time t .

We can derive the optimal amount of intermediate goods produced X_{it} under this program.

Proposition 144 (Social Planner's Level of Intermediate Goods)

The optimal production level of intermediate goods under the social planner's problem is:

$$X_{it}^* = \bar{L} \alpha^{\frac{1}{1-\alpha}} \tag{167}$$

Proof. The first order conditions for the social planner's problem is:

$$\frac{\partial}{\partial X_{it}} : \alpha \bar{L}^{1-\alpha} X_{it}^{\alpha-1} = 1.$$

We can then solve for

$$X_{it}^* = \bar{L} \alpha^{\frac{1}{1-\alpha}}.$$

□

Comparing the two production levels, the decentralised quantity X_{it} is **less** than the planner's quantity X_{it}^* . The monopolist equates marginal revenue to marginal cost and restricts output relative to the efficient level.

We can do a similar exercise for output and consumption.

Theorem 145 (Social Planner's Output and Consumption)

Social planner's level of output and consumption is:

$$Y_t^* = \bar{L} N_i \alpha^{\frac{\alpha}{1-\alpha}} \tag{168}$$

$$C_t^* = \bar{L} N_i (\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}) \tag{169}$$

Compare this to the decentralised equilibrium:

$$Y_t = \bar{L} N_t \alpha^{\frac{2\alpha}{1-\alpha}}$$

$$C_t = Y_t - N_t X_{it}.$$

We see that the output and consumption in the decentralised equilibrium is also lower than the social planner's equilibrium. Hence, the monopolist in restricting output of intermediate goods makes society worse off.

Theorem 146 (Static Inefficiency in Expanding Varieties Model)

The decentralised equilibrium has output and consumption (X_{it}, Y_t, C_t) lower compared to the social planner's problem (X_{it}^*, Y_t^*, C_t^*) :

$$X_{it} < X_{it}^* \quad (170)$$

$$Y_t < Y_t^* \quad (171)$$

$$C_t < C_t^* \quad (172)$$

Dynamic Efficiency

To look at dynamic efficiency, we ask the question: is the speed of new varieties being created sufficiently high? To do so, we again construct the social planner's problem.

Definition 147 (Social Planner's Problem)

The social planner's problem is:

$$\begin{aligned} \max_{\{C_t, N_t\}} \quad & \int_0^\infty e^{-\rho t} \log(C_t) dt \\ \text{s.t.} \quad & \tilde{C}_t = C_t + \eta \dot{N}_t \\ & \tilde{C}_t = \bar{L} N_t \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \right) \end{aligned} \quad (173)$$

where $\tilde{C}_t$ is the static efficient level of consumption.

Now, if we solve this, we get an Euler equation for consumption.

Proposition 148 (Social Planner's Euler Equation for Consumption)

The social planner's Euler equation for consumption is:

$$\frac{\dot{C}_t}{C_t} = \frac{\bar{L} \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \right)}{\eta} - \rho. \quad (174)$$

Proof. Write out the Hamiltonian:

$$\mathcal{H} = \log(C_t) e^{-\rho t} + \lambda_t \left[\frac{\bar{L} N_t \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \right)}{\eta} - C_t \right].$$

Take first order conditions and solve. □

We now compare the Euler equation for consumption under social planner:

$$\frac{\dot{C}_t}{C_t} = \frac{\bar{L} \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \right)}{\eta} - \rho$$

to the Euler equation for consumption under decentralisation:

$$\frac{\dot{c}_t}{c_t} = \frac{1-\alpha}{\alpha\eta} \bar{L} \alpha^{\frac{2}{1-\alpha}} - \rho.$$

Proposition 149 (Dynamic Inefficiency in Expanding Varieties Model)

The consumption growth rate under the social planner is greater than the consumption growth rate under the decentralised equilibrium:

$$\frac{\dot{C}_t}{C_t} > \frac{\dot{c}_t}{c_t}. \quad (175)$$

Therefore, the decentralised distortions make consumption growth too low and reduce welfare.

Hence, we can summarise by saying that the monopoly power reduces both the quantity of intermediate goods produced and the growth in the variety of intermediate goods.

8 Real Business Cycle Model

8.1 Introduction

The real business cycle (RBC) model builds on the Ramsey model. Under perfectly competitive markets, the welfare theorems allow us to study the competitive allocation using either a decentralised formulation or a social planner's problem. One key difference is that, in the RBC model, the household chooses how much labour to supply in every period rather than taking labour supply as exogenous. The household therefore decides how to allocate time between work and leisure each period, in addition to choosing consumption over time.

8.2 Household Problem

We assume that there is a representative household that chooses between consumption c_t and labour l_t .

Definition 150 (Household Intratemporal Problem)

The household has the intratemporal problem of:

$$\max_{c_t, l_t} U(c_t, l_t) = \max_{c_t, l_t} \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \chi l_t \right] \quad (176)$$

where l_t is labour and χ captures the disutility of labour.

We now derive one of the key intuitions for the household. Suppose that we are looking at an indifference curve:

$$dU = 0$$

and therefore, the household is trading off between consumption and leisure as:

$$dU = U_c dc + U_l dl = 0.$$

We can then solve to get the **marginal rate of substitution**:

$$\frac{dc}{dl} = -\frac{U_l}{U_c}.$$

Looking at the other side of the problem, if the household chooses to work one extra hour dl , they will earn a wage w . With this wage w , they can increase their consumption dc . Therefore, we have that the **marginal rate of transformation** is:

$$\frac{dc}{dl} = w.$$

We can now equate the marginal rate of substitution to the marginal rate of transformation to derive the **intratemporal optimality condition**:

$$-\frac{U_l}{U_c} = w.$$

To graphically visualise this, we go from looking at labour l to looking at leisure $1 - l$. We show in Fig. 24 that the agent will choose the leisure-consumption pair at the point where $MRS = MRT$. Now, suppose that the wage rate w is increased to w^* . Then, the MRT rotates around the fixed number of hours (24) and the slope increases to w^* . The price of leisure has now gone up and hence we have a **substitution effect** away from leisure and towards working so that they can increase their consumption. Meanwhile, the agent is getting paid more for each hour they are currently

working so they are richer, as a result, they want to consume more and take more leisure, which is the **income effect**. However, in our models, it will be the case that the substitution effect dominates and hence the agent's leisure will fall as the wage rate increases so that they can work more.

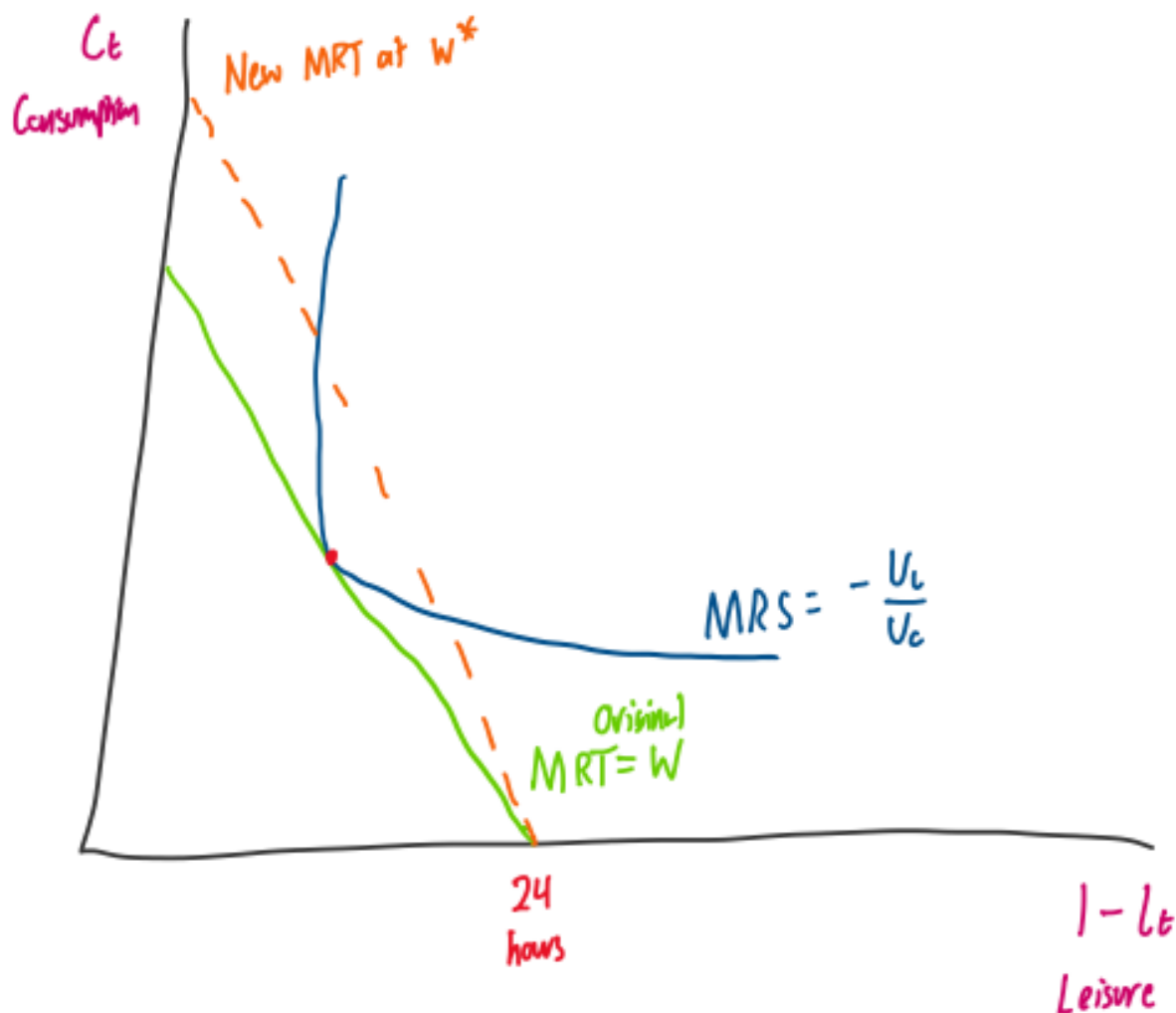


Figure 24: Consumption vs leisure. We see that the MRT rotates around the intersection of the x-axis when the wage is increased to w^* .

We now set up the full household optimisation problem.

Definition 151 (Household Optimisation Problem)

The optimisation problem facing the household is:

$$\begin{aligned} \max_{\{c_t, l_t, a_{t+1}\}} \quad & \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \chi l_t \right] \\ \text{s.t.} \quad & a_{t+1} = R_t a_t + w_t l_t - c_t \\ & \lim_{T \rightarrow \infty} \frac{a_{T+1}}{\prod_{s=0}^T R_s} \geq 0 \end{aligned} \quad (177)$$

where l_t is the time spent working, a_t is the amount of assets to hold at time t , and R_t is the return on assets.

Proposition 152 (Household Euler Equations)

The household's Euler equations are:

$$\beta R_{t+1} c_{t+1}^{-\sigma} = c_t^{-\sigma} \quad (178)$$

$$w_t c_t^{-\sigma} = \chi \quad (179)$$

Proof. The Lagrangian is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \chi l_t \right] + \lambda_t \left[R_t a_t + w_t l_t - c_t - a_{t+1} \right] \right\}$$

Take first order conditions with respect to c_t, a_{t+1}, l_t to get:

$$\beta^t c_t^{-\sigma} - \beta^t \lambda_t = 0$$

$$\beta^{t+1} \lambda_{t+1} R_{t+1} - \beta^t \lambda_t = 0$$

$$-\beta^t \chi + \beta^t w_t \lambda_t = 0.$$

Plug the first FOC into the other 2 equations to get the Euler equation. □

The first Euler equation is the **intertemporal Euler equation** (consumption today vs consumption tomorrow) whilst the second Euler equation is the **intratemporal Euler equation** (marginal disutility of labour vs marginal utility of consumption in a given period).

We can derive the **Euler equation behavior of labour supply** by combining the two equations:

$$\beta R_{t+1} \frac{\chi}{w_{t+1}} = \frac{\chi}{w_t}$$

to get:

$$\beta R_{t+1} \frac{w_t}{w_{t+1}} = 1 \quad (180)$$

which tells us that the determinants of changes in the marginal utility of leisure are the interest rate R_{t+1} and the ratio of future/current wages w_t/w_{t+1} .

8.3 Firm Problem

The firm problem is a standard firm problem. The Representative firm hires labour and rents capital in competitive markets to produce output Y_t .

Definition 153 (Firm Optimisation Problem)

The optimisation problem facing the firm is:

$$\begin{aligned} \max_{k_t, l_t} \quad & y_t - R_t^k k_t - w_t l_t \\ \text{s.t.} \quad & y_t = A_t k_t^\alpha l_t^{1-\alpha} \end{aligned} \quad (181)$$

for $0 < \alpha < 1$.

The first order conditions are standard assuming depreciation of capital δ :

Theorem 154 (First Order Conditions for Firm)

The first-order conditions for the firm are:

$$R_t^k = \alpha A_t \left(\frac{l_t}{k_t} \right)^{1-\alpha} \quad (182)$$

$$w_t = (1 - \alpha) A_t \left(\frac{k_t}{l_t} \right)^\alpha \quad (183)$$

where factor prices equal the marginal products of capital and labour. The household's gross return on capital is $R_t = R_t^k + 1 - \delta$.

We close the model by specifying technological change and capital accumulation.

Definition 155 (Technological Change)

Technological change is given by an AR(1) process:

$$\log(A_{t+1}) = \rho \log(A_t) + \epsilon_t, \quad |\rho| < 1. \quad (184)$$

Proposition 156 (Capital Accumulation)

$$k_{t+1} = (1 - \delta)k_t + y_t - c_t \quad (185)$$

8.4 Equilibrium

We state the equilibrium of the model.

Theorem 157 (Equilibrium in RBC Model)

In equilibrium, the following 7 equations hold:

$$\beta R_{t+1} c_{t+1}^{-\sigma} = c_t^{-\sigma} \quad (\text{Intertemporal Equation})$$

$$w_t c_t^{-\sigma} = \chi \quad (\text{Intratemporal Equation})$$

$$y_t = A_t k_t^\alpha l_t^{1-\alpha} \quad (\text{Production Function})$$

$$R_t = \alpha A_t \left(\frac{l_t}{k_t} \right)^{1-\alpha} + (1 - \delta) \quad (\text{MPK})$$

$$w_t = (1 - \alpha) A_t \left(\frac{k_t}{l_t} \right)^\alpha \quad (\text{MPL})$$

$$\log(A_{t+1}) = \rho \log(A_t) + \epsilon_t \quad (\text{Technological Evolution})$$

$$k_{t+1} = (1 - \delta) k_t + y_t - c_t \quad (\text{Capital Accumulation})$$

whereby we have the seven variables $c_t, k_t, l_t, y_t, R_t, w_t, A_t$.

With this equilibrium model, we can then log-linearise this model.

8.5 Blanchard-Kahn Conditions

In the RBC model, after log-linearising, we usually obtain an expression of the form

$$\mathbb{E}_t[AX_{t+1}] = BX_t \quad (186)$$

What this says is that our future expectations of our variable X_{t+1} , times a coefficient matrix A , depends on our current variables X_t times a coefficient matrix B . We want to be able to solve this system of equations for X_t .

We need to assume that the matrix B is invertible, so that we may multiply by the inverse B^{-1} on both sides

$$\mathbb{E}_t[B^{-1}AX_{t+1}] = X_t$$

We can then redefine the matrix $C = B^{-1}A$ to arrive at

$$\mathbb{E}_t[CX_{t+1}] = X_t \quad (187)$$

Now, what generally happens in the RBC model is that we can divide the matrix X_t up into the set of state and control variables. For example, in the RBC model, the state variables was the productivity shock a_t and current level of capital k_t , whilst the control variable was consumption c_t . Mathematically, we can partition X_t into the set of state variables $w_t = (a_t, k_t)$ and control variables $z_t = (c_t)$ as follows:

$$X_t = \begin{bmatrix} w_t \\ z_t \end{bmatrix} = \begin{bmatrix} a_t \\ k_t \\ c_t \end{bmatrix}$$

Going back to Eq. (186), we can therefore rewrite this system as

$$\mathbb{E}_t \left[C \begin{bmatrix} w_{t+1} \\ z_{t+1} \end{bmatrix} \right] = \begin{bmatrix} w_t \\ z_t \end{bmatrix} \quad (188)$$

Hopefully, you may recall from the Maths crash course that a $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors. Obviously, many matrices do not satisfy this property. To get around this, we can always put a matrix into a **Jordan normal form** which is quite similar.

Theorem 158 (Jordan Decomposition)

A matrix C can be decomposed into its Jordan normal form

$$C = P^{-1}\Lambda P \quad (189)$$

whereby Λ is block diagonal, with Jordan blocks and the eigenvalues of C on its diagonal, whilst the rows of P are the corresponding left generalised eigenvectors.

Remark 159. Note that what we have defined here coincides with ordinary diagonalisation of matrices! It will take a while to explain how Jordan decomposition differs to diagonalisation, but for now, just think of them as the same thing except that you can always Jordan decompose any matrix.

We now have a matrix Λ which contains the associated eigenvalues to the matrix C . We can then state the Blanchard-Kahn conditions.

Theorem 160 (Blanchard-Kahn Conditions)

For the representation $\mathbb{E}_t[CX_{t+1}] = X_t$, the Blanchard-Kahn conditions require the number of roots of C inside the unit circle to equal the number of forward-looking controls. These roots correspond to explosive forward modes because the transition law uses C^{-1} .

Going back to our problem at hand, we had one control variable c_t . Therefore, based on the Blanchard-Kahn conditions, we require that the matrix Λ contain one entry along its diagonal that is less than one (which corresponds to an unstable eigenvalue).

We can take our matrix of eigenvalues, and partition it into block matrices Λ_1 and Λ_2 whereby Λ_1 contains the **stable eigenvalues** and Λ_2 contains the **unstable eigenvalues**

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}$$

and we can do the same for the matrix of eigenvectors P

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

We can plug these new expressions into our system Eq. (188) to get

$$\begin{aligned}\mathbb{E}_t \left[P^{-1} \Lambda P \begin{bmatrix} w_{t+1} \\ z_{t+1} \end{bmatrix} \right] &= \begin{bmatrix} w_t \\ z_t \end{bmatrix} \\ \Leftrightarrow \mathbb{E}_t \left[\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}^{-1} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w_{t+1} \\ z_{t+1} \end{bmatrix} \right] &= \begin{bmatrix} w_t \\ z_t \end{bmatrix} \\ \Leftrightarrow \mathbb{E}_t \left[\begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w_{t+1} \\ z_{t+1} \end{bmatrix} \right] &= \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w_t \\ z_t \end{bmatrix}\end{aligned}$$

We can then redefine the new variables

$$\begin{bmatrix} \tilde{w}_t \\ \tilde{z}_t \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w_t \\ z_t \end{bmatrix}$$

Note that these new variables are linear combinations of our existing variables

$$\begin{cases} \tilde{w}_t = P_{11}w_t + P_{12}z_t \\ \tilde{z}_t = P_{21}w_t + P_{22}z_t \end{cases}$$

These new variables $\tilde{w}_t$ and $\tilde{z}_t$ are our transformed variables for our state and control variables respectively. We can now define our new linear system:

Definition 161 (Jordan Decomposition of Linear System)

Our new linear system after Jordan decomposition is

$$\mathbb{E}_t \left[\begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} \tilde{w}_{t+1} \\ \tilde{z}_{t+1} \end{bmatrix} \right] = \begin{bmatrix} \tilde{w}_t \\ \tilde{z}_t \end{bmatrix} \quad (190)$$

Now, if you look very carefully, what is interesting is that we now have a decoupled system! If we expand out the matrix multiplication, we get a system of linear equations

$$\begin{cases} \mathbb{E}_t [\Lambda_1 \tilde{w}_{t+1}] = \tilde{w}_t \\ \mathbb{E}_t [\Lambda_2 \tilde{z}_{t+1}] = \tilde{z}_t \end{cases}$$

We now look at the expression whereby $\tilde{z}_t$ contains our transformed control variables:

$$\mathbb{E}_t [\Lambda_2 \tilde{z}_{t+1}] = \tilde{z}_t$$

We can invert this and iterate forward to get

$$\mathbb{E}_t [\tilde{z}_{t+j}] = (\Lambda_2^{-1})^j \tilde{z}_t$$

Remember that we said earlier that Λ_2 contains unstable eigenvalues (where modulus is less than 1), this means that this system will explode. Therefore, the only way that we have our system not blowing up is if $\tilde{z}_t = 0$ for all time.

Therefore, we have that

$$\mathbb{E}_t \tilde{z}_{t+j} = 0$$

So then, going back to our earlier equation

$$\tilde{z}_t = P_{21}w_t + P_{22}z_t$$

we have that since $\tilde{z}_t = 0$,

$$0 = P_{21}w_t + P_{22}z_t$$

and solving for z_t , we find that our control vector is

$$z_t = -P_{22}^{-1}P_{21}w_t$$

What we have done is solved for our control variable (in this case c_t) as a linear function of our state variables!!!

We can repeat a similar process to solve for the state variables w_t by analysing $\tilde{w}_t$.

8.6 Solving a First Difference Equation

Many times in working with RBC models, we usually arrive at a first order linear difference equation:

$$x_t = ax_{t-1} + b \quad (191)$$

where a and b are coefficients. Through back substitution, we obtain

$$x_t = a^t x_0 + b \frac{1 - a^t}{1 - a}, \quad a \neq 1.$$

Now, if we have that $|a| < 1$, then our difference equation is **stable**. Then, as $t \rightarrow \infty$, we clearly see that:

$$x^* = \frac{b}{1 - a}. \quad (192)$$

Example 162

Suppose that we had the first difference equation (problem set 4 question 2):

$$v_{t+1} = \frac{1}{\beta(\alpha + \gamma)} v_t - \frac{1}{\alpha + \gamma}.$$

Then, $a = \frac{1}{\beta(\alpha + \gamma)}$ and $b = -\frac{1}{\alpha + \gamma}$. Then, assuming that the system is stable, we can immediately deduce that the solution to this equation is:

$$v^* = \frac{b}{1 - a} = \frac{-\frac{1}{\alpha + \gamma}}{1 - \frac{1}{\beta(\alpha + \gamma)}} = \frac{\beta}{1 - \beta(\alpha + \gamma)}.$$

We have now found a solution!

8.7 Impulse Response Function

We now see what happens to the model as we apply a shock to the system. We assume that the model is currently in steady-state and then see the effects of a shock to the system. We show the results in Fig. 25.

First, to interpret this, recall the equilibrium conditions.

Theorem 163 (Equilibrium in RBC Model)

In equilibrium, the following 7 equations hold:

$$\beta R_{t+1} c_{t+1}^{-\sigma} = c_t^{-\sigma} \quad (\text{Intertemporal Equation})$$

$$w_t c_t^{-\sigma} = \chi \quad (\text{Intratemporal Equation})$$

$$y_t = A_t k_t^\alpha l_t^{1-\alpha} \quad (\text{Production Function})$$

$$R_t = \alpha A_t \left(\frac{l_t}{k_t} \right)^{1-\alpha} + (1 - \delta) \quad (\text{MPK})$$

$$w_t = (1 - \alpha) A_t \left(\frac{k_t}{l_t} \right)^\alpha \quad (\text{MPL})$$

$$\log(A_{t+1}) = \rho \log(A_t) + \epsilon_t \quad (\text{Technological Evolution})$$

$$k_{t+1} = (1 - \delta)k_t + y_t - c_t \quad (\text{Capital Accumulation})$$

whereby we have the seven variables $c_t, k_t, l_t, y_t, R_t, w_t, A_t$.

We explain what is happening in the figures in the following points.

1. We apply a unit shock ϵ_t to the evolution of technology $\ln(A_{t+1}) = \rho \ln(A_t) + \epsilon_t$. The deviation from steady state jumps to 1%. Because $|\rho| < 1$, the effect of the shock eventually dies out.
2. The marginal product of labour has increased as technology has increased:

$$w_t = (1 - \alpha) A_t \left(\frac{k_t}{l_t} \right)^\alpha.$$

This implies that the wage rate increase. Hence, we see a jump in the wage rate.

3. From what we discussed before, due to the increase in the wage rate, we see a substitution effect away from leisure and towards working, so that households can consume more, which is reflected in the intratemporal Euler equation:

$$w_t c_t^{-\sigma} = \chi.$$

This then increases the overall supply of labour. Hence, the amount of hours worked n has increased.

4. As technology has improved temporarily and the labour supply has increased, then output will increase too:

$$y_t = A_t k_t^\alpha l_t^{1-\alpha}.$$

We therefore see an increase in output.

5. With this new output, we want to smooth the consumption of this output. Therefore, we will not consume all of the extra output and instead smooth our consumption.
6. As a result of smoothing our consumption, we invest the remaining unconsumed output into capital k :

$$k_{t+1} = (1 - \delta)k_t + y_t - c_t.$$

Hence, the stock of capital increases.

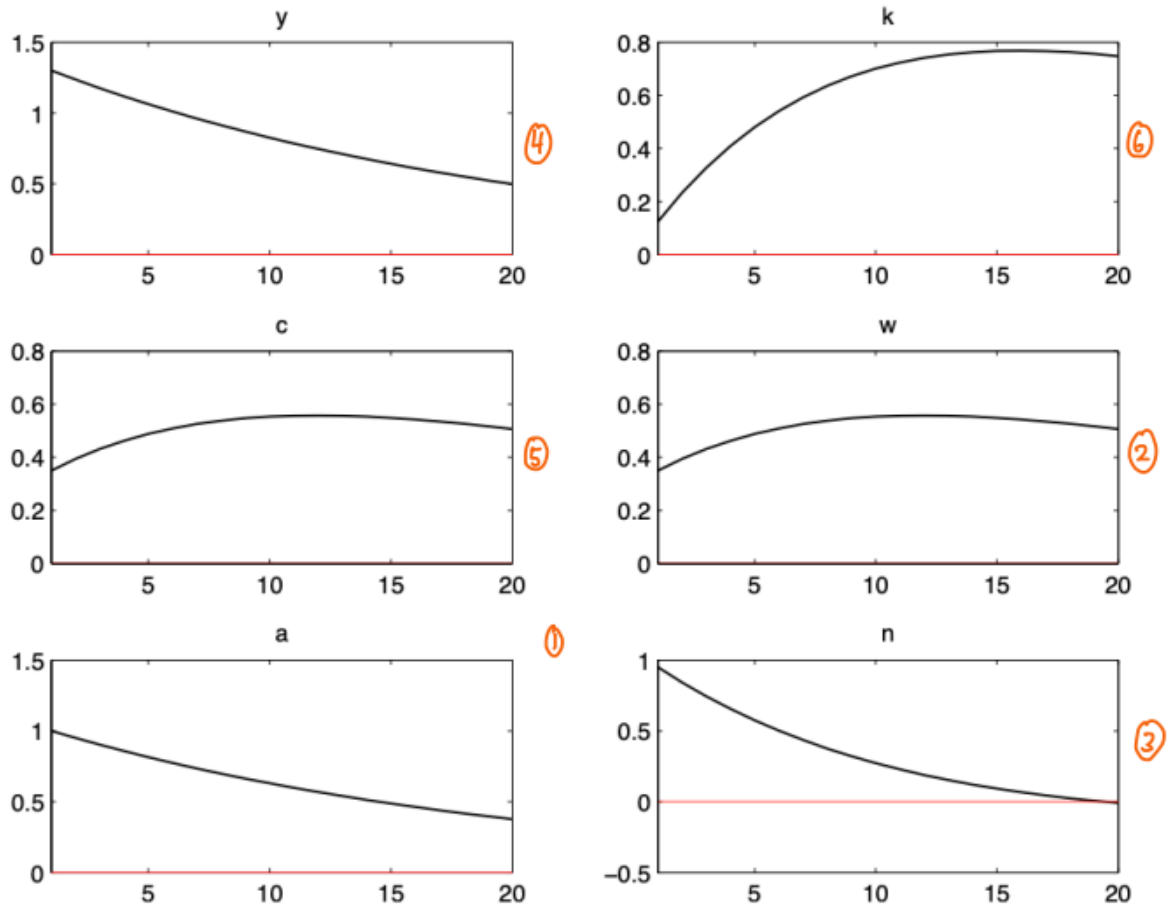


Figure 25: Impulse response function to RBC model.

8.8 Empirical Results of RBC Model

Looking at the results from the model and comparing it to empirical data, we have that:

1. The RBC model with productivity shocks applied is able to generate significant volatility in output, but not as much as is observed in the data.
2. The model predicts only about half of the volatility in labour input observed in the data.
3. Hours and productivity move together in the simulations but not in the data. One difficulty is that individual labour supply is fairly inelastic in practice.
4. Consumption is smoother in the simulations, and much smoother than what is observed in the data. Here, we assume no financial frictions so households are able to borrow freely and smooth consumption.
5. The relative volatility of output and investment is approximately correct.

In other words, the model seems to do reasonably well explaining investment, has a mixed record with consumption and performs poorly with respect to the labour market. Notice the model implies that all variables tend to move together. Given that the model assumes the existence of only one shock this is fairly inevitable.

We now turn to the amplification and propagation mechanisms of the RBC model. Amplification is a model's ability to make output respond by more than the exogenous shock—to turn small shocks into large fluctuations. In the basic RBC model, the main amplification mechanism is the labour-supply response as households substitute between work and leisure. This mechanism is fairly weak. Propagation is a model's ability to make shocks have persistent effects.

The main propagation mechanism in the RBC model is capital accumulation, but it is weak as well. Hence, the time-series properties of output end up looking much like those of technology: output is about as variable and persistent as the technology shock.

Therefore, we have that currently, the RBC model is unable to explain internal propagation that well and it has little amplification. Furthermore, the RBC doesn't explain prices well since wages are quite volatile and asset prices are quite smooth.

Definition 164 (Persistence Problem)

The RBC model is unable to generate persistent shocks from transitory technology shocks.

Definition 165 (Amplification Problem)

The RBC model is unable to create large fluctuations from small shocks to the model.

8.9 Reformulated RBC

We reformulate the model using the notation employed later in the course.

8.9.1 Decentralised Problem

There is a representative household that discounts the future by $\beta < 1$. Every period, the household decides how much labour to supply, N_t , and how much to consume, C_t . We normalise the time endowment to 1, so leisure is $1 - N_t$. We assume that consumption and leisure are separable, so flow utility is

$$u(C_t) + v(1 - N_t).$$

In every period, the household earns wage income $w_t N_t$. Bonds purchased in the previous period, B_{t-1} , pay the gross return $(1 + i_{t-1})$ at time t , and the household chooses new bond holdings B_t . A positive B_t denotes saving and a negative B_t denotes borrowing. The household also rents its capital stock to the firm and earns R_t^K per unit. It chooses consumption C_t and investment I_t , and receives firm profits Π_t . Hence, the household budget constraint is

$$C_t + I_t + B_t = (1 + i_{t-1})B_{t-1} + R_t^K K_{t-1} + w_t N_t + \Pi_t$$

and the evolution of capital will be:

$$K_t = (1 - \delta)K_{t-1} + I_t$$

where δ is the depreciation rate. We can now state the optimisation problem for the household.

Definition 166 (Household Optimisation Problem)

The optimisation problem facing the household is:

$$\begin{aligned} \max_{C_t, N_t, B_t, I_t, K_t} \quad & \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(u(C_t) + v(1 - N_t) \right) \right] \\ \text{s.t.} \quad & C_t + I_t + B_t = (1 + i_{t-1})B_{t-1} + R_t^K K_{t-1} + w_t N_t + \Pi_t \\ & K_t = (1 - \delta)K_{t-1} + I_t \end{aligned} \quad (193)$$

To make life easier, we can combine the budget constraints and get rid of the decision variable for investment to have the problem:

$$\begin{aligned} \max_{C_t, N_t, B_t, K_t} \quad & \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(u(C_t) + v(1 - N_t) \right) \right] \\ \text{s.t.} \quad & C_t + K_t + B_t = (1 + i_{t-1})B_{t-1} + [R_t^K + (1 - \delta)]K_{t-1} + w_t N_t + \Pi_t \end{aligned} \quad (194)$$

We can then form the Lagrangian:

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\left(u(C_t) + v(1 - N_t) \right) + \lambda_t \left((1 + i_{t-1})B_{t-1} + [R_t^K + (1 - \delta)]K_{t-1} + w_t N_t + \Pi_t - C_t - K_t - B_t \right) \right] \quad (195)$$

The FOCs are:

Theorem 167 (First Order Conditions for Household)

$$\frac{\partial \mathcal{L}}{\partial C_t} : \quad u'(C_t) = \lambda_t \quad (196)$$

$$\frac{\partial \mathcal{L}}{\partial N_t} : \quad v'(1 - N_t) = \lambda_t w_t \quad (197)$$

$$\frac{\partial \mathcal{L}}{\partial B_t} : \quad \beta \mathbb{E}_t [\lambda_{t+1} (1 + i_t)] = \lambda_t \quad (198)$$

$$\frac{\partial \mathcal{L}}{\partial K_t} : \quad \beta \mathbb{E}_t [\lambda_{t+1} [R_{t+1}^K + (1 - \delta)]] = \lambda_t \quad (199)$$

We can combine the FOCs together.

Theorem 168 (Optimality Conditions for Household)

$$v'(1 - N_t) = w_t u'(C_t) \quad (200)$$

$$\beta \mathbb{E}_t [u'(C_{t+1}) (1 + i_t)] = u'(C_t) \quad (201)$$

$$\beta \mathbb{E}_t [u'(C_{t+1}) [R_{t+1}^K + (1 - \delta)]] = u'(C_t) \quad (202)$$

The first condition equates the marginal rate of substitution between leisure and consumption to the relative price of leisure, w_t . The second describes the intertemporal trade-off for a bond. The final equation gives the corresponding trade-off for investment in capital.

If we look at the third equation:

$$u'(C_t) = \beta \mathbb{E}_t [u'(C_{t+1})[R_{t+1}^K + (1 - \delta)]]$$

We can apply $\mathbb{E}[X \cdot Y] = \mathbb{E}[X]\mathbb{E}[Y] + \text{Cov}[X, Y]$ to get:

$$\mathbb{E} [u'(C_{t+1}) \cdot [R_{t+1}^K + (1 - \delta)]] = \mathbb{E} [u'(C_{t+1})] \cdot \mathbb{E} [R_{t+1}^K + (1 - \delta)] + \text{Cov}[u'(C_{t+1}), R_{t+1}^K]$$

R_{t+1}^K is the marginal product of capital, as we will see later. The covariance $\text{Cov}[u'(C_{t+1}), R_{t+1}^K]$ is generally negative: when productivity, the return to capital, and consumption are high, marginal utility is low. An asset that pays poorly when marginal utility is high must offer a higher expected return than a safe asset. This covariance term therefore helps determine the **equity premium**; it is not itself the premium.

8.9.2 Firm Problem

The representative firm hires labour and rents capital in competitive markets to produce output Y_t .

Definition 169 (Firm Optimisation Problem)

The optimisation problem facing the firm is:

$$\begin{aligned} \max_{K_{t-1}, N_t} \quad & Y_t - R_t^K K_{t-1} - w_t N_t \\ \text{s.t.} \quad & Y_t = A_t F(K_{t-1}, N_t) \end{aligned} \tag{203}$$

We can construct the **Lagrangian**:

$$\mathcal{L} = A_t F(K_{t-1}, N_t) - R_t^K K_{t-1} - w_t N_t. \tag{204}$$

Finding the first order equations is straightforward.

Theorem 170 (First Order Conditions for Firm)

The first-order conditions for the firm are:

$$\frac{\partial \mathcal{L}}{\partial K_{t-1}} : R_t^K = A_t F_K(K_{t-1}, N_t) \tag{205}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} : w_t = A_t F_N(K_{t-1}, N_t) \tag{206}$$

8.9.3 Closing the Model

We also need to specify how technology A_t evolves. We assume that log technology follows an AR(1) process:

$$\ln(A_t) = \rho_A \ln(A_{t-1}) + \epsilon_t, \quad |\rho_A| < 1. \tag{207}$$

We now consolidate the constraints for both the household and the firm:

$$C_t + I_t + B_t = (1 + i_{t-1})B_{t-1} + R_t^K K_{t-1} + w_t N_t + \Pi_t$$

$$\pi_t = Y_t - R_t^K K_{t-1} - w_t N_t$$

Combining these two equations, we get:

$$C_t + I_t + B_t = (1 + i_{t-1})B_{t-1} + R_t^K K_{t-1} + w_t N_t + Y_t - R_t^K K_{t-1} - w_t N_t$$

We now impose an assumption in equilibrium.

Assumption (Net Supply of Bonds)

The net supply of bonds is zero in equilibrium:

$$B_t = B_{t-1} = 0.$$

This then leads us to:

$$\begin{aligned} \Rightarrow C_t + I_t &= R_t^K K_{t-1} + w_t N_t + Y_t - R_t^K K_{t-1} - w_t N_t \\ \Leftrightarrow C_t + I_t &= Y_t \\ \Leftrightarrow C_t + I_t &= A_t F(K_{t-1}, N_t) \end{aligned}$$

Therefore, we have found another important equation that holds whereby all output is either consumed or invested:

$$A_t F(K_{t-1}, N_t) = C_t + I_t. \quad (208)$$

8.9.4 Equilibrium

We can now state the equilibrium of this model.

Theorem 171 (Equilibrium in RBC Model)

In equilibrium, the following 9 equations hold:

$$\begin{aligned} \beta \mathbb{E}_t [u'(C_{t+1})(A_{t+1} F_K(K_t, N_{t+1}) + 1 - \delta)] &= u'(C_t) && \text{(Capital Market Equilibrium)} \\ \beta \mathbb{E}_t [u'(C_{t+1})(1 + i_t)] &= u'(C_t) && \text{(Bond Market Equilibrium)} \\ v'(1 - N_t) &= u'(C_t) A_t F_N(K_{t-1}, N_t) && \text{(Labour Market Equilibrium)} \\ Y_t &= C_t + I_t && \text{(Investment)} \\ \ln(A_t) &= \rho_A \ln(A_{t-1}) + \epsilon_t && \text{(Evolution of Technology)} \\ w_t &= A_t F_N(K_{t-1}, N_t) && \text{(Factor Marginal Product)} \\ R_t^K &= A_t F_K(K_{t-1}, N_t) && \text{(Factor Marginal Product)} \\ Y_t &= A_t F(K_{t-1}, N_t) && \text{(Output)} \\ K_t &= (1 - \delta) K_{t-1} + A_t F(K_{t-1}, N_t) - C_t && \text{(Capital Accumulation)} \end{aligned}$$

whereby we have the 9 variables $C_t, K_{t-1}, N_t, A_t, w_t, R_t^K, i_t, Y_t, I_t$.

9 Real Business Cycle Model With Labour

9.1 Introduction

We have seen that we had a strong labour supply response in the RBC model due to the intratemporal Euler equation and this was quite important in affecting other parts of the model. In order for us to have this effect, we also needed an elastic labour supply in the model. However, in the data, labour supply is quite inelastic in practice.

In the RBC model, we assume that the disutility of labour is linear, which therefore implies that the marginal disutility of labour is constant. In the data, this is not true either. The individual's labour supply is quite inelastic as people generally want to work 40 hours a week.

We now state the big issue with trying to model labour supply. **Individual labour supply is inelastic whilst aggregate labour supply is elastic.** This poses an issue, as the RBC model requires elastic labour supply to work as we have previously seen. To reconcile this with the model, we need to argue either that the volatility of wages is underestimated or that the elasticity of labour supply is underestimated.

We now look at three different modelling techniques we can use to reconcile these ideas.

9.2 Lotteries in Labour Market - Hansen (1985)

We try to reconcile the inelastic individual labour supply with an elastic aggregate labour supply.

Definition 172 (Individual Utility Function)

The utility function for an individual is:

$$U(c_t, n_t) = \log(c_t) + \alpha \log(1 - n_t) \quad (209)$$

where c_t is consumption and $1 - n_t$ is leisure.

We now need some assumptions on the amount of labour a household will supply. We shall assume that there is a fixed cost to working so households will either work h hours or not at all.

Assumption (Lottery Labour Supply)

The number of hours that a household supplies is a lottery:

$$n_t = \begin{cases} h & \text{w.p. } \pi_t \\ 0 & \text{w.p. } 1 - \pi_t \end{cases}$$

The result of this assumption is that all fluctuations arise due to the extensive margin whereby in boom times, it is not the case that people are working overtime but rather firms are hiring more workers.

Now, we assume that households are perfectly insured so that if a household does not work, they will still be able to consume the same amount c_t as a household that works.

Proposition 173 (Expected Utility for Individual Household)

The expected utility for an individual household is:

$$U(c_t, n_t) = \log(c_t) + \alpha \pi_t \log(1 - h). \quad (210)$$

Proof. Take the household's utility function:

$$U(c_t, n_t) = \log(c_t) + \alpha \log(1 - n_t).$$

Now, with probability π_t , the household will consume c_t and work h so their leisure is $(1 - h)$. With probability $(1 - \pi_t)$, they will consume c_t and work $n_t = 0$, so that their leisure is 1. We therefore have that the expected utility is:

$$\begin{aligned} U(c_t, n_t) &= \pi_t [\log(c_t) + \alpha \log(1 - h)] + (1 - \pi_t) [\log(c_t) + \alpha \log(1)] \\ &= \log(c_t) + \alpha \pi_t \log(1 - h). \end{aligned}$$

□

Proposition 174 (Hours Worked in Economy)

Hours worked in the economy are given by the probability of someone working times the amount they work:

$$n_t = \pi_t h. \quad (211)$$

As our utility function is of Gorman-Polar form (log-utility), we can simply aggregate the individual utility function to get the aggregate utility function.

Proposition 175 (Utility Function for Aggregate Economy)

The utility function for the (aggregate) representative household is:

$$U(c_t, n_t) = \log(c_t) + \frac{\alpha \log(1 - h)}{h} n_t. \quad (212)$$

Proof. First, rewrite the number of hours worked in the economy:

$$n_t = \pi_t h \quad \leftrightarrow \quad \pi_t = \frac{n_t}{h}.$$

Then, plug this into the individual expected utility:

$$\begin{aligned} U(c_t, n_t) &= \log(c_t) + \alpha \pi_t \log(1 - h) \\ &= \log(c_t) + \alpha \frac{n_t}{h} \log(1 - h) \end{aligned}$$

□

We have achieved our goal! **We now have that the individual utility function is a log-utility function whilst the aggregate utility function is linear in hours worked.** This means that the individual utility function will be **concave** whilst the aggregate utility function will be **linear in hours**. As a result, due to diminishing returns from concavity, there will be less substitution between consumption and leisure, so individuals will have a low labour-supply elasticity. Meanwhile, because utility is linear in aggregate hours, aggregate labour supply is highly elastic.

We can fit the model and compare it to the data. We find the following findings.

1. Hours worked are almost as volatile as output.
2. All fluctuations in employment are at the extensive margin. The extensive margin refers to how many people are joining or leaving employment, whilst the intensive margin refers to how many hours each individual works. We

want fluctuations in employment to occur at the extensive margin, as individuals do not change their working hours (the intensive margin).

9.3 Efficiency Wages

This modelling technique tries to use moral hazard considerations in order to generate rigidities in real wages. **The key idea is that labour productivity is related to the wage that is paid. Therefore, firms want to pay workers an optimal amount of wage in order to maximise productivity per unit wage.**

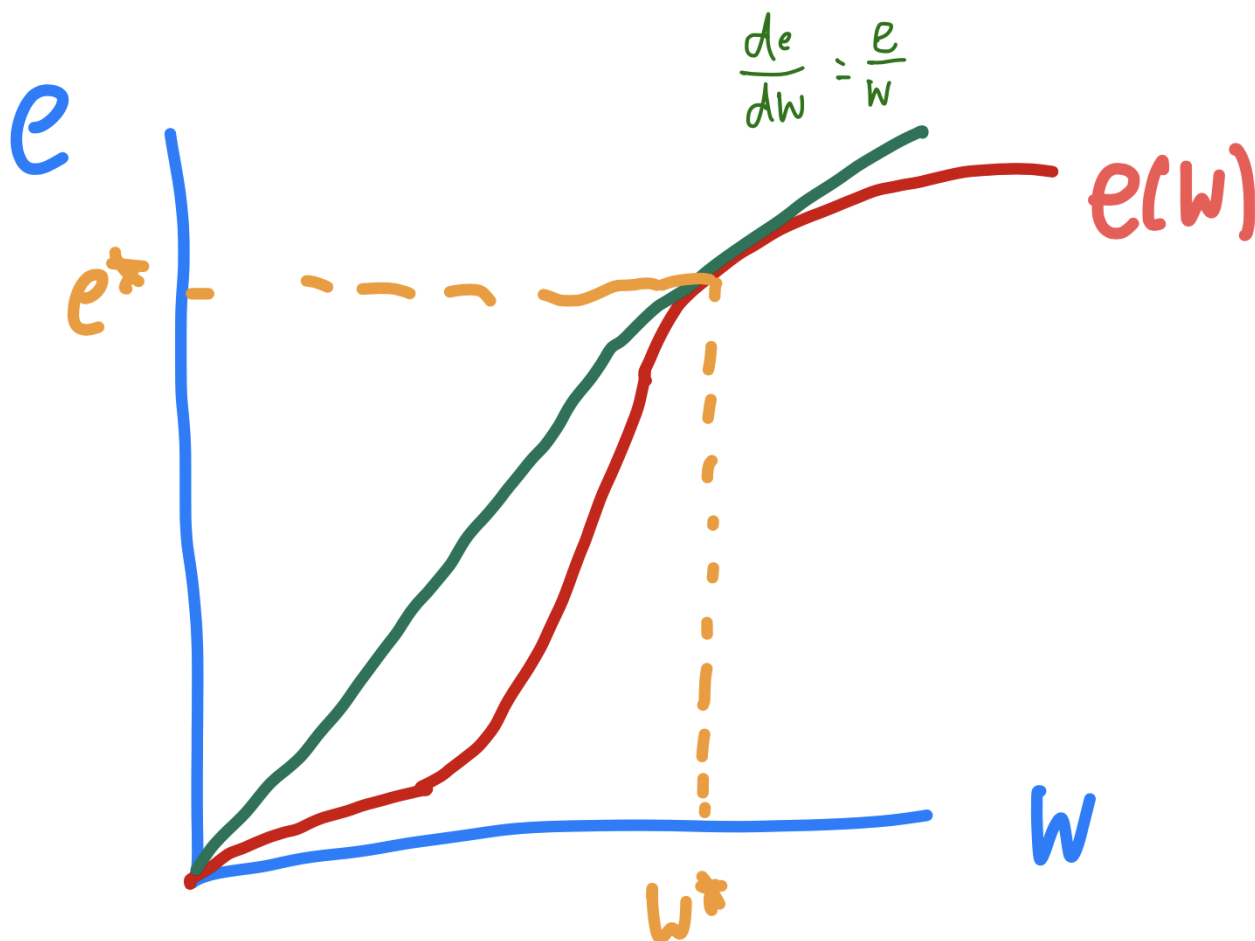


Figure 26: Effort $e(w)$ as a function of wage w . The efficiency wage w^* is the optimal wage to pay.

Graphically, suppose we plot out the amount of effort $e(w)$ that a worker puts in as a function of their wage w . Then, the firm wants to pay the worker the optimal wage w^* to maximise the optimal amount of effort e^* that the worker puts in. We show this in Fig. 26.

We now show mathematically what is happening in Fig. 26. The firm wants to maximise effort per unit of wage:

$$\max_w \frac{e(w)}{w}$$

We can then take first order conditions:

$$\frac{d}{dw} \left[\frac{e(w)}{w} \right] = \frac{we' - e}{w^2} = 0$$

where $e' = \frac{de}{dw}$. We can then re-arrange to get:

$$\frac{de}{dw} w = e$$

and finally get:

$$\frac{\frac{de}{dw}}{e/w} = 1. \quad (213)$$

This equation is known as the **Solow condition** and says that the elasticity of effort with respect to the wage equals one.

We can then re-arrange to derive:

$$\frac{de}{dw} = \frac{e}{w}$$

which tells us the point of tangency to derive the optimal wage-effort pair (w^*, e^*) .

The punchline from this analysis is that during an economic boom, the firm would rather employ an extra person and pay them a wage of w^* rather than pay an existing worker more $w' > w^*$ to incentivise them to work $e' > e^*$ due to diminishing returns. Therefore, **wages are pinned down by the level of effort in this economy**.

Therefore, unemployment that occurs at the wage w^* in Fig. 26 is involuntary, as firms have no incentive to lower wages (since w^* maximises productivity per unit of wage). We now want to give a micro-foundation for why the relationship in Fig. 26 might arise. This is what the Shapiro–Stiglitz model tries to answer.

9.3.1 Shapiro and Stiglitz (1984)

The beginning of the original paper has a nice summary of the model.

Involuntary unemployment appears to be a persistent feature of many modern labor markets. The presence of such unemployment raises the question of why wages do not fall to clear labor markets. In this paper we show how the information structure of employer-employee relationships, in particular the inability of employers to costlessly observe workers' on-the-job effort, can explain involuntary unemployment as an equilibrium phenomenon. Indeed, we show that imperfect monitoring necessitates unemployment in equilibrium. The intuition behind our result is simple. Under the conventional competitive paradigm, in which all workers receive the market wage and there is no unemployment, the worst that can happen to a worker who shirks on the job is that he is fired. Since he can immediately be rehired, however, he pays no penalty for his misdemeanor. With imperfect monitoring and full employment, therefore, workers will choose to shirk. To induce its workers not to shirk, the firm attempts to pay more than the "going wage"; then, if a worker is caught shirking and is fired, he will pay a penalty. If it pays one firm to raise its wage, however, it will pay all firms to raise their wages. When they all raise their wages, the incentive not to shirk again disappears. But as all firms raise their wages, their demand for labor decreases, and unemployment results. With unemployment, even if all firms pay the same wages, a worker has an incentive not to shirk. For, if he is fired, an individual will not immediately obtain another job. The equilibrium unemployment rate must be sufficiently large that it pays workers to work rather than to take the risk of being caught shirking.

Here, we assume that firms pay high real wages in order to induce workers to work hard, as workers do not want to lose a well-paid job.

We now impose assumptions on the utility function of workers and the level of effort they can put in.

Assumption (Utility Function of Workers)

There are N workers with risk-neutral utility:

$$U(w, e) = w - e \quad (214)$$

where w is wage and e is effort.

Assumption (Level of Effort)

Effort e can take on two possible values:

$$e = \begin{cases} e & \text{no shirking} \\ 0 & \text{shirking} \end{cases} \quad (215)$$

A worker who shirks produces nothing. Therefore, the firm wants to ensure there is no shirking and hence tries to monitor the worker's effort. However, there are limitations to this.

Assumption (Monitoring)

The firm can inspect a worker with probability q every period. If a worker is found shirking, they are dismissed.

We also assume that a worker can get fired with a certain probability due to external factors.

Assumption (Exogenous Firing)

There is a probability b every period of a worker being fired.

Hence, we have three different scenarios that a worker can be in:

1. Employed but shirking.
2. Employed but not shirking.
3. Unemployed.

We now want to formulate the worker's problem in terms of dynamic programming. First, recall that from an asset pricing equation, it says that the return on an asset is equal to the payoff of the asset plus future capital gains/losses on the asset. Therefore, we can then think of the return of a worker being employed but shirking is the immediate payoff they get from working and shirking plus the future value of them losing their job (capital loss). The same line of reasoning applies for the other two types of workers. Before we can model this, we need to define the value functions for the workers.

Definition 176 (Value Functions For Worker)

The three value functions for a worker are:

$$V_E^S \equiv \text{Shirking if currently employed} \quad (216)$$

$$V_E^N \equiv \text{Not Shirking if currently employed} \quad (217)$$

$$V_U \equiv \text{Unemployed Worker.} \quad (218)$$

From this, we can *price* the value of the different scenarios for the workers.

Proposition 177 (Asset Pricing Equation for Employed Worker)

Let $r > 0$ be the worker's discount rate. Then, the asset pricing equations for an employed worker are:

$$rV_E^S = w + (b + q)(V_U - V_E^S) \quad (219)$$

$$rV_E^N = (w - e) + (b)(V_U - V_E^N). \quad (220)$$

Proof. The first equation says that the return from being employed and shirking is the immediate wage w plus the probability of being fired (the monitoring probability q plus the exogenous separation probability b) times the capital loss from becoming unemployed, $V_U - V_E^S$.

The second equation says that the return from being employed and not shirking is the immediate wage w less the effort cost e , plus the probability of being fired (the exogenous separation probability b) times the capital loss from becoming unemployed, $V_U - V_E^N$. $\square$

Hence, we can see that if a worker shirks, they get a better payoff but they also increase their chances of losing their job by probability q .

Now, we can figure out how much a firm should pay in order to encourage an employed worker not to shirk. That is, we need $V_E^N \geq V_E^S$. This is known as a **no-shirking condition**.

Proposition 178 (No Shirking Condition)

The no shirking condition is given by:

$$w \geq rV_U + \frac{(r + b + q)e}{q} \equiv \tilde{w} \quad (221)$$

which is the wage w that the firm needs to pay to prevent the worker shirking.

Proof. First, re-arrange the asset pricing equations:

$$V_E^S = \frac{w + (b + q)V_U}{r + b + q}$$

$$V_E^N = \frac{(w - e) + bV_U}{r + b}$$

We then use $V_E^N \geq V_E^S$ and hence if we re-arrange, we get the No shirking condition. $\square$

We can interpret this no shirking condition. A shirker doesn't produce anything so the firm needs to set a wage so that the worker does not shirk. There are 5 factors that affect the minimum wage $\tilde{w}$ to prevent shirking.

1. The greater the effort e for the worker, the higher the wage that the firm needs to pay to discourage them from shirking.
2. The higher the discounted utility V_U from being unemployed, the higher the wage $\tilde{w}$ you need to pay as the worker is not affected as much by being unemployed.
3. As the probability q of being caught shirking decreases, you need to pay them a higher wage to discourage them from shirking.
4. The higher the discount rate, the higher the wage you need to pay the worker. This is because they put less weight on the future and discount it more. The worker is not as worried about losing their job, and hence you need to pay them more.

5. The higher the exogenous separation rate b , the more you need to pay them, since the worker is more likely to lose the job regardless of effort.

We also notice that from this:

$$q(V_E^S - V_U) \geq e$$

which therefore implies that:

$$V_E^S > V_U$$

which says that there has to be a penalty from being laid off for shirking.

The issue with this equation for the no-shirking wage is that it depends on V_U which is an endogenous variable. We want to make the no-shirking wage as a function of only exogenous variables. To do so, we need to analyse the firm's problem as that will help pin down the no shirking wage.

Going forward, we now assume that the wage w is being set at the level $\tilde{w}$ to prevent shirking.

Assumption (Minimum Wage)

The firm pays a wage of $\tilde{w}$ to prevent workers from shirking.

We now look at the firm's problem and solve for the general equilibrium.

Assumption (Firm Structure)

There are M identical firms, each with a production function $f(L)$, where L is the number of non-shirking workers at wage rate $\tilde{w}$.

We normalise the price of output to be 1 so that the production function $f(L)$ coincides with the revenue function. We can then solve for how much labour does a given firm demand and from that, how much do the M firms demand in total.

Proposition 179 (Individual and Aggregate Demand for Labour)

Let $f(L)$ be the production function. Then, the optimal labour demand for firm i is:

$$L_i = (f')^{-1}[\tilde{w}] \quad (222)$$

where $L_i \equiv L_i(\tilde{w})$ is the amount of labour supplied at the no-shirking wage $\tilde{w}$.

The aggregate labour demand is:

$$L = L_1(\tilde{w}) + L_2(\tilde{w}) + \dots + L_M(\tilde{w}) \quad (223)$$

Proof. The firm pays $\tilde{w}$ to prevent shirking and therefore, they demand:

$$L_i \equiv L_i(\tilde{w}).$$

Then, the firm's FOC is that marginal product of labour is equal to the wage so we have that:

$$f'(L_i) = \tilde{w}.$$

We then invert $f'(\cdot)$ to get the individual firm's demand for labour:

$$L_i = (f')^{-1}[\tilde{w}].$$

Then by aggregation, we can just sum up each firm's demand for labour to get aggregate demand for labour. $\square$

The firm's demand for labour:

$$L_i = (f')^{-1}[\tilde{w}]$$

is what pins down the efficiency wage $\tilde{w}$ in the economy.

Now, we want to derive the no shirking wage as a function of only exogenous variables. First, we assume that we are in a flow equilibrium whereby people coming in and out of employment is in equilibrium.

Assumption (Equilibrium Flow of Employment)

Let a be the endogenous equilibrium probability per unit time of being hired out of unemployment. Let N be the total number of workers, L be the number of employed workers and $N - L$ the number of unemployed workers. Then, the equilibrium flow of employment is:

$$bL = a(N - L) \quad (224)$$

where the left-hand side is the flow of workers **out of** employment and the right-hand side is the flow of workers into employment.

Remark 180. Notice that we do not have q in this equation as no workers are shirking since the no shirking wage $\tilde{w}$ is being paid.

From this, we can derive the asset pricing equation for unemployed workers as the unemployment benefits y they receive times the expected payoff from them being hired.

Proposition 181 (Asset Pricing Equation for Unemployed Worker)

The asset pricing equation for an unemployed worker is:

$$rV_U = y + a(V_E^N - V_U). \quad (225)$$

We can now derive the aggregate no shirking wage.

Theorem 182 (Aggregate No Shirking Condition)

The no-shirking wage $\tilde{w}$ as a function of only exogenous and aggregate variables is given by:

$$w \geq e + y + \frac{e}{q} \left(\frac{bN}{N - L} + r \right) \equiv \tilde{w}. \quad (226)$$

Notice that this no shirking wage only depends on aggregate labour and market conditions. We can think of a worker being unemployed as a disciplining device to keep them from shirking.

We can see that as the number of employed workers (L) increases, there are fewer unemployed people. Therefore, to remain in the NSC region, firms will need to pay a higher wage. The intuition is that workers are less worried about finding another job if they get fired, since the labour market is tight, and hence firms need to pay them more to encourage them not to shirk.

We can plot this no shirking condition in Fig. 27. Here, we see that if the firm hired a new worker and pays a wage $w' < \tilde{w}$, then the newly hired worker has an incentive to shirk. Therefore, the firm will not hire any worker below the rate $\tilde{w}$ as the worker cannot credibly commit to not shirking. This is how unemployment arises in this model as wages cannot adjust downwards as workers at this lower wage will shirk.

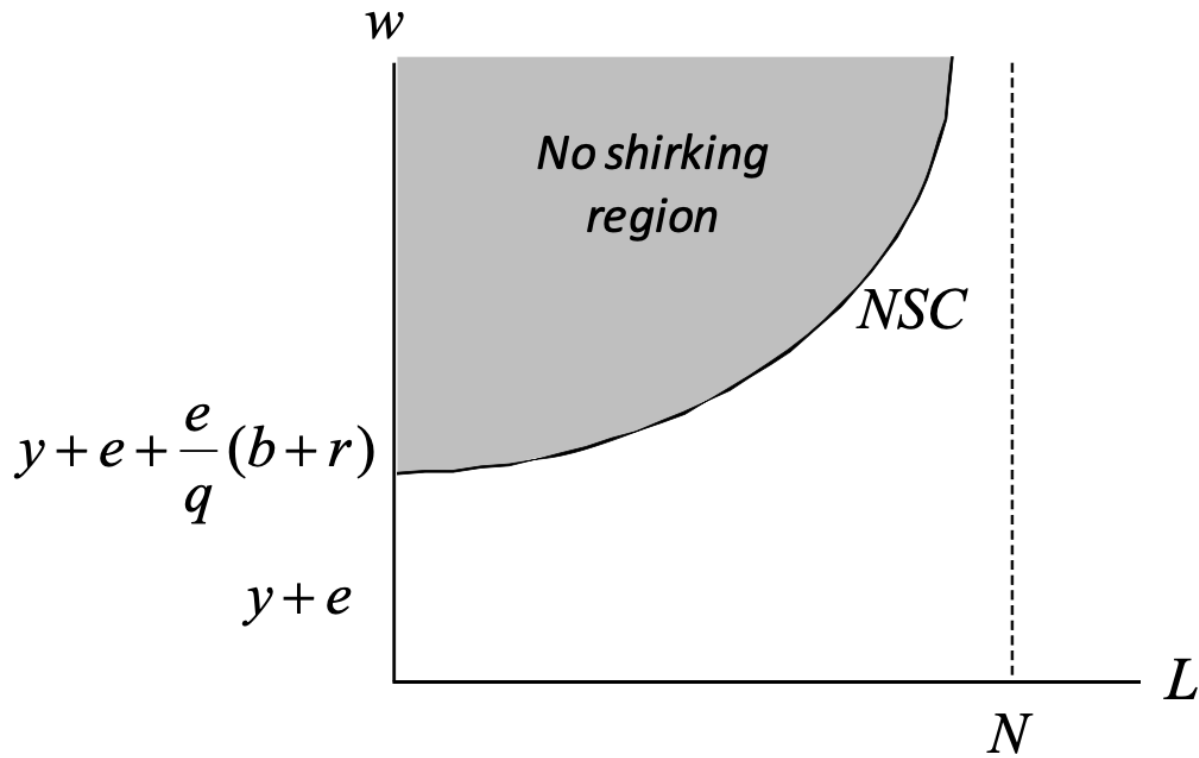


Figure 27: No shirking condition.

Now, we can impose an aggregate labour demand curve on our graph and arrive at Fig. 28. Here, we can see that on the left, for low levels of employment L , labour supply is elastic. As we move to the right, labour supply becomes inelastic. Now, suppose that we face a positive technology shock, so that aggregate labour demand increases. Only aggregate labour demand moves, as the no-shirking condition is unchanged by the technology shock. Then, we can see from the shape of the no-shirking region that employment moves **more** than wages. Therefore, **employment moves around a lot, but wages do not move around much**.

We have just seen that variation in aggregate demand can lead to variation in unemployment because real wages cannot adjust fully when firms must maintain the no-shirking condition. This gives rise to real wage rigidity.

Definition 183 (Real Wage Rigidity)

The real wage rigidity is given by:

$$\hat{w} = \frac{1}{\eta} \hat{y} \quad (227)$$

where η measures the extent of the real rigidity.

Here, we get back the RBC model if $\eta = 1$. For $\eta > 1$, we have a model with real rigidities. Again, real wage is not adjusting fully so unemployment arises.

9.4 Search and Matching

When we look at the data, it is noticeable that even in recessions, a large number of firms have unfilled vacancies whilst there is also a large amount of unemployment. Furthermore, even in booms, we see that some firms are laying

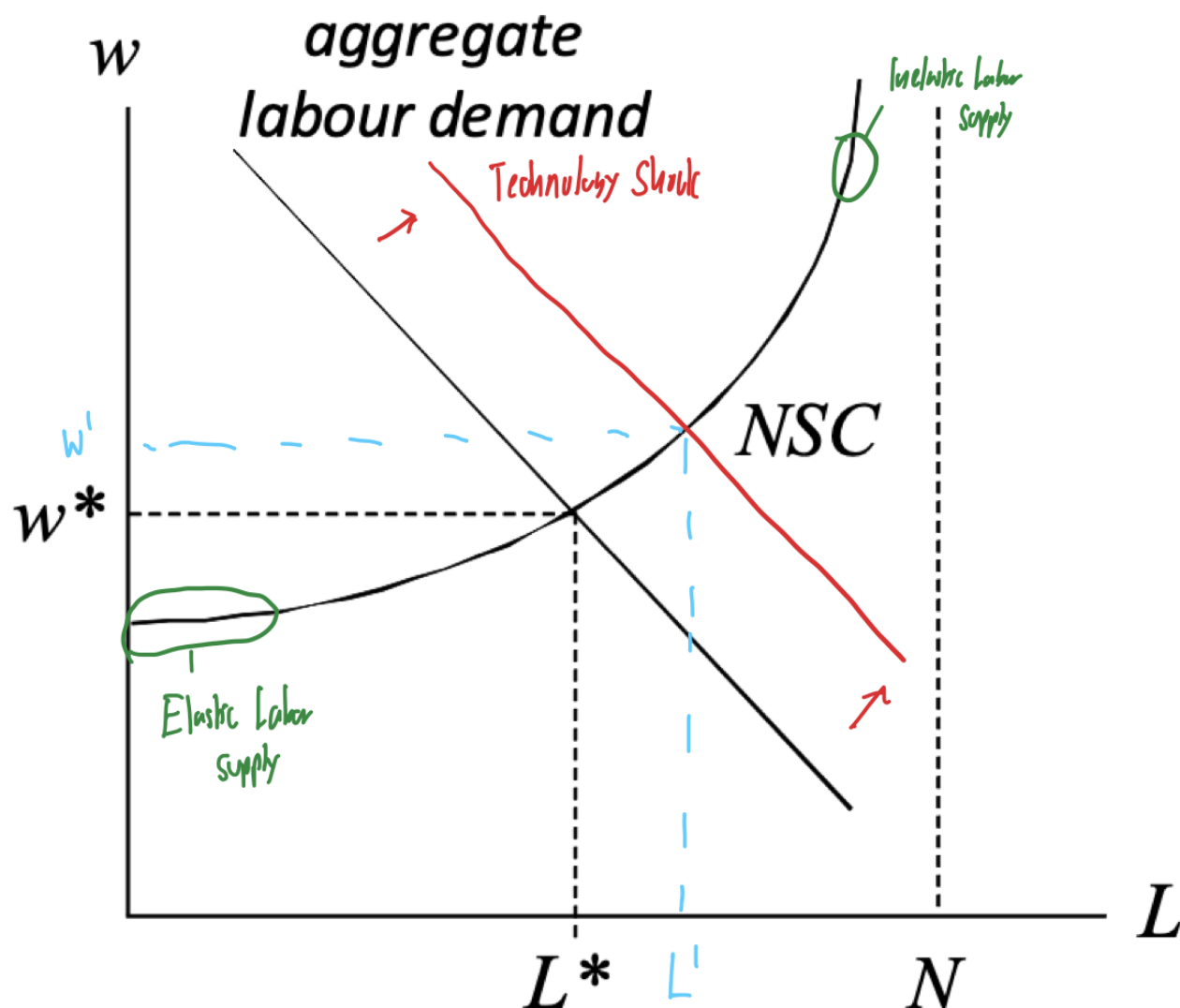


Figure 28: Aggregate demand and no shirking condition.

off workers. All this leads us to focus on flows into and out of unemployment rather than the unemployment level itself.

We now model a non-Walrasian feature of the labour market. We assume that workers and jobs are heterogeneous. As a result, instead of meeting in centralised markets, workers and firms meet in a decentralised, one-on-one fashion and engage in a costly process of matching preferences, skills, and needs. Since this process is not instantaneous, it results in unemployment.

9.4.1 Diamond-Mortensen-Pissarides (2000)

There are three components to the search and matching model:

1. Workers are matched with firms via a matching function.
2. Workers and firms bargain over wage and divide match surplus from costs of searching for a job and costs of

having an open vacancy.

3. Match dissolves with exogenous probability λ .

We normalise the total labour force L to be 1. We denote by v the **number of vacancies** and by u the **number of unemployed workers**. Note that due to the normalisation, u is also the **rate of unemployment**.

Definition 184 (Number of Vacancies and Unemployed)

Let v denote the number of job vacancies and u denote the number of unemployed workers.

Now, we define the following function to match these workers.

Definition 185 (Matching Function)

The matching function is given by $m(u, v)$ which is increasing and concave with respect to each argument. The matching function has constant returns-to-scale.

The matching function $m(u, v)$ tells us how many matches are made in a period, given unemployment u and vacancies v .

Definition 186 (Labour Market Tightness)

Labour market tightness is given by:

$$\theta \equiv \frac{v}{u} \quad (228)$$

where v is the number of vacancies and u is unemployment.

This is a natural definition, as θ is the ratio of vacancies to unemployed workers. The higher θ is, the more vacancies there are relative to the number of unemployed, meaning that many firms are unable to fill their vacancies. For a firm, it is not a good time to hire, as there are few unemployed individuals and a lot of competition from other firms.

Now, to make life easier, we can rewrite the matching function in terms of probabilities.

Definition 187 (Probability of Filling a Vacancy and Finding a Job)

The probability for a firm to fill a vacancy is given by:

$$q(\theta) \equiv \frac{m(u, v)}{v}. \quad (229)$$

The probability of a worker finding a job is given by:

$$\theta q(\theta) \equiv \frac{m(u, v)}{u} \quad (230)$$

where θ is labour market tightness, u is the number of unemployed workers, and v is the number of vacancies.

Proof. We use the property of constant returns to scale of the matching function and the probability of filling a vacancy is the matching function divided by the number of vacancies:

$$\frac{m(u, v)}{v} = m\left(\frac{u}{v}, 1\right) = m\left(\frac{1}{\theta}, 1\right) \equiv q(\theta).$$

Also, the probability of finding a job depends on the matching function and the number of unemployed:

$$\frac{m(u, v)}{u} = m\left(1, \frac{v}{u}\right) \equiv \theta q(\theta).$$

We need some important properties on the probability of filling a vacancy.

Assumption (Probability of Filling A Vacancy)

Let $q(\theta)$ be the probability of filling a vacancy given as a function of labour market tightness θ .

We assume that as the labour market gets tighter, the harder it is to fill a vacancy:

$$q'(\theta) < 0. \quad (231)$$

The probability of finding a job increases as the labour market gets tighter:

$$\frac{d\theta q(\theta)}{d\theta} > 0. \quad (232)$$

9.4.2 Beveridge Curve

We can now construct the Beveridge curve, which relates labour market tightness to the unemployment rate (recall that the rate and level of unemployment are identical here, as the total labour supply is normalised to 1).

Before we do so, we need to think about the dynamics of the rate of unemployment $\dot{u}$. First, the number of people who are transitioning to becoming unemployed is the number of people currently employed $(1 - u)$, times the exogenous probability of being separated:

$$\lambda(1 - u).$$

The flow **out** of unemployment is the number of unemployed workers u times the probability of finding a job $\theta q(\theta)$:

$$u\theta q(\theta).$$

We can combine these together.

Definition 188 (Flow of Unemployment)

The flow of unemployed workers is given by:

$$\dot{u} = \lambda(1 - u) - u\theta q(\theta) \quad (233)$$

where λ is the exogenous separating rate, u is the number of unemployed, $(1 - u)$ is the number currently employed, and $\theta q(\theta)$ is the probability of finding a job.

Remark 189. Notice the - sign as that is the number of people becoming **employed** or flowing **out** of unemployment.

We can derive the Beveridge curve which measures the equilibrium level of unemployment.

Definition 190 (Beveridge Curve)

The Beveridge curve measures the relationship between labour market tightness and unemployment rate and is given by:

$$u = \frac{\lambda}{\lambda + \theta q(\theta)} \quad (234)$$

where λ is the exogenous separating rate and $\theta q(\theta)$ is the probability of finding a job.

Proof. Recall that the flow of unemployment is equal to:

$$\dot{u} = \lambda(1 - u) - \theta q(\theta)u.$$

In equilibrium, we have that the rate of unemployment is 0:

$$\dot{u} = 0.$$

Hence, we can equate our flow of unemployment equation:

$$\lambda(1 - u) - \theta q(\theta)u = 0.$$

Then, we can solve for u to derive the Beveridge curve:

$$\lambda u + \theta q(\theta)u = \lambda$$

and hence:

$$u = \frac{\lambda}{\lambda + \theta q(\theta)}.$$

□

We plot the Beveridge curve in Fig. 29.

The Beveridge curve has nice properties.

Proposition 191 (Properties of Beveridge Curve)

The Beveridge curve has the following properties:

1. The Beveridge curve measures the relationship between unemployment rate and number of vacancies in equilibrium.
2. The Beveridge curve is a downward sloping function.
3. The Beveridge curve is a **convex** function.
4. The Beveridge curve is a function of the matching function $m(u, v)$ and the exogenous job separating rate λ .

The Beveridge curve slopes downwards due to the properties of the matching function. For a high level of vacancies the matching function predicts that unemployment will be low, and so the Beveridge curve slopes down. The Beveridge curve is convex because the matching function is characterised by diminishing marginal returns. The position of the Beveridge curve depends on the job destruction process, and the efficiency of the matching process. Increases in the job destruction process or decreases in matching efficiency involve an outward shift of the Beveridge curve and so a higher level of unemployment for any given level of vacancies.

Also, notice that at points on the left of the Beveridge curve, there is a high number of vacancies. This implies that there is a lot of hiring and a large flow **out** of unemployment. In steady state, this must be matched by a large flow **into** unemployment, so there is a high churn rate as many people lose their jobs. On the right of the Beveridge curve, there are few vacancies, so there is less hiring and fewer people losing their jobs.

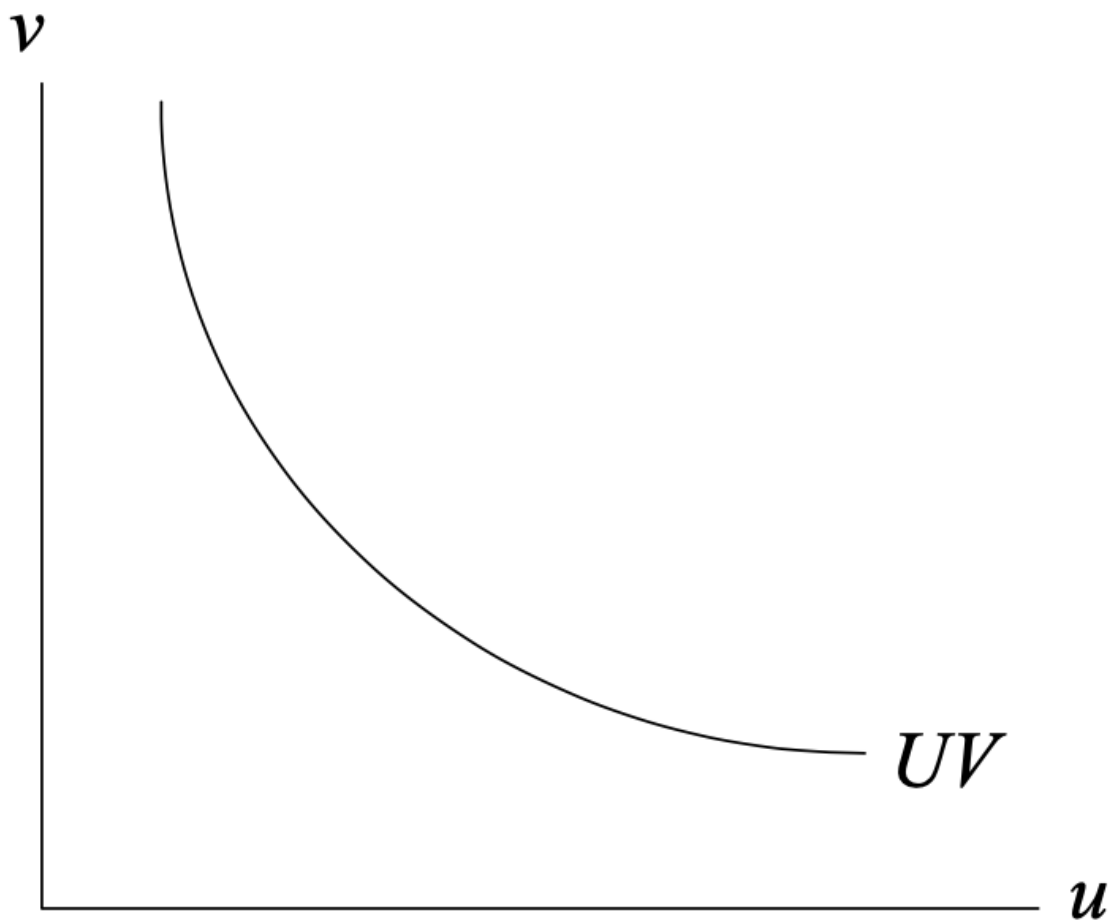


Figure 29: Beveridge curve.

9.4.3 Job Creation Condition

We now want to figure out what is the rate at which jobs will be created. To do so, we need to give microfoundations to when firms will create new jobs. In this setting, we want to analyse the conditions for a firm to post a vacancy.

Definition 192 (Cost of Posting Vacancy)

The cost to the firm of posting a vacancy is c .

Why would the firm want to post a vacancy if there is a cost c associated with it? A filled vacancy will produce output for the firm.

Definition 193 (Output Production from Filled Vacancy)

A filled vacancy will produce a units of real output.

We can now describe the asset pricing conditions for a firm to post a vacancy.

Definition 194 (Present Discounted Value of Expected Profit from Filled Vacancy)

Denote J as the present discounted value of expected profits from an occupied job.

Thus, J measures the present discounted value of profits that the firm makes from a filled vacancy. We also define the value of an open vacancy that is **not filled**.

Definition 195 (Present Discounted Value of Expected Profit from Unfilled Vacancy)

Denote V as the present discounted value of expected profits from an unfilled vacancy.

Now, we can use asset pricing to figure out the value of having a vacancy opening. First, the return r on having an open vacancy is given by:

$$rV.$$

Now, the firm has to pay the cost $-ac$ (cost here is a percentage of output) every period:

$$-ac.$$

However, there is a probability $q(\theta)$ that they will be able to fill this vacancy and if so, they get the difference from transition from not filling the vacancy V to filling the vacancy J :

$$q(\theta)(J - V).$$

We can combine these results to get an asset pricing equation from having a vacancy opening.

Definition 196 (Asset Pricing Equation for Having a Vacancy)

Denote J and V as the present discounted value of expected profit from an occupied job and vacancy opening respectively. Then, the asset pricing equation from having a vacancy V is given by:

$$rV = -ac + q(\theta)(J - V) \quad (235)$$

where the vacancy-posting cost is ac —that is, a fraction c of output a —and $q(\theta)$ is the probability of filling a vacancy.

Remark 197. *The return on a vacancy equals the negative cost of posting it plus the expected capital gain from filling it. Here, the capital gain is the difference between the continuation payoff from a filled vacancy J and that from an unfilled vacancy V .*

In equilibrium, the value of posting a vacancy should be zero; otherwise, firms have an incentive to continue posting vacancies.

Proposition 198 (Equilibrium Value of PDV of Filled Vacancy)

The equilibrium value of the present discounted value of expected profits from a filled vacancy J is given by:

$$J = \frac{ac}{q(\theta)} \quad (236)$$

where the vacancy-posting cost is ac —that is, a fraction c of output a —and $q(\theta)$ is the probability of filling a vacancy.

Proof. In equilibrium, the PDV of posting a vacancy $V = 0$ or else we have arbitrage. Then, from our expression for the asset pricing equation for having a vacancy, this becomes:

$$\begin{aligned} rV &= -ac + q(\theta)(J - V) \\ \leftrightarrow 0 &= -ac + q(\theta)(J - 0) \\ \Rightarrow J &= \frac{ac}{q(\theta)}. \end{aligned}$$

□

We can also derive an asset pricing equation for the value of having a job vacancy filled.

Definition 199 (Asset Pricing Equation for Having a Job Filled)

Denote J as the present discounted value of expected profit from an occupied job. Then, the asset pricing equation from having a vacancy filled is:

$$rJ = a - w - \lambda J \quad (237)$$

where a is output from the filled job, w is the wage paid to the worker, and λ is the exogenous separation rate.

Proof. Here rJ is the return on having a job filled. Now, if the job is filled, the firm gets the output less the wage they pay the employee:

$$a - w.$$

Then, there is a probability λ that the worker is separated so the firm loses the employee:

$$-\lambda J$$

□

We now have the condition relating the wage that firms will pay and the cost of posting vacancies.

Definition 200 (Job Creation Condition)

The job creation condition is given by:

$$\frac{a - w}{r + \lambda} - \frac{ac}{q(\theta)} = 0 \quad (238)$$

where a is the output from a filled job, w is the wage paid, r is the discount rate, λ is the exogenous separation rate, and $q(\theta)$ is the probability of filling a vacancy.

Proof. First, look at the asset pricing equation for having a job filled:

$$rJ = a - w - \lambda J.$$

We can re-arrange and solve for J :

$$J = \frac{a - w}{r + \lambda}.$$

Now, we know that in equilibrium, the PDV of posting a vacancy is 0 or else a firm can continue to post vacancies $V = 0$. Under such a situation, we have already found the equilibrium value PDV of a filled vacancy to be:

$$J = \frac{ac}{q(\theta)}.$$

We can therefore plug this into our arranged asset equation to get:

$$\begin{aligned} J &= \frac{a - w}{r + \lambda} \\ \Leftrightarrow \left(\frac{ac}{q(\theta)} \right) &= \frac{a - w}{r + \lambda} \\ \Rightarrow \frac{a - w}{r + \lambda} - \frac{ac}{q(\theta)} &= 0. \end{aligned}$$

□

So how do we interpret the job creation condition:

$$\frac{a - w}{r + \lambda} - \frac{ac}{q(\theta)} = 0.$$

This says that in equilibrium, job creation (posting a vacancy for the firm) must lead to zero profits or else a firm can continue to post vacancies.

9.4.4 Wage Curve

We now want to pin down wages. For the households, we can construct their asset pricing equations for being employed vs being unemployed. To do so, we first look at the household's problem. Then, we can derive the wage as a result of bargaining between the household and firm.

Definition 201 (Asset Pricing Equation for Unemployed Household)

The value to a household of being unemployed is:

$$rU = z + \theta q(\theta)(E - U) \quad (239)$$

where U is the value of being unemployed, E is the value of being employed, r is the return, z is unemployment benefits and $\theta q(\theta)$ is the probability of finding a job.

This asset pricing equation captures the unemployment benefit times the expected payoff for an unemployed person being matched and finding a job.

We can do something similar for an employed household.

Definition 202 (Asset Pricing Equation for Employed Household)

The value to a household of being employed is:

$$rE = w + \lambda(U - E) \quad (240)$$

where U is the value of being unemployed, E is the value of being employed, w is the wage and where $\lambda(U - E)$ measures the risk that you lose your job.

Now, if a firm and household are matched, they can bargain over the wage. This follows a Nash bargaining problem.

Definition 203 (Nash Wage Bargaining)

The Nash wage bargaining problem is:

$$w = \arg \max_w \left[(E - U)^\beta (J - V)^{(1-\beta)} \right] \quad (241)$$

where w is the wage, β measures the bargaining power of the worker, E is the value to the household of being employed, U is the value to the household of being unemployed, J is the value to the firm of having a filled vacancy, and V is the value to the firm of having an unfilled vacancy.

We can now derive the first order conditions.

Proposition 204 (First Order Condition for Nash Wage Bargaining)

The first-order condition for Nash wage bargaining is:

$$(1 - \beta)E = \beta J + (1 - \beta)U \quad (242)$$

where β is the bargaining power of workers, E is the value of being employed to household, U is the value of being unemployed to household and J is the value to firm for having a vacancy filled.

Sketch. Define:

$$\mathcal{W}_t(w_t) \equiv (E - U)$$

to be the worker's surplus from bargaining, which is a function of wage as E is a function of wage:

$$rE = w + \lambda(U - E).$$

Define:

$$\mathcal{F}_t \equiv J - V$$

to be the firm's surplus from bargaining, which is also a function of wage as J is a function of wage:

$$rJ = a - w - \lambda J.$$

Then rewrite the Nash bargaining problem as:

$$\max_w \left[\mathcal{W}_t^\beta \mathcal{F}_t^{(1-\beta)} \right].$$

Taking the first-order condition and using $\partial E / \partial w = -(\partial J / \partial w)$ gives:

$$\beta \mathcal{W}_t^{\beta-1} \mathcal{F}_t^{1-\beta} - (1 - \beta) \mathcal{W}_t^\beta \mathcal{F}_t^{-\beta} = 0.$$

Therefore, $(1 - \beta)\mathcal{W}_t = \beta\mathcal{F}_t$. Substituting back and imposing $V = 0$ gives the stated condition.

□

By taking first order conditions and imposing $V = 0$, we can get the optimal wage.

Theorem 205 (Optimal Wage from Nash Bargaining)

The optimal wage from Nash wage bargaining is:

$$w = \beta a + (1 - \beta)rU \quad (243)$$

where w is the wage, β is the bargaining power of the worker, a is the output from a filled vacancy, r is the return and U is the value to the worker being unemployed.

Remark 206. When $\beta = 1$, this says that the worker has all the bargaining power and therefore gets a wage of $w = a$ whilst the firm gets zero profits. $\beta = 0$ implies that firms have all the bargaining power and therefore workers receive their reservation wage $w = rU$.

We see that the wage is a function of both the productivity and value of being unemployed. We can now derive the wage curve.

Definition 207 (Wage Curve)

The wage curve is given by:

$$w = \beta a + (1 - \beta)z + \theta\beta ac \quad (244)$$

where β is the bargaining power of the worker, a is the output produced by a filled vacancy, z is the unemployment benefits for the unemployed worker, θ is the labour market tightness, and ac is the cost of posting a vacancy.

Proof. First look at the FOC from Nash bargaining problem:

$$(1 - \beta)E = \beta J + (1 - \beta)U.$$

Rewrite this as:

$$(1 - \beta)(E - U) = \beta J.$$

Now, we know that in equilibrium, the present discounted value for having a job filled is equal to:

$$J = \frac{ac}{q(\theta)}.$$

We plug this into our FOC:

$$(1 - \beta)(E - U) = \beta \frac{ac}{q(\theta)}.$$

Now, plug this into the asset pricing equation for the value of being unemployed:

$$rU = z + \theta q(\theta)(E - U)$$

to become:

$$rU = z + \theta q(\theta) \left[\frac{\beta}{1 - \beta} \frac{ac}{q(\theta)} \right].$$

This becomes:

$$rU = z + \theta \frac{\beta}{1 - \beta} ac.$$

Finally, plug this into the optimal wage from Nash bargaining:

$$w = \beta a + (1 - \beta)rU$$

to get:

$$w = \beta a + (1 - \beta) \left[z + \theta \frac{\beta}{1 - \beta} ac \right]$$

which gives us the wage curve:

$$w = \beta a + (1 - \beta)z + \theta \beta ac.$$

□

If a worker is matched with a firm, a higher vacancy-posting cost c raises the worker's wage because replacing the worker would be more costly for the firm.

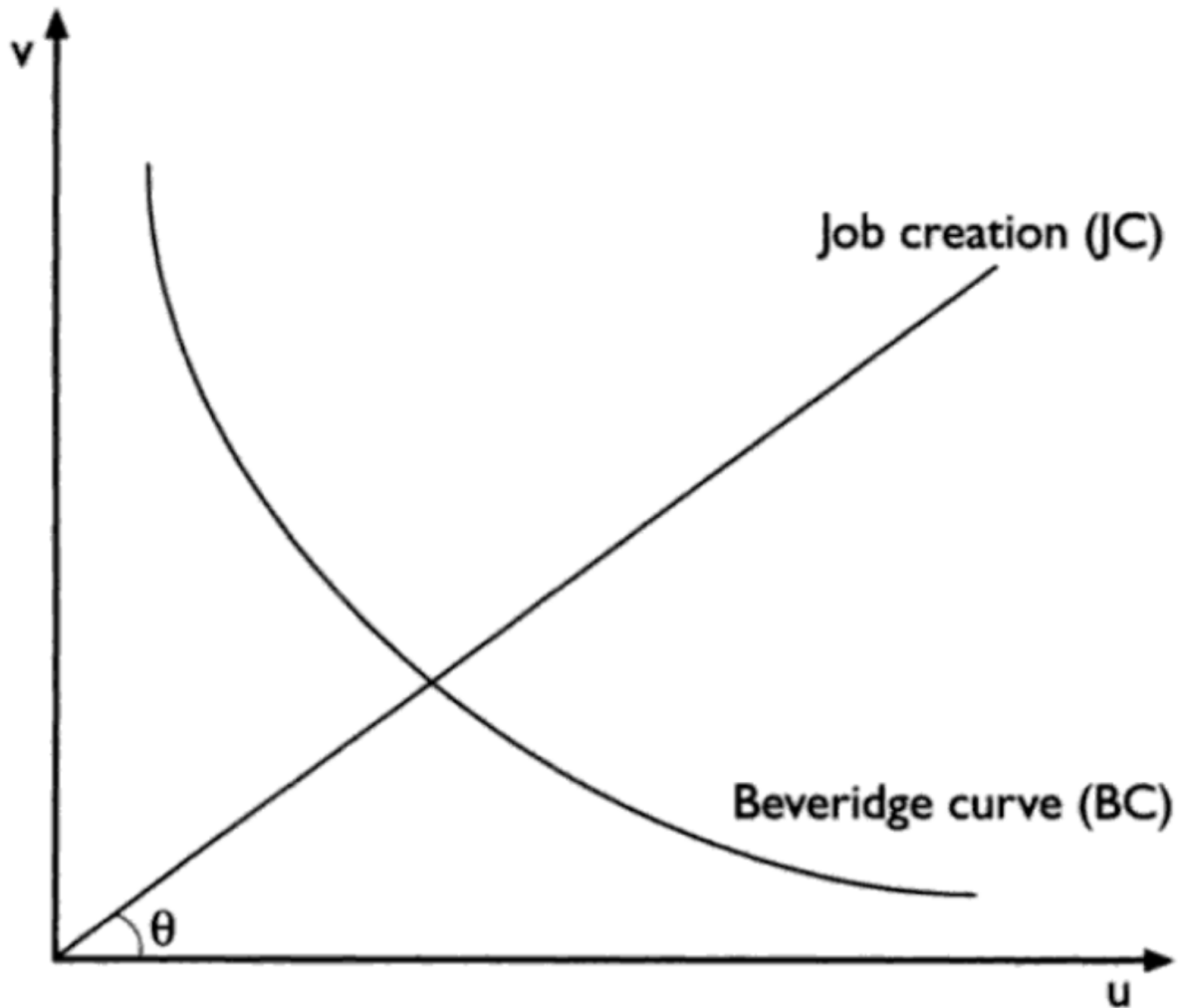


Figure 30: Beveridge curve and job creation curve.

We now bring together our three equations that we derived to get the DMP model.

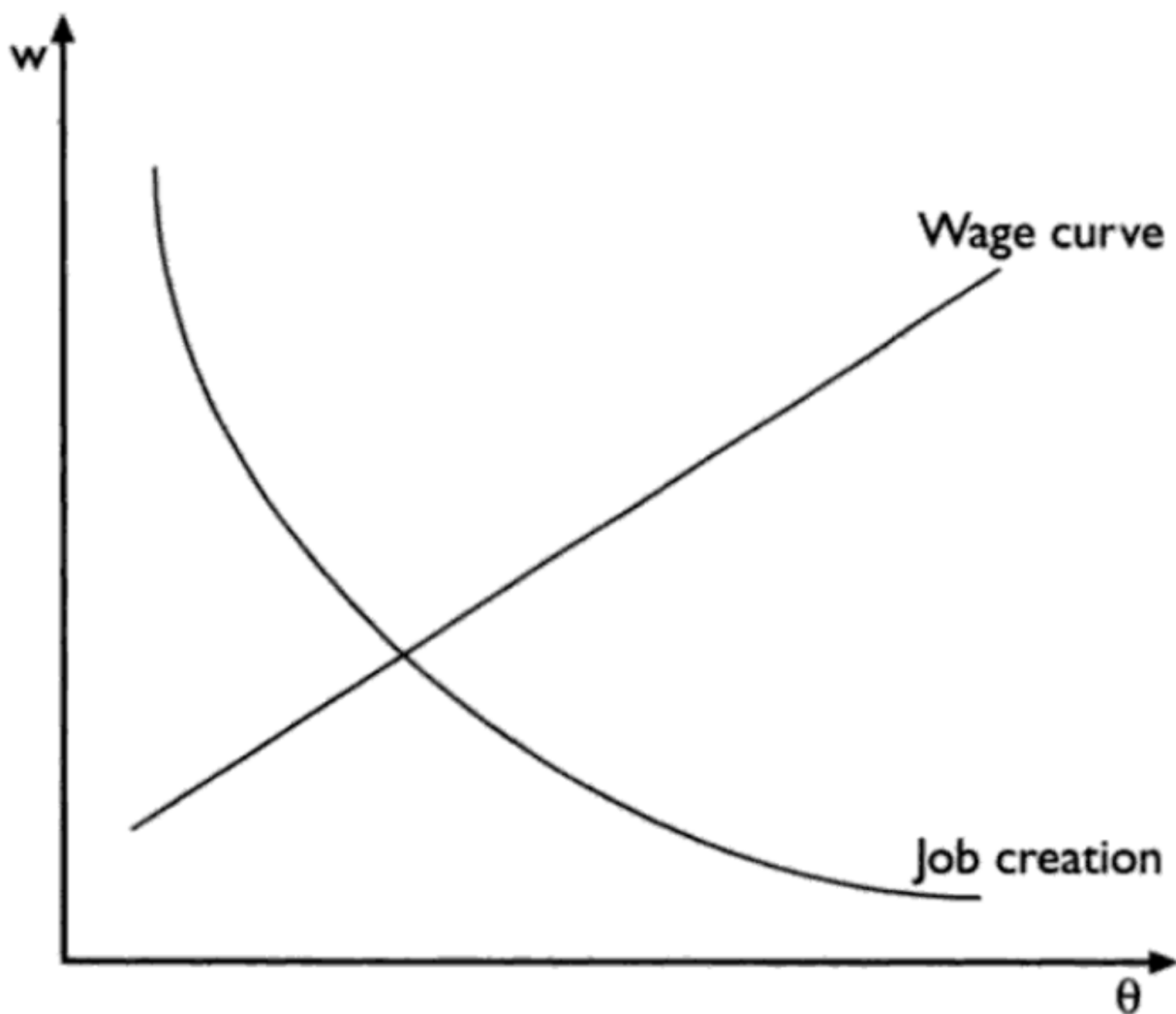


Figure 31: Wage curve and job creation curve.

Proposition 208 (DMP Model)

The Beveridge curve tells us the inflow and outflow of employment. The job creation condition tells us how vacancies are created. The wage curve tells us the wages paid to the worker as a result of the Nash bargaining problem.

$$u = \frac{\lambda}{\lambda + \theta q(\theta)} \quad (\text{Beveridge Curve})$$

$$\frac{a - w}{r + \lambda} - \frac{ac}{q(\theta)} = 0 \quad (\text{Job Creation Condition})$$

$$w = \beta a + (1 - \beta)z + \theta \beta ac. \quad (\text{Wage Curve})$$

where β is the bargaining power of the worker, a is the output produced by a filled vacancy, z is the unemployment benefits for the unemployed worker, θ is the labour market tightness, w is the wage, λ is the exogenous separation rate, $\theta q(\theta)$ is the probability of finding a job, and ac is the cost of posting a vacancy.

We plot the job creation curve against the Beveridge curve in Fig. 30. If unemployment increases, then the labour market is less tight. In a less tight labour market, the worker's bargaining position weakens, so firms can reduce the wage paid to workers. Lower wages create more incentives for firms to post vacancies, which is captured by the upward-sloping job creation curve. We also plot the relationship between the wage curve and job creation in Fig. 31.

We can also do some comparative statics. Suppose we have a productivity shock. Then, productivity a increases, then we can see that the job creation curve shifts out and the wage curve shifts out. Intuitively, as productivity increases, firms want to hire more workers and will pay them more.

Now, suppose that the labor market gets tighter so θ goes up. Then, the job creation curve rotates around to the left (goes up). The Beveridge curve does not move. This means that we have a higher level of employed versus unemployed.

Definition 209 (Shimer Puzzle)

The basic DMP model fails to match the data in terms of the volatility of vacancies v and unemployment u when driven by realistic productivity shocks.

Remark 210. *The reason that we do not get **enough volatility** in this model is because the wage curve shifts AND the job creation curve also shifts, hence helping labour markets to clear, which we do not want.*

The Shimer puzzle states that the standard labour-market matching model predicts fluctuations in the unemployment rate that are much smaller than those actually observed over the business cycle.

To summarise, we have looked at three (unsatisfactory) ways to try to deal with labour markets in the RBC model.

10 Asset Pricing

10.1 Introduction

We are interested in the problem whereby an investor must decide how much to save, how much to consume, and what portfolio of assets to hold. We can derive the first order conditions for this problem and this will lead us to a fundamental equation.

Suppose we are at time t , and we want to price an asset that has a payoff in the next period of x_{t+1} . The payoff of a stock is its new price plus its dividend:

$$x_{t+1} = p_{t+1} + d_{t+1}$$

Here, x_{t+1} is a **random variable**. Now, the consumer wishes to maximise

$$u(c_t) + \beta \mathbb{E}_t[u(c_{t+1})]$$

Generally, the CRRA utility function is used:

$$u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$

Remark 211. Note that c_{t+1} is also a **random variable** as the investor does not know their wealth tomorrow, which means that they don't know their consumption tomorrow.

Now, suppose that the investor is able to buy or sell as much of the payoff x_t at the price p_t . We can denote by b_t the amount of the asset they buy or sell. Let e_t be the endowment that the investor would consume at time t if they did not buy the asset at all. Therefore, we have the optimisation problem:

$$\begin{aligned} \max_{b_t} \quad & u(c_t) + \beta \mathbb{E}_t[u(c_{t+1})] \\ \text{s.t.} \quad & c_t = e_t - p_t b_t \\ & c_{t+1} = e_{t+1} + x_{t+1} b_t \end{aligned} \tag{245}$$

We can take the first order conditions by substituting in the constraints into the objective function and differentiating with respect to b_t :

$$\max_{b_t} u(e_t - p_t b_t) + \beta \mathbb{E}_t[u(e_{t+1} + x_{t+1} b_t)].$$

Then, the first order condition is:

$$-p_t u'(e_t - p_t b_t) + \beta \mathbb{E}_t[x_{t+1} \cdot u'(e_{t+1} + x_{t+1} b_t)] = 0.$$

Substitute back in the constraints and re-arrange to get:

$$\beta \mathbb{E}_t[x_{t+1} \cdot u'(c_{t+1})] = p_t u'(c_t).$$

Now, we can push constants inside the expectation operator:

$$\mathbb{E}_t[\beta u'(c_{t+1}) x_{t+1}] = p_t u'(c_t).$$

Finally, at time t , $u'(c_t)$ is a constant when you take the conditional expectation of it with respect to time t and

therefore we can also push this inside the conditional expectation:

$$\mathbb{E}_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1} \right] = p_t.$$

From this, we get the most fundamental asset pricing equation.

Definition 212 (Fundamental Asset Pricing Equation 1)

Let c_t be consumption, $u(c_t)$ the utility function, p_t the price of an asset and x_t the payoff of an asset. The basic pricing equation is

$$p_t = \mathbb{E}_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1} \right] \quad (246)$$

We can rewrite this in the form

$$p_t u'(c_t) = \mathbb{E}_t \left[\beta u'(c_{t+1}) x_{t+1} \right]$$

The marginal utility loss of consuming less today and buying more of the asset $p_t u'(c_t)$ should equal the discounted marginal utility gain by consuming more of the asset's payoff in the future $\mathbb{E}_t \left[\beta u'(c_{t+1}) x_{t+1} \right]$. If this relationship between the price of the asset p_t and payoff x_t does not hold, then the investor should buy or sell the asset such that it does.

From this, we can see that the asset's price should equal the expected discounted value of the asset's payoff, whereby it is the investor's marginal utility that is used to discount the payoff.

We now look at one of the terms in our fundamental asset pricing equation Eq. (246) and give it a special name due to its significance.

Definition 213 (Stochastic Discount Factor)

The stochastic discount factor is defined to be

$$\mathcal{M}_{t,t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} \quad (247)$$

We will soon see why this is important. First, we can now rewrite our fundamental asset pricing equation Eq. (246) as

$$p_t = \mathbb{E}_t \left[\mathcal{M}_{t,t+1} x_{t+1} \right] \quad (248)$$

where x_{t+1} is the payoff of the asset, $\mathcal{M}_{t,t+1}$ is our stochastic discount factor, and p_t is the price of the asset today. To think about the stochastic discount factor $\mathcal{M}_{t,t+1}$, it is helpful to think of it as being the **marginal rate of substitution** whereby $\mathcal{M}_{t,t+1}$ is the rate at which an investor is willing to substitute consumption at time $t + 1$ for consumption at time t . Therefore, if we have an asset that has a payoff tomorrow x_{t+1} , the price of the asset today p_t is the expected value of the payoff multiplied by how willing we are to substitute consumption tomorrow for consumption today! Note that this means that as we choose a different utility function $u(\cdot)$, we will get a different stochastic discount factor $\mathcal{M}_{t,t+1}$ as we have a different preference for substituting consumption tomorrow for today, and hence a different price for our asset!

10.2 Example: Pricing Equities

We can also rewrite our basic asset pricing equation by switching from payoffs x_{t+1} to returns R_{t+1} . For example, for stocks, the one-period payoff x_{t+1} is the new price of the stock p_{t+1} plus dividends d_{t+1} , which means our return on

stocks is

$$R_{t+1} = \frac{p_{t+1} + d_{t+1}}{p_t}$$

Therefore, we can take:

$$p_t = \mathbb{E}_t \left[\mathcal{M}_{t,t+1} x_{t+1} \right]$$

and plug in $x_{t+1} = p_{t+1} + d_{t+1}$ and divide through by p_t :

$$1 = \mathbb{E}_t \left[\mathcal{M}_{t,t+1} \frac{p_{t+1} + d_{t+1}}{p_t} \right]$$

We can then replace $\frac{p_{t+1} + d_{t+1}}{p_t}$ by our returns variable R_{t+1} . This gives us a new form of the asset pricing equation.

Proposition 214 (Fundamental Asset Pricing Equation 2)

Another form of the fundamental asset pricing equation is

$$1 = \mathbb{E}_t \left[\mathcal{M}_{t,t+1} R_{t+1} \right] \quad (249)$$

Remark 215. Note that this equation holds more generally than just for stocks. In fact, it holds for any financial asset once we take into account how its return R_{t+1} is defined.

We now state an important expression we can derive.

10.3 The Risk-Free Rate

Theorem 216 (Formula for the risk-free rate)

The risk-free rate R^f is the inverse of the expectation of the stochastic discount factor:

$$R_{t+1}^f = \frac{1}{\mathbb{E}_t[\mathcal{M}_{t,t+1}]} \quad (250)$$

Proof. First, recall that

$$1 = \mathbb{E}_t \left[\mathcal{M}_{t,t+1} R_{t+1} \right]$$

and we have that $R_{t+1} = R_{t+1}^f$ is the guaranteed risk-free rate. This is a constant. Then, the result follows trivially

$$1 = \mathbb{E}_t \left[\mathcal{M}_{t,t+1} \right] R_{t+1}^f$$

□

10.3.1 The Equilibrium Risk-Free Rate under no uncertainty

We now look at a specific application of the risk-free rate formula. Suppose we have power utility, so that $u'(c) = c^{-\gamma}$. Furthermore, suppose that there is no uncertainty. Then, the stochastic discount factor is given by:

$$\mathcal{M}_{t,t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} = \beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma}$$

Then, plug this into:

$$R_{t+1}^f = \frac{1}{\mathbb{E}_t[\mathcal{M}_{t,t+1}]}$$

to get:

$$R_{t+1}^f = \frac{1}{\mathbb{E}_t[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma}]}$$

Now, we make the big assumption and **remove any uncertainty from the model** so that the expectation term disappears. From this, we can arrive at the following.

Proposition 217 (Risk-Free Rate without Uncertainty)

Assuming a CRRA utility function and without any uncertainty in the model, we have that the risk-free rate is given by:

$$R_{t+1}^f = \frac{1}{\beta} \left(\frac{c_{t+1}}{c_t} \right)^{\gamma} \quad (251)$$

We can see **three** effects relating the risk-free rate to the economy:

1. If people are impatient (low β), then interest rates are high. This is because people want to consume now and hence we need a high rate of interest to persuade them to save. They need to be induced to want to save.
2. If interest rates will be high, then consumption growth is high c_{t+1}/c_t . People are persuaded to consume less now and consume more in the future. Alternatively, we can say that if we know that consumption will grow in the future, we want to smooth consumption so we consume more today c_t . This will cause the rate of interest R_f to increase today.
3. The more that agents care about smoothing consumption by having a higher γ , then the higher the interest rate will be. This is because it takes a higher rate of interest to induce the agent to shift their consumption profile due to their preference for consumption smoothing.

Before we proceed further, we need to define and explore what a log-normally distributed random variable is.

10.3.2 Log-Normally Distributed Random Variable

We now define what a log-normal random variable is.

Definition 218 (Log-Normal Distribution)

Let Y be a log-normal random variable. This means that

$$X = \log(Y) \sim \mathcal{N}(\mu, \sigma^2). \quad (252)$$

We have that the probability density function of X is the Gaussian probability density function:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\} \quad (253)$$

for any $x \in \mathbb{R}$.

Alternatively, Y has the probability density function:

$$f_Y(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(\ln(y) - \mu)^2}{2\sigma^2} \right\} \quad (254)$$

for $y > 0$.

Remark 219. To think about this, we say that Y is log-normally distributed if $\log(Y)$ is normally distributed.

Therefore, we can say that either $Y \sim \text{Lognormal}(\mu, \sigma^2)$ or $\log(Y) \sim \mathcal{N}(\mu, \sigma^2)$.

Proposition 220 (Moments of a Log-Normally Distributed Random Variable)

Let $X = \log(Y) \sim \mathcal{N}(\mu, \sigma^2)$, or equivalently $Y \sim \text{Lognormal}(\mu, \sigma^2)$. Then the first moment is:

$$\mathbb{E}[e^X] = \mathbb{E}[Y] = e^{\mu + \frac{1}{2}\sigma^2}. \quad (255)$$

The second moment is:

$$\mathbb{E}[e^{2X}] = \mathbb{E}[Y^2] = e^{2\mu + 2\sigma^2}. \quad (256)$$

The scaled first moment is:

$$\mathbb{E}[e^{-\gamma X}] = \mathbb{E}[Y^{-\gamma}] = e^{-\gamma\mu + \frac{1}{2}\gamma^2\sigma^2}. \quad (257)$$

10.3.3 The Equilibrium Risk-Free Rate under uncertainty

Now we add uncertainty back into the model. We have that the risk-free rate is the inverse of the stochastic discount factor:

$$R_{t+1}^f = \frac{1}{\mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \right]}$$

Look at the denominator:

$$\mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \right]$$

We can rewrite this by taking exponential and logs:

$$\beta \mathbb{E}_t \left[\exp \left(-\gamma \log \left\{ \frac{c_{t+1}}{c_t} \right\} \right) \right]$$

We now make a new assumption.

Assumption 221 (Consumption Growth is Log-Normal)

Consumption growth is log-normally distributed:

$$\log\left\{\frac{c_{t+1}}{c_t}\right\} \sim \mathcal{N}(\mu_t, \sigma_t^2) \quad (258)$$

We can then evaluate the expectation in the denominator.

Proposition 222 (Expectation of Consumption Growth)

Assume that consumption growth is log-normally distributed. Then, we have that:

$$\mathbb{E}_t \left[\exp \left(-\gamma \log \left\{ \frac{c_{t+1}}{c_t} \right\} \right) \right] = \exp \left(-\gamma \mu_t + 0.5 \gamma^2 \sigma_t^2 \right) \quad (259)$$

Proof. Recall that if $X = \log(Y) \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = \exp(X)$. Here, $X = \log \left\{ \frac{c_{t+1}}{c_t} \right\}$, so by our earlier discussion:

$$\mathbb{E}[e^{-\gamma X}] = \exp \left(-\gamma \mu + \frac{1}{2} \gamma^2 \sigma^2 \right).$$

□

We therefore have that

$$\beta \mathbb{E}_t \left[\exp \left(-\gamma \log \left\{ \frac{c_{t+1}}{c_t} \right\} \right) \right] = \beta \exp \left(-\gamma \mu_t + 0.5 \gamma^2 \sigma_t^2 \right)$$

Therefore, we have that the equilibrium risk-free rate is:

$$R_{t+1}^f = \frac{1}{\beta \exp \left(-\gamma \mu_t + 0.5 \gamma^2 \sigma_t^2 \right)}$$

We now take logs on both sides.

Definition 223 (Risk-Free Rate with Uncertainty)

Assuming a CRRA utility function, log-normally distributed consumption growth, and uncertainty in the model, we have that the risk-free rate is given by:

$$\log(R_{t+1}^f) = -\log(\beta) + \gamma \mu_t - 0.5 \gamma^2 \sigma_t^2 \quad (260)$$

whereby $\log \left\{ \frac{c_{t+1}}{c_t} \right\} \sim \mathcal{N}(\mu_t, \sigma_t^2)$.

We go back to our risk-free rate **without** uncertainty and compare the expression we had there:

$$R_{t+1}^f = \frac{1}{\beta} \left(\frac{c_{t+1}}{c_t} \right)^\gamma$$

We can take logs of both sides and make the approximation that consumption growth is equal to its mean in order to get:

$$\log(R_{t+1}^f) = -\log(\beta) + \gamma \mu_t$$

This is very similar to our expression for the risk-free rate with uncertainty:

$$\log(R_{t+1}^f) = -\log(\beta) + \gamma\mu_t - 0.5\gamma^2\sigma_t^2$$

except we have this new term $0.5\gamma^2\sigma_t^2$.

Definition 224 (Precautionary Savings)

The variance term:

$$0.5\gamma^2\sigma_t^2 \quad (261)$$

captures **precautionary savings**. When consumption is more volatile, people are more worried about future low consumption states and therefore they want to save more. This drives down interest rates.

10.4 Pricing Risky Assets in Terms of Consumption

We now look at the price and returns for risky assets. Recall our fundamental equation

$$p_t = \mathbb{E}_t[\mathcal{M}_{t,t+1}x_{t+1}]$$

then, using the formula for covariance $\text{cov}(x, y) = \mathbb{E}(xy) - \mathbb{E}[x]\mathbb{E}[y]$, we can rewrite our fundamental equation as

$$p_t = \mathbb{E}_t[\mathcal{M}_{t,t+1}]\mathbb{E}_t[x_{t+1}] + \text{Cov}(\mathcal{M}_{t,t+1}, x_{t+1})$$

Now, recall that the risk-free rate is the inverse of the expected stochastic discount factor

$$p_t = \frac{\mathbb{E}_t[x_{t+1}]}{R_{t+1}^f} + \text{Cov}(\mathcal{M}_{t,t+1}, x_{t+1})$$

Now, also substitute in the definition for the stochastic discount factor and we arrive at our pricing equation in terms of consumption.

Definition 225 (Pricing Equation in terms of consumption)

The price of an asset in terms of consumption and payoffs is given by

$$p_t = \frac{\mathbb{E}_t[x_{t+1}]}{R_{t+1}^f} + \frac{\text{Cov}(\beta u'(c_{t+1}), x_{t+1})}{u'(c_t)} \quad (262)$$

Remark 226. Recall that we can pull constants out of covariances $\text{cov}(ax, y) = a \text{cov}(x, y)$. Here, $1/u'(c_t)$ is a constant and therefore we can pull it outside of our covariance term.

This is a really important relationship and tells us a lot. We can see that the current marginal utility of consumption $u'(c_t)$ and the covariance of future marginal utility with the asset payoff $\text{Cov}(\beta u'(c_{t+1}), x_{t+1})$ affect the price of the asset. Let us see why.

1. We assume that the utility function is concave and has diminishing marginal utility. Therefore, as current consumption c_t increases, marginal utility $u'(c_t)$ decreases. For a procyclical payoff, whose covariance with future marginal utility is negative, this makes the negative risk adjustment larger in magnitude and lowers the asset price p_t .
2. Suppose that the asset has a high payoff x_{t+1} precisely when future consumption c_{t+1} is high. Concavity implies that $u'(c_{t+1})$ is then low, so the covariance between the payoff and marginal utility is negative, lowering the asset

price. Intuitively, an asset that pays off when you are already consuming a lot is not very useful as insurance, so its price must be lower to induce investors to hold it. If instead x_{t+1} negatively covaries with future consumption, it is a good form of insurance. It pays off when $u'(c_{t+1})$ is high, so the covariance term is positive and the price p_t increases.

We can also derive the pricing equation in terms of returns. Let us look at our basic pricing equation in terms of returns

$$1 = \mathbb{E}_t[\mathcal{M}_{t,t+1} R_{t+1}]$$

Apply the formula for covariance and using the fact that the risk-free rate is the inverse of the expected stochastic discount factor

$$1 = \mathbb{E}_t[\mathcal{M}_{t,t+1}] \mathbb{E}_t[R_{t+1}] + \text{Cov}(\mathcal{M}_{t,t+1}, R_{t+1})$$

$$\mathbb{E}[R_{t+1}] - R_{t+1}^f = -R_{t+1}^f \text{Cov}(\mathcal{M}_{t,t+1}, R_{t+1})$$

Definition 227 (Return Equation in terms of consumption)

The return of an asset in terms of consumption and payoffs is given by

$$\mathbb{E}[R_{t+1}] - R_{t+1}^f = -\frac{\text{Cov}[u'(c_{t+1}), R_{t+1}]}{\mathbb{E}[u'(c_{t+1})]} \quad (263)$$

This says that all assets have an expected return equal to the risk-free return plus a risk adjustment. Assets whose returns covary positively with consumption growth make consumption more volatile and hence must offer higher expected returns in order to induce investors to hold them.

11 Stochastic Lucas Tree Model

11.1 Introduction

We are interested in pricing an asset (the Lucas tree) which pays out an unknown dividend every period. As we shall soon see, what we price this asset is based on our intertemporal marginal rate of substitution, which comes in via the stochastic discount factor.

11.2 Solving the Stochastic Lucas Tree

We wish to maximise our expected present discounted utility. Let D_t be the dividend paid out in period t . Let z_t be the share of the tree you own in period t . The value of the shares we purchase for the next period (at the current market price) plus current consumption is constrained by the value of the shares we own plus the dividends we receive this period. We now state the problem mathematically.

Definition 228 (Stochastic Lucas Tree Problem)

Let c_t be consumption, z_t be the fraction of shares of the tree you own, P_t be the price of the tree, and D_t be the dividends generated all at time t . The stochastic Lucas tree problem is then the following problem:

$$\begin{aligned} \max_{\{c_t, z_{t+1}\}_{t=0}^{\infty}} \quad & \mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \\ \text{s.t.} \quad & P_t z_{t+1} + c_t \leq P_t z_t + D_t z_t \end{aligned} \quad (264)$$

Here, future dividends D_{t+j} for $j \geq 1$ are seen as random variables. Our consumption c_t and number of shares we purchase in the next period z_{t+1} are our control variables.

First, we set up the present-value Lagrangian

$$\mathcal{L} = \mathbb{E}_t \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] + \sum_{t=0}^{\infty} \mathbb{E}_t \left[\lambda_t [P_t z_t + D_t z_t - P_t z_{t+1} - c_t] \right] \quad (265)$$

We then take first order conditions

$$\frac{\partial \mathcal{L}}{\partial c_t} : \mathbb{E}_t [\beta^t u'(c_t) - \lambda_t] = 0 \quad (266)$$

$$\frac{\partial \mathcal{L}}{\partial z_{t+1}} : \mathbb{E}_t [\lambda_{t+1} P_{t+1} + \lambda_{t+1} D_{t+1} - \lambda_t P_t] = 0 \quad (267)$$

By equation Eq. (266), variables at time t are known at time t , therefore we can drop the expectations operator and conclude

$$\lambda_t = \beta^t u'(c_t) \quad (268)$$

$$\lambda_{t+1} = \beta^{t+1} u'(c_{t+1}) \quad (269)$$

Looking at Eq. (267), we can pull the term $\lambda_t P_t$ outside of the expectations operator (since it is known at time t) and move it to the other side:

$$\mathbb{E}_t [\lambda_{t+1} P_{t+1} + \lambda_{t+1} D_{t+1}] = \lambda_t P_t$$

We can then plug in our expressions for the current value Lagrange multipliers to get:

$$\mathbb{E}_t [\beta^{t+1} u'(c_{t+1}) P_{t+1} + \beta^{t+1} u'(c_{t+1}) D_{t+1}] = \beta^t u'(c_t) P_t$$

We can then divide through by $\beta^t P_t$ on both sides noting that we can push P_t inside the expectations operator since it is known at time t . Then, factorise by the marginal utility at time period $t + 1$ and arrive at the Stochastic euler equation.

Definition 229 (Stochastic Euler Equation (Consumption))

The stochastic Euler equation for the Stochastic Lucas tree is given by:

$$u'(c_t) = \beta \mathbb{E}_t \left[u'(c_{t+1}) \left(\frac{D_{t+1} + P_{t+1}}{P_t} \right) \right] \quad (270)$$

Under equilibrium, we may assume that the supply of shares of the trees in the economy is fixed (or that there is just one tree) and therefore $z_t = z_{t+1} = 1$. Furthermore, it can be shown that the agent will consume all the dividends at every period $c_t = D_t$.

Theorem 230 (Equilibrium level of consumption)

Under the assumption of inelastic supply of shares of trees $z_t = z_{t+1} = 1$, then consumption in every period will equal dividends produced every period

$$c_t = D_t$$

Proof. Look at the (binding) budget constraint

$$P_t z_{t+1} + c_t = P_t z_t + D_t z_t$$

Let $z_t = z_{t+1} = 1$

$$P_t + c_t = P_t + D_t$$

It immediately follows that

$$c_t = D_t$$

every period. □

Therefore, we could also switch out C_t for D_t to get a different expression for the Stochastic Euler equation:

Definition 231 (Stochastic Euler Equation (Dividends))

The stochastic Euler equation in terms of dividends for the Stochastic Lucas tree is given by:

$$u'(D_t) = \beta \mathbb{E}_t \left[u'(D_{t+1}) \left(\frac{D_{t+1} + P_{t+1}}{P_t} \right) \right] \quad (271)$$

Remark 232. *In equilibrium, there is no storage technology for transferring our dividends across time. This explains why we consume everything each period.*

We now want to find the price of the Stochastic Lucas tree. First, let us go back to our original Stochastic Euler equation:

$$u'(c_t) = \beta \mathbb{E}_t \left[u'(c_{t+1}) \left(\frac{D_{t+1} + P_{t+1}}{P_t} \right) \right]$$

Multiply out the P_t and expand out the brackets inside the expectations term

$$u'(c_t) P_t = \beta \mathbb{E}_t \left[u'(c_{t+1}) D_{t+1} \right] + \beta \mathbb{E}_t \left[u'(c_{t+1}) P_{t+1} \right] \quad (272)$$

This is a condition that holds across all time t . Therefore, we also know that by iterating forward one step

$$u'(c_{t+1})P_{t+1} = \beta \mathbb{E}_{t+1} [u'(c_{t+2})D_{t+2}] + \beta \mathbb{E}_{t+1} [u'(c_{t+2})P_{t+2}]$$

We can plug this expression into our expectations term on the right hand side of Eq. (272) to get:

$$u'(c_t)P_t = \beta \mathbb{E}_t [u'(c_{t+1})D_{t+1}] + \beta \mathbb{E}_t [\beta \mathbb{E}_{t+1} [u'(c_{t+2})D_{t+2}] + \beta \mathbb{E}_{t+1} [u'(c_{t+2})P_{t+2}]]$$

We will need to use a known result from probability theory.

Theorem 233 (Law of iterated expectations)

For a random variable known at time $t + 2$, X_{t+2} we have that

$$\mathbb{E}_t [\mathbb{E}_{t+1} [X_{t+2}]] = \mathbb{E}_t [X_{t+2}]$$

Remark 234. This says that today's forecast of tomorrow's forecast is today's forecast!

Therefore, we have that our earlier expression

$$u'(c_t)P_t = \beta \mathbb{E}_t [u'(c_{t+1})D_{t+1}] + \beta^2 \mathbb{E}_t [\mathbb{E}_{t+1} [u'(c_{t+2})D_{t+2}]] + \beta^2 \mathbb{E}_t [\mathbb{E}_{t+1} [u'(c_{t+2})P_{t+2}]]$$

will now become

$$u'(c_t)P_t = \beta \mathbb{E}_t [u'(c_{t+1})D_{t+1}] + \beta^2 \mathbb{E}_t [u'(c_{t+2})D_{t+2}] + \beta^2 \mathbb{E}_t [u'(c_{t+2})P_{t+2}]$$

We can repeat this again by subbing into the right hand side

$$u'(c_{t+2})P_{t+2} = \beta \mathbb{E}_{t+2} [u'(c_{t+3})D_{t+3}] + \beta \mathbb{E}_{t+2} [u'(c_{t+3})P_{t+3}]$$

and applying the law of iterated expectations to get:

$$\begin{aligned} u'(c_t)P_t &= \beta \mathbb{E}_t [u'(c_{t+1})D_{t+1}] + \beta^2 \mathbb{E}_t [u'(c_{t+2})D_{t+2}] \\ &\quad + \beta^3 \mathbb{E}_t [u'(c_{t+3})D_{t+3}] + \beta^3 \mathbb{E}_t [u'(c_{t+3})P_{t+3}] \end{aligned}$$

By induction, we can see that for any T iterations, we will get the expression:

$$u'(c_t)P_t = \sum_{s=1}^T \beta^s \mathbb{E}_t [u'(c_{t+s})D_{t+s}] + \beta^T \mathbb{E}_t [u'(c_{t+T})P_{t+T}] \quad (273)$$

The relevant **transversality condition** is

$$\lim_{T \rightarrow \infty} \mathbb{E}_t [\beta^T u'(c_{t+T})P_{t+T}] = 0.$$

Therefore, for Eq. (273), sending $T \rightarrow \infty$ and using the transversality condition, we end up with

$$u'(c_t)P_t = \sum_{s=1}^{\infty} \beta^s \mathbb{E}_t [u'(c_{t+s})D_{t+s}]$$

We can then divide through by $u'(c_t)$ on both sides, and push the β^s terms back inside the brackets to get

$$P_t = \sum_{s=1}^{\infty} \mathbb{E}_t \left[\beta^s \frac{u'(c_{t+s})}{u'(c_t)} D_{t+s} \right]$$

We have arrived at our desired result now.

Definition 235 (Price of Stochastic Lucas Tree)

The price of a Stochastic Lucas tree is

$$P_t = \sum_{s=1}^{\infty} \mathbb{E}_t \left[\beta^s \frac{u'(c_{t+s})}{u'(c_t)} D_{t+s} \right] \quad (274)$$

or equivalently

$$P_t = \sum_{s=1}^{\infty} \mathbb{E}_t \left[\mathcal{M}_{t,t+s} D_{t+s} \right] \quad (275)$$

where the **stochastic discount factor** is given by

$$\mathcal{M}_{t,t+s} = \beta^s \frac{u'(c_{t+s})}{u'(c_t)}$$

Remark 236. *The price of a stock equals the present discounted value of all future dividends! Here, the discount rate is the stochastic discount factor, which discounts future dividends based on the agents' intertemporal marginal rate of substitution!*

We can also view the effects of uncertainty on the pricing equation. Recall that

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] + \text{Cov}(X, Y)$$

. Apply this to the right hand side of

$$P_t = \sum_{s=1}^{\infty} \mathbb{E}_t \left[\beta^s \frac{u'(c_{t+s})}{u'(c_t)} D_{t+s} \right]$$

to get

$$P_t = \sum_{s=1}^{\infty} \mathbb{E}_t \left[\beta^s \frac{u'(c_{t+s})}{u'(c_t)} \right] \mathbb{E}_t[D_{t+s}] + \sum_{s=1}^{\infty} \text{Cov}_t \left(\beta^s \frac{u'(c_{t+s})}{u'(c_t)}, D_{t+s} \right)$$

We see that we can interpret the price as two components:

1. The first component is the present discounted value of dividends;
2. The second component is the sum of risk-adjustments on how does the asset act as an insurance. If the covariance is positive, then the asset pays out a high dividend D_t when marginal utility is high u' , which only occurs when consumption is low c_t due to the concavity of the utility function. Therefore, this causes the price of the asset to rise!

11.3 Lucas Tree With Growing Dividends

Let us consider a setup with log utility $u(c_t) = \log(c_t)$ in which markets clear (all dividends are consumed, so $D_t = c_t$) and dividends grow at a constant rate g :

$$D_{t+1} = (1 + g)D_t \quad (276)$$

We want to price this asset now. Go back to the Stochastic Euler equation for consumption we derived earlier:

$$u'(c_t) = \beta \mathbb{E}_t \left[u'(c_{t+1}) \left(\frac{D_{t+1} + P_{t+1}}{P_t} \right) \right]$$

Rewrite this as:

$$P_t = \mathbb{E}_t \left[\frac{\beta u'(c_{t+1})}{u'(c_t)} (D_{t+1} + P_{t+1}) \right]$$

By log-utility assumption, we have:

$$P_t = \mathbb{E}_t \left[\frac{\beta c_t}{c_{t+1}} (D_{t+1} + P_{t+1}) \right]$$

and then by market clearing, we have

$$P_t = \mathbb{E}_t \left[\frac{\beta D_t}{D_{t+1}} (D_{t+1} + P_{t+1}) \right]$$

We rewrite the discount factor as

$$\beta = \frac{1}{1 + \delta}$$

and plug in the growth rate of dividends $D_{t+1} = (1 + g)D_t$ to get:

$$\begin{aligned} P_t &= \mathbb{E}_t \left[\frac{\beta D_t}{D_{t+1}} (D_{t+1} + P_{t+1}) \right] \\ &= \mathbb{E}_t \left[\frac{\beta D_t D_{t+1}}{D_{t+1}} + \frac{\beta D_t P_{t+1}}{D_{t+1}} \right] \\ &= \frac{D_t}{1 + \delta} + \frac{1}{(1 + g)(1 + \delta)} \mathbb{E}_t [P_{t+1}] \end{aligned}$$

We therefore have that

$$P_t = \frac{D_t}{1 + \delta} + \frac{1}{(1 + g)(1 + \delta)} \mathbb{E}_t [P_{t+1}]$$

and iterating one step forward:

$$P_{t+1} = \frac{D_{t+1}}{1 + \delta} + \frac{1}{(1 + g)(1 + \delta)} \mathbb{E}_{t+1} [P_{t+2}]$$

We can combine these expressions to get:

$$P_t = \frac{D_t}{(1 + \delta)} + \frac{D_t}{(1 + \delta)^2} + \frac{1}{((1 + g)(1 + \delta))^2} \mathbb{E}_t [P_{t+2}]$$

We can repeat this procedure and get:

$$P_t = \frac{D_t}{(1 + \delta)} + \frac{D_t}{(1 + \delta)^2} + \frac{D_t}{(1 + \delta)^3} + \frac{1}{((1 + g)(1 + \delta))^3} \mathbb{E}_t [P_{t+3}]$$

We therefore have the general expression for T periods:

$$P_t = \sum_{s=1}^T \frac{D_t}{(1 + \delta)^s} + \frac{\mathbb{E}_t [P_{t+T}]}{((1 + g)(1 + \delta))^T}$$

By a no-bubble condition, we assume that

$$\lim_{T \rightarrow \infty} \frac{\mathbb{E}_t [P_{t+T}]}{((1 + g)(1 + \delta))^T} = 0$$

We therefore have the price for a Lucas tree with growing dividends under log-utility.

Proposition 237 (Price of Stochastic Lucas Tree with Growing Dividends Under log Utility)

The price for a Lucas tree with growing dividends under log-utility is given by:

$$P_t = \sum_{s=1}^{\infty} \frac{D_t}{(1+\delta)^s}. \quad (277)$$

11.4 Stochastic Lucas Tree Numerical Example

We now give a specific example. Let us choose the CRRA utility function

$$u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$

Furthermore, let the (random variable) dividends take on one of three possible values

$$D_t \in \{0.5, 1, 1.5\}$$

with a 0.5 probability measure that the dividend remains the same and a 0.25 probability measure that it switches to one of the other two dividends:

$$\pi_{i,j} = \begin{cases} 0.5 & \text{if } j = i \\ 0.25 & \text{for } j \neq i \end{cases}$$

We now look at our Stochastic Euler equation for our stochastic Lucas tree:

$$u'(c_t) = \beta \mathbb{E}_t \left[u'(c_{t+1}) \left(\frac{D_{t+1} + P_{t+1}}{P_t} \right) \right]$$

We then rewrite this and substitute in $c_t = D_t$ (under equilibrium assumptions)

$$u'(D_t)P_t = \beta \mathbb{E}_t \left[u'(D_{t+1}) (D_{t+1} + P_{t+1}) \right]$$

We know that

$$u'(D_t) = D_t^{-\gamma} \quad \text{and} \quad u'(D_{t+1}) = D_{t+1}^{-\gamma}$$

so we then have

$$\begin{aligned} D_t^{-\gamma} P_t &= \beta \mathbb{E}_t \left[D_{t+1}^{-\gamma} (D_{t+1} + P_{t+1}) \right] \\ \Leftrightarrow D_t^{-\gamma} P_t &= \beta \mathbb{E}_t \left[D_{t+1}^{1-\gamma} + D_{t+1}^{-\gamma} P_{t+1} \right] \end{aligned}$$

We can now price our stochastic Lucas tree based on the equation:

$$D_t^{-\gamma} P_t = \beta \mathbb{E}_t \left[D_{t+1}^{1-\gamma} + D_{t+1}^{-\gamma} P_{t+1} \right] \quad (278)$$

Let D^1, D^2, D^3 denote when dividends D_t are equal to either $\{0.5, 1, 1.5\}$ respectively. Compactly, this is

$$D_t \in \{D^1, D^2, D^3\} = \{0.5, 1, 1.5\}$$

Let P^1, P^2, P^3 denote the share prices when D_t equal D^1, D^2, D^3 respectively.

Furthermore, let $\pi_{1,1}, \pi_{1,2}, \pi_{1,3}$ denote the probability of the dividends either remaining the same (at 0.5), switching from 0.5 to 1, or switching from 0.5 to 1.5.

Let us examine the case where dividends are $D_t = D^1$. We then have that

$$(D^1)^{-\gamma} P^1 = \beta \mathbb{E}_t \left[(D^1)^{1-\gamma} + (D^1)^{-\gamma} P^1 \right]$$

We can then apply the expectation operator, whereby we multiply the term in the bracket by the respectively probabilities of switching to a different dividend

$$(D^1)^{-\gamma} P^1 = \beta \pi_{1,1} \left[(D^1)^{1-\gamma} + (D^1)^{-\gamma} P^1 \right] + \beta \pi_{1,2} \left[(D^2)^{1-\gamma} + (D^2)^{-\gamma} P^2 \right] + \beta \pi_{1,3} \left[(D^3)^{1-\gamma} + (D^3)^{-\gamma} P^3 \right]$$

We can do a similar line of reasoning for the other cases (D^2, P^2) and (D^3, P^3) respectively

$$(D^2)^{-\gamma} P^2 = \beta \pi_{2,1} \left[(D^1)^{1-\gamma} + (D^1)^{-\gamma} P^1 \right] + \beta \pi_{2,2} \left[(D^2)^{1-\gamma} + (D^2)^{-\gamma} P^2 \right] + \beta \pi_{2,3} \left[(D^3)^{1-\gamma} + (D^3)^{-\gamma} P^3 \right]$$

$$(D^3)^{-\gamma} P^3 = \beta \pi_{3,1} \left[(D^1)^{1-\gamma} + (D^1)^{-\gamma} P^1 \right] + \beta \pi_{3,2} \left[(D^2)^{1-\gamma} + (D^2)^{-\gamma} P^2 \right] + \beta \pi_{3,3} \left[(D^3)^{1-\gamma} + (D^3)^{-\gamma} P^3 \right]$$

Theorem 238 (Linear System of Pricing Equations for Stochastic Lucas Tree)

Therefore, we have that for our specific example of the Stochastic Lucas tree, the set of pricing equations

$$\begin{aligned} (D^1)^{-\gamma} P^1 &= \beta \pi_{1,1} \left[(D^1)^{1-\gamma} + (D^1)^{-\gamma} P^1 \right] + \beta \pi_{1,2} \left[(D^2)^{1-\gamma} + (D^2)^{-\gamma} P^2 \right] + \beta \pi_{1,3} \left[(D^3)^{1-\gamma} + (D^3)^{-\gamma} P^3 \right] \\ (D^2)^{-\gamma} P^2 &= \beta \pi_{2,1} \left[(D^1)^{1-\gamma} + (D^1)^{-\gamma} P^1 \right] + \beta \pi_{2,2} \left[(D^2)^{1-\gamma} + (D^2)^{-\gamma} P^2 \right] + \beta \pi_{2,3} \left[(D^3)^{1-\gamma} + (D^3)^{-\gamma} P^3 \right] \\ (D^3)^{-\gamma} P^3 &= \beta \pi_{3,1} \left[(D^1)^{1-\gamma} + (D^1)^{-\gamma} P^1 \right] + \beta \pi_{3,2} \left[(D^2)^{1-\gamma} + (D^2)^{-\gamma} P^2 \right] + \beta \pi_{3,3} \left[(D^3)^{1-\gamma} + (D^3)^{-\gamma} P^3 \right] \end{aligned}$$

Remark 239. Note that this is a linear system (in terms of prices)!!! This makes it easy to solve!

We now plug into our system of pricing equations the following values given to us

$$D^1 = 0.5$$

$$D^2 = 1$$

$$D^3 = 1.5$$

$$\pi_{i,j} = 0.5 \quad i = j$$

$$\pi_{i,j} = 0.25 \quad i \neq j$$

We therefore have the set of pricing equations:

$$\begin{aligned}(D^1)^{-\gamma} P^1 &= \beta 0.5 \left[(0.5)^{1-\gamma} + (0.5)^{-\gamma} P^1 \right] + \beta 0.25 \left[(1.0)^{1-\gamma} + (1.0)^{-\gamma} P^2 \right] + \beta 0.25 \left[(1.5)^{1-\gamma} + (1.5)^{-\gamma} P^3 \right] \\(D^2)^{-\gamma} P^2 &= \beta 0.25 \left[(0.5)^{1-\gamma} + (0.5)^{-\gamma} P^1 \right] + \beta 0.5 \left[(1.0)^{1-\gamma} + (1.0)^{-\gamma} P^2 \right] + \beta 0.25 \left[(1.5)^{1-\gamma} + (1.5)^{-\gamma} P^3 \right] \\(D^3)^{-\gamma} P^3 &= \beta 0.25 \left[(0.5)^{1-\gamma} + (0.5)^{-\gamma} P^1 \right] + \beta 0.25 \left[(1.0)^{1-\gamma} + (1.0)^{-\gamma} P^2 \right] + \beta 0.5 \left[(1.5)^{1-\gamma} + (1.5)^{-\gamma} P^3 \right]\end{aligned}$$

Now, we can set $\beta = 0.97$. Furthermore, we can vary the level of relative risk aversion γ across different values. Once we specify β and γ , we can solve the system of equations for P^1, P^2, P^3 .

For different levels of γ , with a $\beta = 0.97$, we have the following prices:

γ	P^1	P^2	P^3
0.5	16.5	23.5	28.8
1.0	12.0	24.0	36.0
2.0	7.4	29.3	65.6

To interpret this, the stock prices are more volatile (wider gap between P^1 and P^3) as we increase the coefficient of relative risk aversion γ . Furthermore, stock prices are procyclical as the higher the dividend, the higher the asset price as $P^3 > P^2 > P^1$.

12 Consumption-Based Capital Asset Pricing Model

12.1 Motivation

We continue our theme on being able to link macroeconomics to financial markets. In particular, we want to be able to price securities that allows us to transfer resources over time. We can price such assets using optimality conditions.

We will look at pricing 4 classes of assets:

1. Pricing bonds with a short maturity;
2. Pricing bonds with a long maturity;
3. Pricing housing stock;
4. Pricing stocks.

To clear up some financial jargon, a bond is a financial instrument that pays interest (coupons) until its maturity date, when it repays the principal amount. For example, you can buy a 10-year bond with a \$1,000 principal and a 10% annual coupon. This means that you receive \$100 every year for 10 years and, in the tenth year, also receive the \$1,000 principal. The maturity of such a bond is 10 years. Another term sometimes used for maturity is tenor.

Whilst we try to price these 4 classes of assets, we also end up stumbling across 4 notable concepts in the macro-finance literature:

1. Permanent income hypothesis;
2. Term premium puzzle;
3. Volatility puzzle;
4. Equity premium puzzle.

12.2 Pricing Short Tenor Bonds

We are interested in setting up the household portfolio problem whereby an agent needs to choose between consumption c_t , buying (real) bonds b_t , and buying stocks s_t in order to maximise their lifetime expected discounted utility subject to a budget flow constraint.

Fleshing this out more, there exists a one-period risk-free (government) bond b_t which pays a gross return r_t . The government can levy taxes T_t on the household in order to pay for their bonds. This is the short tenor bond we wish to eventually price. We also have a **tree** in the economy, which we can buy s_t shares of. The cost of buying shares of this tree is p_t and we get back dividends y_t proportional to the number of shares of the tree we own. Bringing all this together, we have that in every period, the agent will spend on consumption, buying bonds, and buying shares of the tree:

$$c_t + b_t + p_t s_t.$$

The income they receive is the principal plus interest on bonds, the dividends and share price appreciation of the tree less government taxes:

$$r_{t-1} b_{t-1} + (p_t + y_t) s_{t-1} - T_t.$$

Finally, the stochastic part of this model is the stream of dividends y_t that is paid out. With this, we can now state the household portfolio problem.

Definition 240 (Household Portfolio Problem)

The household portfolio problem is given by:

$$\begin{aligned} \max_{\{c_t, b_t, s_t\}_{t=0}^{\infty}} \quad & \mathbb{E}_t \sum_{i=0}^{\infty} \left[\beta^i u(c_{t+i}) \right] \\ \text{s.t.} \quad & c_t + b_t + p_t s_t = r_{t-1} b_{t-1} + (p_t + y_t) s_{t-1} - T_t \end{aligned} \quad (279)$$

Remark 241. Note that for now, we leave what is in the government taxation term T_t as undefined as we will later see, we can give an expression for this as the government's binding budget constraint.

12.2.1 Pricing Short Tenor Bonds: Deriving Pricing Equations

With our household portfolio problem, we now wish to derive the pricing equations. In order to do so, we require to also venture into how to use stochastic Lagrangians.

$$\begin{aligned} \max_{\{c_t, b_t, s_t\}_{t=0}^{\infty}} \quad & \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i u(c_{t+i}) \right] \\ \text{s.t.} \quad & c_t + b_t + p_t s_t = r_{t-1} b_{t-1} + (p_t + y_t) s_{t-1} - T_t \end{aligned} \quad (280)$$

The newest thing that we have now is the $\mathbb{E}_t$ operator, which can be interpreted as taking expectations with respect to the information set $\mathcal{I}_t$ (more on this later).

Just like for Hamiltonians, we can write our Lagrangians in two forms whereby we substitute our expression for T_t into the budget constraint.

Definition 242 (Present Valued Lagrangian)

The present-value Lagrangian is given by

$$\mathcal{L} = \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t) + \mathbb{E} \sum_{t=0}^{\infty} \lambda_t [r_{t-1} b_{t-1} + (p_t + y_t) s_{t-1} - T_t - c_t - b_t - p_t s_t] \quad (281)$$

Alternatively, we could have written the current-value Lagrangian.

Definition 243 (Current Valued Lagrangian)

$$\mathcal{L} = \mathbb{E} \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + \lambda_t [r_{t-1} b_{t-1} + (p_t + y_t) s_{t-1} - T_t - c_t - b_t - p_t s_t] \right\} \quad (282)$$

The difference between them is how we interpret the Lagrange multiplier λ_t . In the present-value Lagrangian, λ_t can be thought of as the marginal increase in the objective function $\sum_{t=0}^T \beta^t u(c_t)$ resulting from relaxing the constraint in period t . In the current-value Lagrangian, it has the same interpretation except that it measures the marginal increase in instantaneous utility $u(c_t)$.

We can use either method and the difference will be where your discount term β will appear (they will switch in which first order condition they appear in).

We will work with the Present Valued Lagrangian. My tip when taking first order conditions is to take them as you normally would and add an expectation operator $\mathbb{E}_t$ at the front!

Pricing Short Tenor Bonds: First Order Conditions

We can now state the three first order conditions:

Proposition 244 (First Order Conditions)

The first order conditions of the problem are given by:

$$\frac{\partial \mathcal{L}}{\partial c_t} : \mathbb{E}_t[\beta^t u'(c_t) - \lambda_t] = 0 \quad (283)$$

$$\frac{\partial \mathcal{L}}{\partial b_t} : \mathbb{E}_t[\lambda_{t+1} r_t - \lambda_t] = 0 \quad (284)$$

$$\frac{\partial \mathcal{L}}{\partial s_t} : \mathbb{E}_t[\lambda_{t+1}(p_{t+1} + y_{t+1}) - \lambda_t p_t] = 0 \quad (285)$$

Let us flesh out where Eq. (284) came from. We want to differentiate

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t) + \mathbb{E} \sum_{t=0}^{\infty} \lambda_t [r_{t-1} b_{t-1} + (p_t + y_t) s_{t-1} - T_t - c_t - b_t - p_t s_t]$$

with respect to b_t . We can see that we have a b_{t-1} term and a b_t on the right hand side of our expression. This immediately tells us that we will have a b_t term when $t = t + 1$ and when $t = t$. That is, we have that b_t will appear in two periods:

$$\lambda_t [r_{t-1} b_{t-1} + (p_t + y_t) s_{t-1} - T_t - c_t - b_t - p_t s_t] + \lambda_{t+1} [r_t b_t + (p_{t+1} + y_{t+1}) s_t - T_{t+1} - c_{t+1} - b_{t+1} - p_{t+1} s_{t+1}]$$

Differentiating this with respect to b_t and taking expectations $\mathbb{E}_t$ and setting it equal to 0 will yield us Eq. (284).

A similar argument holds for Eq. (285) whereby we can see that s_{t-1} and s_t appears on the right hand side of our Lagrangian. Therefore, we have that s_t will appear at $t = t$ and $t = t + 1$:

$$\lambda_t [r_{t-1} b_{t-1} + (p_t + y_t) s_{t-1} - T_t - c_t - b_t - p_t s_t] + \lambda_{t+1} [r_t b_t + (p_{t+1} + y_{t+1}) s_t - T_{t+1} - c_{t+1} - b_{t+1} - p_{t+1} s_{t+1}]$$

Differentiating this with respect to s_t and taking expectations $\mathbb{E}_t$ and setting it equal to 0 will yield us Eq. (285).

12.2.2 Pricing Short Tenor Bonds: Finding Pricing Equations Through The Euler Equations

We now inspect the first order condition Eq. (284):

$$\frac{\partial \mathcal{L}}{\partial b_t} : \mathbb{E}_t[\lambda_{t+1} r_t - \lambda_t] = 0$$

We state an important result we will need.

Theorem 245 (Conditional expectation of random variable)

The conditional expectation of a random variable that is known at time t is the random variable itself

$$\mathbb{E}_t[X_t] = \mathbb{E}[X_t | \mathcal{I}_t] = X_t \quad (286)$$

For those not familiar, $\mathbb{E}[X_t | \mathcal{I}_t]$ means to take expectation of X_t based on the information we know at time t . Intuitively, this makes sense! If we know what has happened at time t , then X_t is no longer *random* so its expectation is itself!

Going back to our first order equation Eq. (284), we can now take out λ_t as that will be known at time t

$$\mathbb{E}_t[\lambda_{t+1}r_t - \lambda_t] = 0 \quad (287)$$

$$\leftrightarrow \mathbb{E}_t[\lambda_{t+1}r_t] = \lambda_t \quad (288)$$

We now visit our other first order equation Eq. (283) and re-arrange too

$$\mathbb{E}_t[\beta^t u'(c_t) - \lambda_t] = 0 \quad (289)$$

$$\leftrightarrow \mathbb{E}_t[\beta^t u'(c_t)] = \lambda_t \quad (290)$$

$$\beta^t u'(c_t) = \lambda_t \quad (291)$$

where we used the fact that c_t and λ_t are known at time t . We can also iterate this equation one step forward

$$\mathbb{E}_{t+1}[\beta^{t+1} u'(c_{t+1}) - \lambda_{t+1}]$$

and using our rule on expectation

$$\lambda_{t+1} = \beta^{t+1} u'(c_{t+1}) \quad (292)$$

Going back to Eq. (288)

$$\mathbb{E}_t[\lambda_{t+1}r_t] = \lambda_t$$

we now plug in Eq. (291) and Eq. (292)

$$\mathbb{E}_t[\beta^{t+1} u'(c_{t+1})r_t] = \beta^t u'(c_t)$$

The final trick we can do is divide through by β^t and $u'(c_t)$ (which we can do as one is a constant and the other is known at time t) in order to get:

$$\mathbb{E}_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} r_t\right] = 1 \quad (293)$$

This is our first Euler equation and this is actually the **bond pricing equation**.

We can now do something similar to solve for the other pricing equation. Looking at Eq. (285), we have that

$$\mathbb{E}_t[\lambda_{t+1}(p_{t+1} + y_{t+1}) - \lambda_t p_t] = 0 \quad (294)$$

$$\mathbb{E}_t[\lambda_{t+1}(p_{t+1} + y_{t+1})] = \lambda_t p_t \quad (295)$$

$$\mathbb{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t} \frac{p_{t+1} + y_{t+1}}{p_t}\right] = 1 \quad (296)$$

$$(297)$$

Plugging in our expression Eq. (291) and Eq. (292), we arrive at the second Euler equation which is the **stock pricing equation**:

$$\mathbb{E}_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \frac{p_{t+1} + y_{t+1}}{p_t}\right] = 1 \quad (298)$$

We therefore summarise the two pricing equations we have found.

Proposition 246 (Pricing Equations)

The bond pricing equation is given by

$$1 = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} r_t \right]. \quad (299)$$

The stock pricing equation is given by

$$1 = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \frac{p_{t+1} + y_{t+1}}{p_t} \right]. \quad (300)$$

Remark 247. *You may have noticed that we did not touch our first FOC Eq. (283). We shall come back to this FOC later on!*

Let us now look at our bond pricing equation and rewrite it as:

$$\frac{1}{r_t} = \mathbb{E}_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \right].$$

As we saw earlier, the term inside the expectation is actually the stochastic discount factor $\mathcal{M}_{t,t+1}$ so we therefore have that

$$\frac{1}{r_t} = \mathbb{E}_t [\mathcal{M}_{t,t+1}]$$

The bond pricing equation states that the inverse of the interest on the bond is the stochastic discount factor, which is the expectation of the discounted ratio of marginal utilities. A rough explanation to the stochastic discount factor is that it looks at how much we value future consumption relative to current consumption. You may wonder why we are looking at the inverse of the bond interest rate but there is a good finance intuition for this. **The price of a bond has an inverse relation to interest rates.** That is, as interest rates go up, the price of a bond falls and vice versa. Therefore, we could also write our equation as

$$P_t^B(r_t) = \mathbb{E}_t [\mathcal{M}_{t,t+1}]$$

where P_t^B is the price of our government bond as a function of the bond's interest rate via an inverse relationship.

We can also look at our stock pricing equation

$$1 = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \frac{p_{t+1} + y_{t+1}}{p_t} \right].$$

We can push the β inside and notice that we now have the stochastic discount factor $\mathcal{M}_{t,t+1}$ again. Furthermore, we can also interpret the second term as the return on a stock

$$r_t^e = \frac{p_{t+1} + y_{t+1}}{p_t}$$

which means we can write our stock pricing equation as

$$1 = \mathbb{E}_t [\mathcal{M}_{t,t+1} r_t^e]$$

This is just our standard **fundamental asset pricing equation** in disguise again! See section 10 for more on this.

12.2.3 Pricing Short Tenor Bonds: Equilibrium Level of Consumption

So far, the FOCs we have derived—Eq. (283), Eq. (284), and Eq. (285)—still contain undetermined c_t terms. We now use the equilibrium conditions to determine them.

The first equilibrium condition that we will have is that the government will face a binding budget constraint. At every period t , the government needs to pay back $r_{t-1}b_{t-1}$. They have two options to pay this back: they can issue new bonds b_t or they can levy taxes T_t . Hence, the government has the budget constraint

$$r_{t-1}b_{t-1} \leq b_t + T_t$$

Using a bit hand-waving, we can say that in equilibrium, this budget constraint will bind and we therefore have that

$$r_{t-1}b_{t-1} = b_t + T_t$$

We can then re-arrange this so that we have an expression for government taxation:

$$T_t = r_{t-1}b_{t-1} - b_t$$

Theorem 248 (Equilibrium level of government taxation)

In equilibrium, the level of government taxation is given by:

$$T_t = r_{t-1}b_{t-1} - b_t. \quad (301)$$

We also state another assumption that we have been implicitly making within this model regarding the supply of stocks.

Assumption 249 (Supply of tree stocks)

The supply of stocks is fixed in this economy

$$s_t = 1, \quad (302)$$

for all time periods t .

With these two new additional results, we can now derive the equilibrium level of consumption in this economy.

Theorem 250 (Equilibrium level of consumption)

The equilibrium level of consumption is:

$$c_t = y_t. \quad (303)$$

Proof. First, recall the household's budget constraint:

$$c_t + b_t + p_t s_t = r_{t-1}b_{t-1} + (p_t + y_t)s_{t-1} - T_t$$

We know that in equilibrium, the government's budget constraint binds so we can plug in our expression for T_t and cancel out terms.

$$\begin{aligned} c_t + b_t + p_t s_t &= r_{t-1}b_{t-1} + (p_t + y_t)s_{t-1} - T_t \\ \Leftrightarrow c_t + b_t + p_t s_t &= r_{t-1}b_{t-1} + (p_t + y_t)s_{t-1} - (r_{t-1}b_{t-1} - b_t) \\ \Leftrightarrow c_t + p_t s_t &= p_t s_{t-1} + y_t s_{t-1} \end{aligned}$$

We now use the fact that the stock supply is always fixed at 1 to have that $s_{t-1} = s_t = 1$

$$\begin{aligned} c_t + p_t s_t &= p_t s_{t-1} + y_t s_{t-1} \\ \Leftrightarrow c_t + p_t(1) &= p_t(1) + y_t(1) \end{aligned}$$

Therefore, we can conclude that

$$c_t = y_t.$$

□

Therefore, in equilibrium, consumption is equal to dividends. The only available consumption good in this economy is the dividend payment and there is no transfer technology. This result will be very useful for the next section when we do some empirical work.

12.2.4 Pricing Short Tenor Bonds: Permanent Income Hypothesis

In the Hall (JPE, 1978) paper, Hall turns the FOCs on their head. That is, instead of using consumption to derive interest rates, Hall observes real-world interest rates and asks whether they are consistent with the consumption dynamics implied by our FOCs. In order to do so, we need to make some further assumptions.

Assumption 251 (Constant Short Term Interest Rates)

We assume that the risk-free interest rate is constant:

$$r_t = r. \quad (304)$$

Recall that in the previous section, we have derived the bond-pricing equation to be:

$$\frac{1}{r_t} = \mathbb{E}_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \right].$$

Recall that the issue was that we do not know what c_t was. However, we have shown that the equilibrium level of consumption is:

$$c_t = y_t$$

We can therefore plug in y_t to get:

$$\frac{1}{r_t} = \mathbb{E}_t \left[\beta \frac{u'(y_{t+1})}{u'(y_t)} \right].$$

Now, we can invoke the assumption that $r_t = r$ is a constant and we can then re-arrange our equation to get:

$$r = \frac{1}{\beta \mathbb{E}_t \left[\frac{u'(y_{t+1})}{u'(y_t)} \right]}$$

We now impose another assumption on this.

Assumption 252 (Ratio of marginal utilities of income is constant)

The ratio of marginal utilities is 1, i.e. $\frac{u'(y_{t+1})}{u'(y_t)} = 1$. This implies that

$$r = \frac{1}{\beta} \quad (305)$$

Moving on, we now impose an assumption on our utility function.

Assumption 253 (Quadratic Utility Function)

We assume that the utility function is a quadratic:

$$u(c_t) = -\frac{1}{2}(\bar{c} - c_t)^2 \quad (306)$$

where we assume that $c_t < \bar{c}$ so that marginal utility $u' > 0$.

Remark 254. One way to think of this is that if we had an utility function $u(\cdot)$, we can take a second-order Taylor approximation of it around the point $\bar{c}$ and only keep the second order term in order to get:

$$u(c_t) \approx u(\bar{c}) + u'(\bar{c})(\bar{c} - c_t) + \frac{1}{2}u''(\bar{c})(\bar{c} - c_t)^2.$$

Now, go back to the bond pricing equation that we have derived earlier in the last section:

$$1 = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} r_t \right]$$

Substitute in $r_t = r$ and re-write this as

$$\mathbb{E}_t[u'(c_{t+1})] = \frac{u'(c_t)}{\beta r}$$

From our assumption of the quadratic utility function, we have:

$$u(c_t) = -\frac{1}{2}(\bar{c} - c_t)^2 \Rightarrow u'(c_t) = \bar{c} - c_t$$

Plugging this into u' , we then arrive at:

Definition 255 (New Bond Pricing Equation)

Our modified bond-pricing equation is now:

$$\mathbb{E}_t[\bar{c} - c_{t+1}] = \frac{(\bar{c} - c_t)}{\beta r} \quad (307)$$

We now want to turn this new bond-pricing equation into something we can test empirically. First, rearrange it to obtain

$$\mathbb{E}_t[c_{t+1}] = \bar{c} \left(1 - \frac{1}{\beta r} \right) + \frac{c_t}{\beta r}.$$

We can think of the left hand side of the above equation as what is our forecast for the variable c_{t+1} at time t . With that interpretation, we can then say that our forecast for a variable is the actual realised value for that variable plus some forecast error:

$$c_{t+1} = \mathbb{E}_t[c_{t+1}] + \epsilon_{t+1}$$

where ϵ_{t+1} is the forecast error. That is, ϵ_{t+1} is the part that cannot be predicted using the information available at time t . Equivalently, ϵ_{t+1} is orthogonal to all variables in the time- t information set. Substituting the conditional expectation above gives:

$$c_{t+1} = \bar{c} \left(1 - \frac{1}{\beta r} \right) + \frac{c_t}{\beta r} + \epsilon_{t+1}.$$

We now write this as a statistical model. Define the intercept $\beta_0 = \bar{c}(1 - 1/(\beta r))$ and the slope coefficient $\beta_1 = \frac{1}{\beta r}$ to get

$$c_{t+1} = \beta_0 + \beta_1 c_t + \epsilon_{t+1}$$

This is a very famous result which we will define.

Definition 256 (Random Walk Hypothesis of Consumption)

The random walk hypothesis of consumption is:

$$c_{t+1} = \beta_0 + \beta_1 c_t + \epsilon_{t+1} \quad (308)$$

Remark 257. *This equation says that future consumption is a function of current consumption plus a deterministic growth trend plus an error term.*

We can now state the hypothesis test we wish to conduct on our random walk hypothesis of consumption equation.

Proposition 258 (Hypothesis Testing the Random Walk Hypothesis of Consumption)

The null hypothesis for testing the random walk hypothesis of consumption is

$$H_0 : \beta_1 = 1 \quad (309)$$

Proof. Recall that $\beta_1 = \frac{1}{\beta r}$. Furthermore, recall that under general equilibrium conditions, we said that

$$r = \frac{1}{\beta}$$

Therefore, we have that

$$\frac{1}{\beta r} = 1$$

under general equilibrium. □

Bringing all this together, we are saying that if the general equilibrium conditions of our theory holds, then β_1 should be equal to 1. If this is indeed the case, then future consumption is

$$c_{t+1} = \beta_0 + c_t + \epsilon_{t+1}$$

which is a **random walk with a drift term**. That is, our future consumption c_{t+1} is our current consumption c_t plus a growth trend β_0 (either increasing or decreasing every period) and an error term ϵ_{t+1} !

So now looking at the Hall paper, he uses a CRRA utility function and have the equation

$$c_t^{-1/\sigma} = \gamma c_{t-1}^{-1/\sigma} + \epsilon_t$$

TABLE 1
REGRESSION RESULTS FOR THE BASIC MODEL, 1948–77
 $c_t^{-1/\sigma} = \gamma c_{t-1}^{-1/\sigma} + \varepsilon_t$

Equation	σ	Constant	γ	SE	R^2	D-W Statistic
1.0	.2	...	.983 (.003)	.000735	.9964	2.06
1.2	1.0	...	.996 (.001)	.00271	.9985	1.83
1.3	−1.0	−.014	1.011 (.003)	.0146	.9988	1.70

NOTE.—The numbers in parentheses in these and subsequent regressions are standard errors.

We see that the slope coefficient γ is close to 1 and not statistically different from one. Therefore, it seems like we fail to reject the null hypothesis that $H_0 : \gamma = 1$. Therefore, this is good news for our theory of consumption being a random walk!!!

So why do we get a random walk in consumption? Suppose we face a fixed interest rate. We wish to smooth consumption because marginal utility is diminishing. Therefore, whatever news we receive about the present value of lifetime income, we try to spread its effect on consumption across our lifetime. Hence, shocks to consumption arise from news about the expected value of lifetime income.

12.2.5 Pricing Short Tenor Bonds: Campbell and Mankiw (1990)

The test by Hall was not *powerful*, that is, for technical reasons, the probability that the test of rejecting a false null hypothesis H_0 was low. This is due to the fact that there will be issues with the test statistic that he employed (the Durbin-Watson statistic) if there is endogeneity in the lag variables. The takeaway is that we should be suspicious of failing to reject the null hypothesis that $H_0 : \beta = 1$.

With this background, Campbell and Mankiw (JBES, 1990) proposed that current income should be a good prediction of consumption growth. We now give the setup of the model.

Suppose that there are two types of consumers. A fraction λ live *hand to mouth*: if they receive any additional income, they consume it all. The other consumers engage in consumption smoothing, as we saw in the last section.

We denote the fraction of the population who consume *hand to mouth* as λ . We can then write that these agents consume c_{1t} all their income y_{1t} every period and we can write this as a fraction of aggregate income y_t :

$$c_{1t} = y_{1t} = \lambda y_t \quad (310)$$

Writing this in growth terms, we subtract $c_{1,t-1}$ on both sides to have

$$\Delta c_{1t} = \Delta y_{1t} = \lambda \Delta y_t \quad (311)$$

For the agents who smooth their consumption, recall that their consumption is a random walk:

$$c_{2t} = \beta_0 + \beta_1 c_{2,t-1} + \epsilon_t \quad (312)$$

We then subtract $c_{2,t-1}$ on both sides to get it in growth terms and arrive at

$$\Delta c_{2t} = \beta_0 + \epsilon_t \quad (313)$$

We can then define aggregate consumption as the sum of consumptions of the two agents:

$$c_t = c_{1t} + c_{2t}$$

Therefore, we can define the aggregate consumption regression.

Definition 259 (Aggregate Consumption Regression)

The aggregate consumption regression is

$$\Delta c_t = \beta_0 + \lambda \Delta y_t + \epsilon_t. \quad (314)$$

Proof. Add the two equations Eq. (311) and Eq. (313) together. □

We can then test whether the coefficient λ equals zero. If a significant fraction of agents consume hand to mouth ($\lambda \neq 0$), then a statistically significant fraction of agents do not smooth consumption. If we fail to reject $\lambda = 0$, this is consistent with all agents being consumption smoothers, as in Hall's original formulation, and therefore with the random-walk hypothesis of consumption.

Definition 260 (Hypothesis Testing of Generalised Consumption Regression)

The hypothesis test for the **generalized consumption regression** is

$$H_0 : \lambda = 0.$$

A big issue is that we cannot estimate β_0 and λ through OLS. The reason is that ϵ_t is orthogonal to variables at **t-1** and earlier but not to variables at time **t**. Therefore, we do not have exogeneity of the error term and regressor as we do not have that:

$$\text{Cov}(\epsilon_t, \Delta y_t) \neq 0.$$

To deal with this, we need to use an **instrumental-variable regression**. We do not need the details here, but we will give a high-level overview. We can follow the procedure covered in the Econometrics course. In particular, we can run an auxiliary regression

$$\Delta y_t = \gamma_0 + \gamma_1 z_{1,t-1} + \gamma_2 z_{2,t-1} + u_t \quad (315)$$

where Δy_t is our endogenous variable and $z_{1,t-1}$ and $z_{2,t-1}$ are instruments that are informative about Δy_t (they covary with it) **and** are uncorrelated with the structural consumption error ϵ_t . We can then retrieve the fitted value $\Delta \hat{y}_t$ and run our original regression with $\Delta \hat{y}_t$ as the regressor. In the paper, lagged changes in income and consumption, Δy_{t-1-j} and Δc_{t-1-j} for $j \geq 1$, were used as instruments. We could also regress consumption growth on the instruments:

$$\Delta c_t = \gamma_0 + \gamma_1 z_{1,t-1} + \gamma_2 z_{2,t-1} + u_t \quad (316)$$

We now look at the result of the regressions in the table. The first line shows the results from doing an ordinary OLS

regression. The subsequent lines shows the results for doing regressions using a different combination of instrumental variables.

Table 2. Basic Permanent-Income Model (scaled levels)

Row	Instruments (Z)	OLS regressions on Z		λ estimate (standard error)	Test of restrictions
		ΔC equation	ΔY equation		
1	None (OLS)	—	—	.296 (.044)	—
2	$\Delta Y_{t-2}, \dots, \Delta Y_{t-4}$	-.005 (.210)	.021 (.013)	.330 (.161)	-.023 (.986)
3	$\Delta Y_{t-2}, \dots, \Delta Y_{t-6}$	.020 (.109)	.042 (.022)	.422 (.141)	-.034 (.984)
4	$\Delta C_{t-2}, \dots, \Delta C_{t-4}$	.018 (.124)	.048 (.009)	.361 (.126)	-.014 (.526)
5	$\Delta C_{t-2}, \dots, \Delta C_{t-6}$	.084 (.001)	.082 (.024)	.476 (.130)	-.016 (1.000)
6	$\Delta i_{t-2}, \dots, \Delta i_{t-4}$	.060 (.000)	.025 (.005)	.637 (.144)	-.015 (.591)
7	$\Delta i_{t-2}, \dots, \Delta i_{t-6}$	.118 (.000)	.086 (.001)	.552 (.140)	-.026 (.516)
8	$\Delta Y_{t-2}, \dots, \Delta Y_{t-4},$ $\Delta C_{t-2}, \dots, \Delta C_{t-4},$ S_{t-2}	.007 (.239)	.084 (.000)	.317 (.110)	-.042 (.909)
9	$\Delta Y_{t-2}, \dots, \Delta Y_{t-4},$ $\Delta C_{t-2}, \dots, \Delta C_{t-4},$ $\Delta i_{t-2}, \dots, \Delta i_{t-4}$ S_{t-2}	.075 (.000)	.110 (.000)	.410 (.087)	-.035 (.159)

Looking at the column titled " λ estimate", this is what we are interested in, as it estimates λ from the data. Recall that the null hypothesis $H_0 : \lambda = 0$ implies that all agents are consumption smoothers and that the random-walk hypothesis holds. The first row shows that a naive OLS regression gives $\lambda = 0.3$, meaning that 30% of agents live hand to mouth and consume their income immediately. With different instrumental variables, the estimates of λ are even higher than 30%, with some estimates near 50%. This implies that the OLS estimate of λ is downward biased because of the endogeneity of Δy_t . Thus, an even larger fraction λ of agents **do not** smooth their consumption.

The columns labelled "OLS regressions on Z" report coefficients from the auxiliary regressions in Eq. (315) and Eq. (316). We see that the coefficients are statistically different from zero, which suggests that the instruments are informative.

To summarise, we have shown in this paper that a substantial fraction of agents do not engage in consumption smoothing and in fact live hand to mouth!

12.3 Pricing Long Tenor Bonds

We are now interested in adding in a long tenor bond into our model. Suppose that we only have two financial assets now: a short-term risk-free bond b_t that matures in one period and a long-term risk-free bond $b_{L,t}$ that matures in L periods. We make the assumption that the long-term bond cannot be traded in the secondary market. That is, once you buy a 10-year bond, you MUST hold it for 10-years and cannot sell it to anyone else in the interim. We denote $r_{L,t}$ to be the **per-period** return on the long-term bond at time t. That is, you get a return of $r_{L,t}$ every period for the L periods you hold the long-term bond. Therefore, we can write this as $(r_{L,t-L})^L b_{L,t-L}$ as this is the total return you earned for holding an L-period bond that you bought L periods ago.

We can now set up our household portfolio problem:

Definition 261 (Household Portfolio Problem: Long-Term Bonds)

The household portfolio problem is given by:

$$\begin{aligned} \max_{\{c_t, b_t, b_{L,t}\}_{t=0}^{\infty}} \quad & \mathbb{E}_t \sum_{i=0}^{\infty} \left[\beta^i u(c_{t+i}) \right] \\ \text{s.t.} \quad & c_t + b_t + b_{L,t} = r_{t-1} b_{t-1} + (r_{L,t-L})^L b_{L,t-L} + y_t \end{aligned} \quad (317)$$

Remark 262. Note that the long term bond on the left hand side only shows up when it has matured since we cannot trade these bonds before they mature.

We now set up our Lagrangian:

$$\mathcal{L} = \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i u(c_{t+i}) + \lambda_t \left[r_{t-1} b_{t-1} + (r_{L,t-L})^L b_{L,t-L} + y_t - c_t - b_t - b_{L,t} \right] \right] \quad (318)$$

Proposition 263 (First Order Conditions)

The first-order conditions of Eq. (318) are given by:

$$\frac{\partial \mathcal{L}}{\partial c_t} : \mathbb{E}_t \left[\beta^t u'(c_t) - \lambda_t \right] = 0 \quad (319)$$

$$\frac{\partial \mathcal{L}}{\partial b_t} : \mathbb{E}_t \left[\lambda_{t+1} r_t - \lambda_t \right] = 0 \quad (320)$$

$$\frac{\partial \mathcal{L}}{\partial b_{L,t}} : \mathbb{E}_t \left[\lambda_{t+L} (r_{L,t})^L - \lambda_t \right] = 0 \quad (321)$$

We can now state our first bond-pricing equation.

Proposition 264 (Short-Bond Pricing Equation)

The short-bond pricing equation is

$$1 = \beta r_t \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right] \quad (322)$$

Proof. We can rewrite Eq. (319) as

$$\lambda_t = \beta^t u'(c_t)$$

and iterate forward to get:

$$\lambda_{t+1} = \beta^{t+1} u'(c_{t+1}).$$

We also rewrite Eq. (320) as

$$\mathbb{E}_t \left[\lambda_{t+1} r_t \right] = \lambda_t$$

and plugging in our Lagrange multipliers as:

$$\mathbb{E}_t \left[\beta^{t+1} u'(c_{t+1}) r_t \right] = \beta^t u'(c_t)$$

We then cancel out the β terms and pull r_t out of the expectation operator as it is known at time t to get our desired

result:

$$1 = \beta r_t \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right]$$

□

We can now state our other bond-pricing equation.

Proposition 265 (Long-Bond Pricing Equation)

The long-bond pricing equation is

$$1 = \beta^L (r_{L,t})^L \mathbb{E}_t \left[\frac{u'(c_{t+L})}{u'(c_t)} \right] \quad (323)$$

Proof. We can rewrite Eq. (319) as

$$\lambda_t = \beta^t u'(c_t)$$

and iterate forward to get:

$$\lambda_{t+L} = \beta^{t+L} u'(c_{t+L}).$$

Plug these expressions into our third FOC Eq. (321) to get

$$\mathbb{E}_t \left[\beta^{t+L} u'(c_{t+L}) (r_{L,t})^L \right] = \beta^t u'(c_t).$$

Divide through and re-arrange to get our desired result.

□

Remark 266. *An interesting thing to note about our long-bond pricing equation is that it looks at the marginal utility between when we purchase the bond at time t against the marginal utility for at when it matures L periods from now.*

We now want to derive a relationship between the short-term and long-term bonds and their interest rates. The gap between the short-term and long-term interest rates is known as the **term premium**. A very interesting observation is that in bond markets, **the per period interest of long-term bonds is higher than the short-term bond**. This is known as the **term premium puzzle**.

Definition 267 (Term Premium Puzzle)

The per period interest rate of long-term bonds is higher than the one period return of short-term bonds.

We now wish to investigate this by imposing some models.

Denote $\lambda_t \equiv u'(c_t)$ to be the marginal utility (this is **not** a Lagrange multiplier!). Looking at our two bond pricing equations

$$1 = \beta r_t \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right]$$

$$1 = \beta^L (r_{L,t})^L \mathbb{E}_t \left[\frac{u'(c_{t+L})}{u'(c_t)} \right]$$

We can log-linearise them around the points $\beta r = 1$ and $\beta^L r_L^L = 1$ (do not worry too much about this) to get:

$$\hat{\lambda}_t = \hat{r}_t + \mathbb{E}_t \hat{\lambda}_{t+1} \quad (324)$$

$$\hat{\lambda}_t = L \hat{r}_{L,t} + \mathbb{E}_t \hat{\lambda}_{t+L} \quad (325)$$

The hats on our variables tell us that the variables are actually in logarithmic form i.e. $\hat{\lambda}_t = \log(\lambda_t)$. What these equations are telling us is that the marginal utility today is equal to the short term interest rate plus the expected marginal utility of the future.

Looking at our short-term bond Euler equation Eq. (324), we can shift it forward one period to get:

$$\hat{\lambda}_{t+1} = \hat{r}_{t+1} + \mathbb{E}_{t+1} \hat{\lambda}_{t+2}$$

We then plug this back into Eq. (324) to get:

$$\hat{\lambda}_t = \hat{r}_t + \mathbb{E}_t \hat{r}_{t+1} + \mathbb{E}_t \hat{\lambda}_{t+2}$$

where we used the fact that $\mathbb{E}_t [\mathbb{E}_{t+1} \hat{\lambda}_{t+2}] = \mathbb{E}_t \hat{\lambda}_{t+2}$.

We repeat this iterative procedure for $L - 1$ periods to get

$$\hat{\lambda}_t = \mathbb{E}_t \left(\sum_{j=0}^{L-1} \hat{r}_{t+j} \right) + \mathbb{E}_t \hat{\lambda}_{t+L} \quad (326)$$

This has a nice interpretation. This says that the marginal utility of today is equal to the sum of expected short term interest rates as every period plus the expected future marginal utility L periods from now.

We now replace the left hand side of Eq. (325) with what we just derived to get our desired result.

Definition 268 (Pure Expectation Hypothesis of Interest Rates)

The pure expectation hypothesis of interest rates for a bond that matures in L periods is

$$\hat{r}_{L,t} = \frac{1}{L} \mathbb{E}_t \left(\sum_{j=0}^{L-1} \hat{r}_{t+j} \right). \quad (327)$$

Remark 269. *This has a useful interpretation: the long-term interest rate is the expected average of short-term interest rates over the life of the bond.*

The issue though is that in the data, this is sadly not the case and that there should be a term premium as there is usually a difference in short-term rates and the long-term rate. We fix this by adding in a premium term u_t to get:

Definition 270 (Pure Expectation Hypothesis of Interest Rates with a Term Premium)

The pure expectation hypothesis of interest rates for a bond that matures in L periods is

$$\hat{r}_{L,t} = \frac{1}{L} \mathbb{E}_t \left(\sum_{j=0}^{L-1} \hat{r}_{t+j} \right) + u_t. \quad (328)$$

Here, the term u_t measures the **term premium**. This measures the **excess compensation for holding long-term bonds** instead of a series of short-term bonds. One good intuition for why this is the case is that by holding a series of short-term bonds, you have more convenience as you can decide what to do with your investment every period whereas it is not attractive if you are forced to hold a bond for L periods without being able to sell it in the interim. As a result, investors demand a higher return on these long-term bonds. We can then test the term premium by doing hypothesis testing on u_t . However, u_t is unobserved so we can do some fancy tricks like taking expectations which we need not concern ourselves!

12.4 Pricing Housing Stock

We now look at a different asset: **housing**. The key difference is that housing is a **real durable** good. We denote h_t to be the square meters of the house we own and this provides us a service flow utility. We denote q_t to be the price of a square meter of the house. Note that as we do not have money in this model, q_t is measured in *consumption terms*, that is, $q_t = 1$ means the price is 1 unit of consumption. The housing stock we own can appreciate in value every period and hence we can treat this as income. Additionally, we can also buy short-term bonds b_t in this model. We finally have that we receive an exogenous income w_t at every period. We can then write our household optimisation program:

Definition 271 (Household Portfolio Problem: Housing)

The household portfolio problem is given by:

$$\begin{aligned} \max_{\{c_t, b_t, h_t\}_{t=0}^{\infty}} \quad & \mathbb{E}_t \sum_{i=0}^{\infty} \left[\beta^i u(c_{t+i}, h_{t+i}) \right] \\ \text{s.t.} \quad & c_t + b_t + q_t h_t = r_{t-1} b_{t-1} + q_t h_{t-1} + w_t \end{aligned} \quad (329)$$

We can write out the present-value Lagrangian:

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \left[\beta^t u(c_t, h_t) + \lambda_t \left\{ r_{t-1} b_{t-1} + q_t h_{t-1} + w_t - c_t - b_t - q_t h_t \right\} \right]. \quad (330)$$

We can then take the first order condition with respect to c_t :

$$\frac{\partial \mathcal{L}}{\partial c_t} : \mathbb{E}_t [\beta^t u'_c(c_t, h_t) - \lambda_t] = 0$$

and using the fact that c_t, h_t is known at time t , and iterating forward, we have that

$$\lambda_t = \beta^t u'_c(c_t, h_t)$$

$$\lambda_{t+1} = \beta^{t+1} u'_c(c_{t+1}, h_{t+1})$$

We will keep this in mind for later. We also take the first order condition with respect to h_t :

$$\frac{\partial \mathcal{L}}{\partial h_t} : \mathbb{E}_t [\beta^t u'_h(c_t, h_t) + \lambda_{t+1} q_{t+1} - \lambda_t q_t] = 0$$

We collect our derived first order conditions.

Definition 272 (First Order Conditions for Pricing Housing)

The first order conditions for the household portfolio problem involving housing stock are:

$$\frac{\partial \mathcal{L}}{\partial c_t} : \mathbb{E}_t [\beta^t u'_c(c_t, h_t) - \lambda_t] = 0 \quad (331)$$

$$\frac{\partial \mathcal{L}}{\partial h_t} : \mathbb{E}_t [\beta^t u'_h(c_t, h_t) + \lambda_{t+1} q_{t+1} - \lambda_t q_t] = 0 \quad (332)$$

Remark 273. Note that we did not take the FOC with respect to b_t as we do not need it for what we want to do.

We re-arrange:

$$\mathbb{E}_t [\beta^t u'_h(c_t, h_t) + \lambda_{t+1} q_{t+1}] = \lambda_t q_t$$

and plug in our expressions λ_t, λ_{t+1} as a function of the marginal utility of consumption:

$$\mathbb{E}_t \left[\beta^t u'_h(c_t, h_t) + \beta^{t+1} u'_c(c_{t+1}, h_{t+1}) q_{t+1} \right] = \beta^t u'_c(c_t, h_t) q_t$$

We then divide through by $\beta^t u'_c(c_t, h_t)$ on both sides to get our Euler equation for housing.

Definition 274 (Euler Equation for Housing)

The Euler equation for housing is

$$q_t = \frac{u'_h(c_t, h_t)}{u'_c(c_t, h_t)} + \beta \mathbb{E}_t \left[\frac{u'_c(c_{t+1}, h_{t+1})}{u'_c(c_t, h_t)} q_{t+1} \right]. \quad (333)$$

Let us interpret this new Euler equation

$$q_t = \underbrace{\frac{u'_h(c_t, h_t)}{u'_c(c_t, h_t)}}_{(*)} + \underbrace{\beta \mathbb{E}_t \left[\frac{u'_c(c_{t+1}, h_{t+1})}{u'_c(c_t, h_t)} q_{t+1} \right]}_{**}$$

The price of a house q_t consists of two components:

1. The **service yield component** of housing which is denoted by (*)
2. The **discounted expected-resale component**, denoted by (**).

The service-yield component is the direct utility you get from owning housing (you can live in it), whilst the discounted expected-resale component reflects the fact that you can sell the housing stock in the future.

Looking at the service yield component, we see that the numerator contains the marginal utility of housing. The higher your marginal utility of housing, the higher the price of the house q_t would be. However, we are working with a model in real terms (i.e. in units of consumption rather than money) so we need to divide by the marginal utility of consumption in order to get the **consumption value** of the house. This means that if your marginal utility of consumption is also high, this will drive **down** the price of the house as you would rather consume rather than buy housing.

We now look at the discounted expected-resale component. We can resell the house tomorrow, so the numerator is the units of consumption we receive, q_{t+1} , times the marginal utility of consumption tomorrow, $u'_c(c_{t+1}, h_{t+1})$. We then divide by the marginal utility of consumption today and discount by β to express it in current consumption units.

So the question is now how do we measure this equation in the data. Typically, you can measure the service yield component of this Euler equation by looking at housing rents, that is, how much are people willing to pay to live in housing. With this in hand, we come across a puzzle in the data as we see that rents are relatively stable but housing prices fluctuate dramatically (think of the great financial crisis!). This is known as the **volatility puzzle**.

Definition 275 (Volatility Puzzle)

Housing rents are stable but the price of housing fluctuates dramatically.

A major issue with this model is that it fails to generate the volatility we observe in the data, as service flows (measured by housing rents) are generally quite stable.

12.5 Pricing Equities

We now return back to the model setup that we saw when pricing short-term bonds in Section 12.2. However, we now wish to be able to price stocks. Recall the problem that we set up previously.

Definition 276 (Household Portfolio Problem)

The household portfolio problem is given by:

$$\begin{aligned} \max_{\{c_t, b_t, s_t\}_{t=0}^{\infty}} \quad & \mathbb{E}_t \sum_{i=0}^{\infty} \left[\beta^i u(c_{t+i}) \right] \\ \text{s.t.} \quad & c_t + b_t + p_t s_t = r_{t-1} b_{t-1} + (p_t + y_t) s_{t-1} - T_t \end{aligned} \quad (334)$$

With the associated first order conditions.

Proposition 277 (First Order Conditions)

The first order conditions of the problem are given by:

$$\frac{\partial \mathcal{L}}{\partial c_t} : \mathbb{E}_t [\beta^t u'(c_t) - \lambda_t] = 0 \quad (335)$$

$$\frac{\partial \mathcal{L}}{\partial b_t} : \mathbb{E}_t [\lambda_{t+1} r_t - \lambda_t] = 0 \quad (336)$$

$$\frac{\partial \mathcal{L}}{\partial s_t} : \mathbb{E}_t [\lambda_{t+1} (p_{t+1} + y_{t+1}) - \lambda_t p_t] = 0 \quad (337)$$

We then derived the following pricing equations:

Proposition 278 (Pricing Equations)

The bond pricing equation is given by

$$1 = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} r_t \right]. \quad (338)$$

The stock pricing equation is given by

$$1 = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \frac{p_{t+1} + y_{t+1}}{p_t} \right]. \quad (339)$$

Let us look at the stock pricing equation:

$$1 = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \frac{p_{t+1} + y_{t+1}}{p_t} \right]$$

Suppose that we had a risk-neutral utility function: $u(c_t) = a + b c_t$. This means that there is no curvature in the utility function so that the marginal utility is a constant $u'(c_t) = b$. If we plug this into our stock pricing equation, we have that the ratio of marginal utilities cancel out and we end up with:

$$1 = \beta \mathbb{E}_t \left[\frac{p_{t+1} + y_{t+1}}{p_t} \right]$$

Divide by β on both sides and rewrite $r^S = \frac{p_{t+1} + y_{t+1}}{p_t}$ to be the return on the stock to get the following:

Definition 279 (Expected return on stocks under risk-neutral utility)

The expected return on a stock r^s under a risk-neutral utility function is

$$\mathbb{E}_t[r^s] = \frac{1}{\beta} \quad (340)$$

Now look at the bond pricing equation:

$$1 = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} r_t \right].$$

Assuming risk-neutral utility function, the marginal utilities also cancel out and we can divide the β term on both sides to get:

$$\mathbb{E}_t[r] = \frac{1}{\beta}$$

This is exactly the same expression for the expected return on a stock!

Proposition 280 (Equality of Expected Returns)

Under a risk-neutral utility function, the expected return on a stock is equal to the expected return on short-term bonds:

$$\mathbb{E}_t[r^s] = \frac{1}{\beta} = \mathbb{E}_t[r] \quad (341)$$

This makes sense! Under risk-neutrality, the expected return of risky-stocks is only the inverse of the discount factor β . This means that the expected returns are equal since we are not pricing in any risk (since we are risk neutral). Under risk-neutrality, we evaluate a safe and a risky asset by only its first moment, which is the mean return β^{-1} .

This is clearly wrong. Stocks earn a much higher return than bonds when we look at the data! We need to move away from risk-neutrality to explain why do stocks earn so much. So let us begin again.

12.5.1 Generalised Price of Stocks

Let us look at the stock-pricing equation again:

$$1 = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \frac{p_{t+1} + y_{t+1}}{p_t} \right]$$

We can push β inside the expectation term and define the **stochastic discount factor**:

$$\mathcal{M}_{t,t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}. \quad (342)$$

We then also multiply p_t on both sides of the stock-pricing equation. After doing that, we arrive at the expression:

$$p_t = \mathbb{E}_t \left[\mathcal{M}_{t,t+1} (p_{t+1} + y_{t+1}) \right] \quad (343)$$

We can iterate this equation one step forward to get:

$$p_{t+1} = \mathbb{E}_{t+1} \left[\mathcal{M}_{t+1,t+2} (p_{t+2} + y_{t+2}) \right] \quad (344)$$

We can plug Eq. (344) into Eq. (343) to get

$$\begin{aligned}
p_t &= \mathbb{E}_t \left[\mathcal{M}_{t,t+1} \left\{ \mathbb{E}_{t+1} \left[\mathcal{M}_{t+1,t+2} (p_{t+2} + y_{t+2}) \right] + y_{t+1} \right\} \right] \\
&= \mathbb{E}_t \left[\mathcal{M}_{t,t+1} \mathbb{E}_{t+1} \left[\mathcal{M}_{t+1,t+2} p_{t+2} \right] + \mathcal{M}_{t,t+1} \mathbb{E}_{t+1} \left[\mathcal{M}_{t+1,t+2} y_{t+2} \right] + \mathcal{M}_{t,t+1} y_{t+1} \right] \\
&= \mathbb{E}_t \left[\mathcal{M}_{t,t+1} \mathbb{E}_{t+1} \left[\mathcal{M}_{t+1,t+2} p_{t+2} \right] \right] + \mathbb{E}_t \left[\mathcal{M}_{t,t+1} \mathbb{E}_{t+1} \left[\mathcal{M}_{t+1,t+2} y_{t+2} \right] \right] + \mathbb{E}_t \left[\mathcal{M}_{t,t+1} y_{t+1} \right] \\
&= \mathbb{E}_t \left[\mathcal{M}_{t,t+1} \mathcal{M}_{t+1,t+2} p_{t+2} \right] + \mathbb{E}_t \left[\mathcal{M}_{t,t+1} \mathcal{M}_{t+1,t+2} y_{t+2} \right] + \mathbb{E}_t \left[\mathcal{M}_{t,t+1} y_{t+1} \right]
\end{aligned}$$

where we used the law of iterated expectations. Now, notice that

$$\mathcal{M}_{t,t+1} \mathcal{M}_{t+1,t+2} = \beta \frac{u'(c_{t+1})}{u'(c_t)} \cdot \beta \frac{u'(c_{t+2})}{u'(c_{t+1})} = \beta^2 \frac{u'(c_{t+2})}{u'(c_t)} = \mathcal{M}_{t,t+2}.$$

So therefore we have that

$$p_t = \mathbb{E}_t \left[\mathcal{M}_{t,t+1} y_{t+1} \right] + \mathbb{E}_t \left[\mathcal{M}_{t,t+2} y_{t+2} \right] + \mathbb{E}_t \left[\mathcal{M}_{t,t+2} p_{t+2} \right]$$

We can repeat this process for T periods to get:

$$p_t = \sum_{s=1}^T \mathbb{E}_t \left[\mathcal{M}_{t,t+s} y_{t+s} \right] + \mathbb{E}_t \left[\mathcal{M}_{t,t+T} p_{t+T} \right]$$

Now, we impose a no bubble-condition so that

$$\lim_{T \rightarrow \infty} \mathbb{E}_t \left[\mathcal{M}_{t,t+T} p_{t+T} \right] = 0.$$

Therefore, when we send $T \rightarrow \infty$, we end up with

$$p_t = \sum_{j=1}^{\infty} \mathbb{E}_t \left[\mathcal{M}_{t,t+j} y_{t+j} \right] + 0.$$

Definition 281 (Generalised Solution for Stock Price)

The generalised stock price is:

$$p_t = \sum_{j=1}^{\infty} \mathbb{E}_t \left[\mathcal{M}_{t,t+j} y_{t+j} \right]. \quad (345)$$

Remark 282. This equation says that the stock price is the discounted sum of dividend payments y_{t+j} , where discounting is done using the stochastic discount factor $\mathcal{M}_{t,t+j}$.

We can apply this generalised stock price to our model under risk-neutrality if we wanted to.

Proposition 283 (Generalised Stock Price under Risk Neutrality)

Under risk-neutrality $u(c_t) = a + bc_t$, the price of a stock is:

$$p_t = \sum_{j=1}^{\infty} \mathbb{E}_t \left[\beta^j y_{t+j} \right] \quad (346)$$

Proof. Note that under the assumption of risk-neutrality, we have that

$$\mathcal{M}_{t,t+j} = \beta^j \frac{u'(c_{t+j})}{u'(c_t)} = \beta^j \frac{b}{b} = \beta^j$$

and therefore we have that the stock price is:

$$p_t = \sum_{j=1}^{\infty} \mathbb{E}_t \left[\beta^j y_{t+j} \right]$$

□

Remark 284. Here, under risk-neutrality, the price of a stock is the discounted value of future dividends, where the discount factor is β rather than the stochastic discount factor $\mathcal{M}_{t,t+s}$.

12.5.2 Equity Premium

We now want to test the theory we have developed so far on data. First, let us go back to the stock-pricing equation again:

$$1 = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \frac{p_{t+1} + y_{t+1}}{p_t} \right]$$

Let us define

$$r_{t+1}^s \equiv \frac{p_{t+1} + y_{t+1}}{p_t}$$

to be the **gross return on stocks**. We therefore can write the stock-pricing equation as

$$1 = \mathbb{E}_t \left[\mathcal{M}_{t,t+1} r_{t+1}^s \right]$$

where $\mathcal{M}_{t,t+1}$ is the stochastic discount factor.

Now, recall the bond-pricing equation:

$$1 = \mathbb{E}_t \left[\mathcal{M}_{t,t+1} r_t \right].$$

Subtract one from another:

$$\mathbb{E}_t \left[\mathcal{M}_{t,t+1} r_{t+1}^s \right] - \mathbb{E}_t \left[\mathcal{M}_{t,t+1} r_t \right] = 1 - 1$$

which will give us:

$$\mathbb{E}_t \left[\mathcal{M}_{t,t+1} (r_{t+1}^s - r_t) \right] = 0$$

We now define

$$r_{t+1}^e \equiv r_{t+1}^s - r_t$$

to be the **excess return of stocks over bonds**. These are the returns that a stock earns over and above the safe rate of return. We therefore get a nice result.

Proposition 285 (No Arbitrage Asset Pricing)

There is no arbitrage under the stochastic discount factor $\mathcal{M}_{t,t+1}$:

$$\mathbb{E}_t \left[\mathcal{M}_{t,t+1} r_{t+1}^e \right] = 0 \tag{347}$$

Remark 286. This says that the excess return of all stocks over bonds evaluated at the stochastic discount factor should be zero. That is, there is no room for arbitrage. Under what is known as **risk-neutral expectations**, all assets earn the same returns.

Taking our no arbitrage asset pricing equation, we can apply the result that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] + \text{Cov}(X, Y)$ to get:

$$\mathbb{E}\left[\mathcal{M}_{t,t+1}\right]\mathbb{E}_t\left[r_{t+1}^e\right] + \text{Cov}(\mathcal{M}_{t,t+1}, r_{t+1}^e) = 0.$$

Remark 287. Notice that we have switched from conditional to unconditional expectations. This follows by taking unconditional expectations of the conditional pricing equation and applying the law of iterated expectations.

The issue is that we don't know what the stochastic discount factor is when looking at the data. However, we can replace as if you recall from the bond-pricing equation that:

$$1 = \mathbb{E}\left[\mathcal{M}_{t,t+1}r_t\right] \leftrightarrow \mathbb{E}\left[\mathcal{M}_{t,t+1}\right] = \mathbb{E}\left[\frac{1}{r_t}\right]$$

we can now derive a testable equation for us to check.

Definition 288 (Testable Relationship for Excess Return of Stocks Over Bonds)

Our expression for the excess return of stocks over bonds is given by:

$$\mathbb{E}\left[r_{t+1}^e\right] = \frac{-\text{Cov}(\mathcal{M}_{t,t+1}, r_{t+1}^e)}{\mathbb{E}\left[\frac{1}{r_t}\right]} \quad (348)$$

This is a very important relationship, and we will use it to test the data. Recall that we want the model to generate a positive equity premium, as stocks earn a higher average return than bonds. From our equation, the sign of the expected excess return depends on the covariance between excess returns and the stochastic discount factor. Stocks are generally procyclical: they tend to have high returns in good times, when consumption is high. With a concave utility function, high consumption means low marginal utility and hence a low stochastic discount factor. Stock excess returns therefore covary negatively with $\mathcal{M}_{t,t+1}$. This negative covariance, together with the minus sign in the equation, generates a **positive equity premium**.

12.5.3 Equity Premium: Mehra and Prescott (1985)

We now turn to the Mehra and Prescott 1985 paper to test the equity premium relationship we have just derived. To explain an equity premium, recall that we need a **negative** covariance between the stochastic discount factor and the real return on stocks. Therefore, we need to choose a utility function in order to compute this. In the paper, they picked a CRRA utility function:

$$u(c_t) = \frac{c_t^{1-\sigma}}{(1-\sigma)}$$

which gives us our testable relationship for excess returns being:

$$\mathbb{E}\left[r_{t+1}^e\right] = \frac{-\text{Cov}\left(\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}, r_{t+1}^e\right)}{\mathbb{E}\left[\frac{1}{r_t}\right]} \quad (349)$$

They made the approximation (from the data) that

$$\mathbb{E}\left[\frac{1}{r_t}\right] \approx 1$$

which means they tested the relationship:

$$\mathbb{E}\left[r_{t+1}^e\right] = -Cov\left(\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}, r_{t+1}^e\right) \quad (350)$$

U.S. Annual Data 1889-1978 (Mehra and Prescott, 1985)

Variable	Mean	Variance-Covariance		
		r_t^s	r_t	c_{t+1}/c_t
r_t^s	1.070	0.0274	0.00104	0.00219
r_t	1.010		0.00308	-0.000193
c_{t+1}/c_t	1.018			0.00127

This paper took the average annual return of stocks over nearly the course of 100 years. In the table, r_t^s represents average annual stock returns. We see that they computed average stock returns to be 7 % per annum with an annual variance is 2.74, which is around 50% annual standard deviation (very high!). We see that the safe return r_t has a 1% per annum return. Therefore, we see a 6% equity premium between returns on stocks and return on safe assets.

Now, look at the covariance between stock returns and the stochastic discount factor in the top right corner, which is a covariance of 0.00219, which we approximate to 0.002. So we have an excess return of 6% and a covariance of 0.002. Looking at our relationship, we can approximate this relationship as follows and compute what the coefficient of relative risk aversion σ has to be in order for our model to be reconciled with the data:

$$0.06 = 0.002(\sigma)$$

$$\Rightarrow \sigma \approx 30.$$

A coefficient of relative risk aversion $\sigma = 30$ is **extremely high**. This is the conclusion of the Mehra and Prescott paper.

Definition 289 (Equity Premium Puzzle)

To match the equity premium found in the data, our model requires an unbelievably high level of risk aversion σ .

So what is wrong with increasing σ ? In Weil (1989) paper, Weil discovered what is known as the **risk-free rate puzzle**.

Definition 290 (Risk-Free Rate Puzzle)

If we increase σ to match the empirical equity premium, the risk-free rate implied by our model is too high.

To summarise, we can see that for our model which culminated in the excess return equation:

$$\mathbb{E}\left[r_{t+1}^e\right] = -Cov\left(\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}, r_{t+1}^e\right) \quad (351)$$

given reasonable parameter values, our model gave a poor fit of

1. Equity premium: The equity premium our model implied is way too small compared to what the data said. If we tried to match the data, then the risk aversion parameter σ that is implied is way too high.
2. Risk-free rate: If we do set the risk aversion parameter σ to be high, this then meant that the risk-free rate implied by our model is way too high compared to the data.
3. Volatility of stocks: Our model also failed to replicate the large volatility of stock prices despite dividends being stable.

This next little part (can be skipped) is just a look at the Hansen and Singleton (1982) try to see if we could fix the model by running a GMM estimation on the Euler equation:

$$\mathbb{E}_t\left[\mathcal{M}_{t,t+j}r_{t+j}^i\right] = 1$$

where i indexes assets. They then estimated a GMM system using j assets:

$$\mathbb{E}_t\begin{bmatrix} \mathcal{M}_{t,t+1}r_{t+1}^1 - 1 \\ \vdots \\ \mathcal{M}_{t,t+1}r_{t+1}^j - 1 \end{bmatrix} = \mathbf{0}.$$

They found that the Euler equation is a **poor fit**.

12.6 Summary

From all this, we have a big issue as we are unable to explain the equity premium seen in the data from our model. Furthermore, econometric testing suggests that the Euler equations we derived in our models is actually a poor fit for the data. This is a big issue as Euler equations are fundamental to many macroeconomic models. Furthermore, we see a big tension between macro models and asset pricing models if this is not resolved.

This will be our pursuit over the next few lectures is to construct a model that can solve the equity premium puzzle! Fun! In particular, we shall look at the following methods to fix the model:

1. Exotic preferences: Habits and Epstein-Zin recursive preferences;
2. Disaster risk models.

13 Exotic Preferences

13.1 Utility with External Habits

We now look into extending upon our utility functions in order to fix the equity premium puzzle. The first example we will look at is incorporating habit formation into our models. By this, our utility depends on consumption C_t as usual, but now it doesn't depend on the absolute level of consumption but our consumption relative to a *habit* or a *subsistence* level X_t . Therefore, it is now how close our consumption is to this habit level is what gives us utility. If we are exceeding our habit level, then we derive positive utility from this. One way to interpret this is if we view X_t as a subsistence level we need to survive, so therefore, the higher we are above this subsistence level, the better off we are.

Definition 291 (Utility with External Habits)

The utility function with external habits is given by:

$$U(C_t) = \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j \frac{(C_{t+j} - X_{t+j})^{1-\gamma}}{1-\gamma} \right] \quad (352)$$

whereby X_t is an external habit.

Given this utility function with external habits, we are then able to see that the relative risk aversion of such a utility function actually varies over time with an interesting interpretation.

Proposition 292 (Time Varying Relative Risk Aversion)

The time varying relative risk aversion is given by:

$$RRA_t = \frac{\gamma}{\left(\frac{C_t - X_t}{C_t} \right)} \quad (353)$$

whereby $\frac{(C_t - X_t)}{C_t}$ is known as the **surplus ratio**.

Proof. First, recall the definition of relative risk aversion under an external habit:

$$RRA = - \frac{C_t \cdot U''(C_t)}{U'(C_t)} \quad (354)$$

We take our utility function and take the first and second derivative:

$$\begin{aligned} \frac{\partial U}{\partial C_t} &= \beta^t (C_t - X_t)^{-\gamma} \\ \frac{\partial^2 U}{\partial C_t^2} &= -\beta^t \gamma (C_t - X_t)^{-\gamma-1} \end{aligned}$$

Plug these into our formula for the relative risk aversion:

$$\begin{aligned}
\text{RRA} &= -\frac{C_t \cdot U''(C_t)}{U'(C_t)} \\
&= \frac{C_t \cdot \beta^t \gamma (C_t - X_t)^{-\gamma-1}}{\beta^t (C_t - X_t)^{-\gamma}} \\
&= C_t \gamma (C_t - X_t)^{-1} \\
&= C_t \frac{\gamma}{C_t - X_t} \\
&= \frac{\gamma}{\left(\frac{C_t - X_t}{C_t}\right)}
\end{aligned}$$

□

We can interpret this relative risk aversion as follows. First, suppose that our consumption level C_t is close to the habit level X_t . This causes the surplus ratio $\frac{C_t - X_t}{C_t}$ to get smaller. This then causes the relative risk aversion to rise. Therefore, as agents consume close to their subsistence levels, this causes their risk aversion to rise.

It is through this mechanism that this model is able to generate time-varying risk premia and can help us to try fit the data. However, it does seem very arbitrary where does this habit come from and as we will later see, it is not clear that this is actually helpful in explaining the data.

13.2 Utility with Internal Habits - BCF (2001)

We quickly look at another habit model based on the influential paper by Boldrin, Christiano, and Fisher (AER, 2001) paper. Here, we can state their formulation of habits.

Definition 293 (Utility function with Internal and External Habit)

The utility function with both an internal and external habit is given by:

$$V_t = \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s [\ln(C_{t+s} - bC_{t+s-1})] - H_{t+s} \right] \quad (355)$$

whereby $b \geq 0$ is an index of the degree of **internal** habit and H_{t+s} is the **external habit**.

Here, we have that utility depends on how much consumption exceeds prior consumption and to our subsistence/habitual level. We are only happy if consumption is greater relative to last period.

Proposition 294 (Marginal Utility under Internal and External Habits)

The marginal utility under this habit formulation is given by:

$$\frac{\partial V_t}{\partial C_t} = \frac{1}{C_t - bC_{t-1}} - \mathbb{E}_t \left[\frac{\beta b}{C_{t+1} - bC_t} \right] \quad (356)$$

$$\frac{\partial^2 V_t}{\partial C_t^2} = -\frac{1}{(C_t - bC_{t-1})^2} - \mathbb{E}_t \left[\frac{\beta b^2}{(C_{t+1} - bC_t)^2} \right] \quad (357)$$

We can see that this will generate for us a time-varying risk aversion. If we denote the marginal utility of consumption to be:

$$\Delta_t \equiv \frac{1}{C_t - bC_{t-1}} - \mathbb{E}_t \left[\frac{\beta b}{C_{t+1} - bC_t} \right]$$

then we can see that our fundamental asset pricing equation can now be written as

$$1 = \mathbb{E}_t \left[\mathcal{M}_{t,t+1} r_{t+1}^i \right] = \mathbb{E}_t \left[\beta \frac{\Delta_{t+1}}{\Delta_t} r_{t+1}^i \right]. \quad (358)$$

This formulation of internal habits introduces an extra state variable, past consumption, into the stochastic discount factor. Including habits causes agents to want to smooth surplus consumption $C_t - bC_{t-1}$ and, as a result, they will hold risky assets only in exchange for a sufficiently high premium.

TABLE 1—FINANCIAL STATISTICS^a

Statistic ^b	Data ^c	Two-sector		One-sector	
		$b = 0.73$	$b = 0$	$b = 0.9$	$b = 0$
Er_t^f	1.19 (0.81)	1.20	1.58	1.58	1.58
$E(r_{t+1}^e - r_t^f)$	6.63 (1.78)	6.63	0.02	0.001	0.001
σ_{r^f}	5.27 (0.74)	24.6	0.62	0.38	0.46
σ_{r^e}	19.4 (1.56)	18.4	0.57	0.40	0.48
$E(r_{t+1}^e - r_t^f)$ σ_{r^e}	0.34 (0.09)	0.36	0.03	0.002	0.002
$\sigma_{P_{k'}}$	8.56 (0.85)	12.1	0.29	0	0
$\rho(Y, P_{k'})$	0.30 (0.08)	0.16	0.16	na ^d	na

^a Results for the models are based on 500 replications of sample size 200.

^b σ_x denotes the standard deviation of variable x , and $\rho(x, y)$ denotes the correlation between variable x and variable y . Rates of returns are annualized and in percent terms before statistics are computed.

^c This column contains estimates (standard errors in parentheses) based on U.S. data. The sample period for the asset-return estimates is 1892–1987 and these estimates are taken from Cecchetti et al. (1993). Our empirical analogue for $P_{k'}$ is the S&P 500 composite (DRI database mnemonic FSPCOM). The output measure and the procedure for estimating $\sigma_{P_{k'}}$ and $\rho(Y, P_{k'})$ is described in Table 3, note ^c.

^d This abbreviates “not applicable.”

We now look at the table from the paper. Looking at the Two-sector column (ignore what the Two-Sector is referring to) we can see that when $b = 0$, so that there are no habits, we see that there is no equity risk premium (the 0.02 in the second row under column $b = 0$). If we look at the cell to the left, where $b = 0.73$, we see that we now have a high equity risk premium of 6.63! This is fantastic! However, an issue is that if we look down at the volatility of the risk-free rate σ_{r^f} , this has a high value of 24.6 and is in fact higher than the volatility of equity returns σ_{r^e} which is 18.4. This is a big issue since it doesn’t make sense nor does it match the empirical facts. Therefore, these habit models have an issue with having a high volatility of the risk-free rate.

13.3 Epstein-Zin Preferences

13.3.1 Motivation

So far, we have used the **fundamental asset pricing equation** multiple times

$$\mathbb{E}_t [\mathcal{M}_{t,t+1} r_{t+1}^i] = 1 \quad (359)$$

whereby $\mathcal{M}_{t,t+1}$ is the **stochastic discount factor**

$$\mathcal{M}_{t,t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

Previously, we saw that if utility is CRRA with parameter σ , it is given by:

$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

, then the stochastic discount factor will be

$$\mathcal{M}_{t,t+1} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \quad (360)$$

When we choose the coefficient of relative risk aversion σ , we have a lot of trouble explaining the equity risk premium (i.e. the expected return less the risk-free rate, $\mathbb{E}_t[r_{t+1}^e] - r_{t+1}^f$). Furthermore, we fail to replicate the high volatility of stock prices despite stable dividends (recall that we derived the stock price as the discounted sum of expected future dividends).

This poses a big issue between macroeconomics and asset pricing as this leads to a conflict between *real* and *financial* variables. Furthermore, the euler equation is integral to DSGE models so for it to fail this badly when it comes to financial variables is a big issue!

We give a rough sketch of what one of the big issues is. First, we need a few definitions.

Definition 295 (Time separable utility function)

A utility function $u(\cdot)$ is **time separable** if the past or future consumption **does not** affect the current level of utility $u(c_t)$.

With time-separable utility functions, we have always expressed them in the form

$$V_0 = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where we can see that at each time period t , only the consumption level c_t at time t is what gives us utility.

Definition 296 (Elasticity of Intertemporal Substitution)

The Elasticity of Intertemporal Substitution (EIS) is the willingness of individuals to substitute consumption **over time**.

We now state an important result.

Proposition 297 (EIS and RRA)

With **time-separable** utility functions, the elasticity of intertemporal substitution (EIS) is the inverse of relative risk aversion (RRA):

$$EIS = \frac{1}{RRA} \quad (361)$$

We have a really important corollary from this.

Corollary 298

The family of CRRA utility functions has both **constant** relative risk aversion and a constant **elasticity of intertemporal substitution**. With the parameterisation above, RRA is σ and EIS is $1/\sigma$.

The way I think about this is that we have these two separate notions in conflict. The EIS measures how we substitute our consumption **inter-temporally** whilst the RRA looks at how we allocate between assets **intra-temporally**. It is somewhat strange that the same parameter σ is governing both of these seemingly different effects.

Therefore, one proposed method is depart away from the CRRA utility function

$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

One well known *exotic preference* we can use instead of CRRA are the Epstein-Zin preferences, which will allow us to have two separate parameters for the EIS and the RRA.

13.3.2 Introduction

As we discussed earlier, if we have an utility function that is *time-separable*, we end up with the relative risk aversion (RRA) and elasticity of intertemporal substitution (EIS) being the inverse of one another. That suggests to us that we should move to a new utility function that is **not** time-separable.

Fortunately, we can draw inspiration from dynamic programming (yay!) and use the recursive formulations employed there when designing our recursive (and hence non-separable) utility functions. Let $u(\cdot)$ denote the utility function and $V(\cdot, \cdot)$ denote the **value function** that we should be familiar from in dynamic programming (i.e. our current utility + future discounted utility). Jumping into it, we will just state what our value function (program) will look like:

$$V_t(c_t, V_{t+1}) = (1 - \beta) \underbrace{u(c_t)}_{(*)} + \beta \underbrace{\mathbb{E}_t \left[V_{t+1}(c_{t+1}, V_{t+2}) \right]}_{(**)} \quad (362)$$

Let us break this down. The first term (*) says that our value function V_t depends on the utility we get from our current consumption $u(c_t)$. The second term (**) says that our value function at $t + 1$ will be a function of future consumption c_{t+1} and the value function in the period after that V_{t+2} . Note that future consumption is not known in advance, and is therefore random, and hence why we need to take $\mathbb{E}_t$ of the value function. Also, I sometime refer V_{t+1} as the **continuation value** of the program as this is the value we will get if we continue following our optimisation procedure one time period at a time. Finally, we can think of $(1 - \beta)$ as a weight on current period's utility relative to the β discounted expected future value function V_{t+1} .

Remark 299. Note that the continuation value V_{t+1} is actually a function of next period's consumption and the utility function in the period after that i.e. $V_{t+1}(c_{t+1}, V_{t+2})$. However, for ease of notation, we usually suppress the arguments and just write V_{t+1} . Do not forget that it is still a function though!

After that introduction, and we are now ready to define more things that we will need! To paint a roadmap, we will need to define two things first that is used in the Epstein-Zin preferences: (1) the certainty equivalent operator and (2) the time aggregator.

13.3.3 Certainty Equivalent Operator

Definition 300 (Certainty Equivalent Operator/Risk Aggregator)

Let V_{t+1} be the **continuation value** of our value function. The certainty equivalent operator $\mu(\cdot)$ is defined by

$$\mu(V_{t+1}) = G^{-1}\left(\mathbb{E}_t[G(V_{t+1})]\right) \quad (363)$$

whereby $G(\cdot)$ is a **concave function**.

This may look scary, but let us break it down. First, we will explain mechanically what we are doing and then explain *why*. The certainty-equivalent operator applies four steps in one go:

1. Take our continuation value V_{t+1} ;
2. Apply our concave function $G(V_{t+1})$;
3. Take expectation of our transformed continuation value $\mathbb{E}_t[G(V_{t+1})]$;
4. Apply the inverse of our concave function to our expectation $G^{-1}\left(\mathbb{E}_t[G(V_{t+1})]\right)$.

An example we could use for our concave function is the log function $G(x) = \log(x)$ and $G^{-1}(x) = \exp(x)$, which means our certainty equivalent operator would be:

$$\mu(V_{t+1}) = \exp\left(\mathbb{E}_t[\log(V_{t+1})]\right)$$

Now let us start to give a flavour for why we are doing this. There are two ways that I think about it. The first is using the idea of certainty equivalence from Microeconomics. Recall that $V_{t+1}(c_{t+1}, V_{t+2})$ is unknown as we don't know what future consumption c_{t+1} will be. So let us first think about a stochastic consumption c_t and a utility function $u(c_t)$. The expected utility is obviously $\mathbb{E}_t[u(c_t)]$. **The certainty equivalent is the level of constant (certain) consumption level $\bar{c}$ that yields the same utility as the expected utility.** Using the mathematical definition of certainty equivalence from microeconomics, we have:

$$\begin{aligned} u(\bar{c}) &= \mathbb{E}_t[u(c_t)] \\ \rightarrow \bar{c} &= u^{-1}[\mathbb{E}_t[u(c_t)]] \end{aligned}$$

For example, if $u(c_t) = \log(c_t)$, then we have

$$\bar{c} = \exp[\mathbb{E}_t[\log(c_t)]]$$

So now, if we go back to our value function and our certainty equivalent operator:

$$\mu(V_{t+1}) = G^{-1}\left(\mathbb{E}_t[G(V_{t+1})]\right)$$

This can be seen as the certainty equivalent of tomorrow's value function or the constant value of $\bar{V}$ that yields the same as the expected value of V_{t+1} .

The second way I think about the certainty-equivalent operator is that we apply a **concave** function $G(\cdot)$ to the random variable V_{t+1} and then take its expectation. By **Jensen's inequality**, this transformed expectation satisfies

$$\mathbb{E}_t[G(V_{t+1})] \leq G(\mathbb{E}_t[V_{t+1}])$$

Then, when we apply the inverse G^{-1} , we will not get back the same result. However, we will get back the same result

if there was *no uncertainty*.

We will soon see that when we define Epstein-Zin preferences, our certainty equivalent operator will be the power function:

$$\mu(X) = G^{-1}\left(\mathbb{E}_t[G(X)]\right) = \left[\mathbb{E}_t[X^{1-\alpha}]\right]^{\frac{1}{1-\alpha}} \quad (364)$$

whereby

$$G(x) = x^{1-\alpha} \quad \text{and} \quad G^{-1}(x) = x^{\frac{1}{1-\alpha}}$$

On a tangential point, for those with measure theory knowledge, $\mu(\cdot)$ is a **certainty equivalent** function that maps a random variable V_{t+1} that is measurable with respect to information at $t+1$ into a random variable that is measurable with respect to this period's information.

13.3.4 Time Aggregator

We now define the other (and much easier part) of the Epstein-Zin preferences. First, recall that a function $f(\cdot)$ is homogenous of degree 1 (hod1) if

$$f(\lambda x) = \lambda f(x)$$

for all constants λ .

We can now define the second part of the EZ preference.

Definition 301 (Time Aggregator)

Let V_t be the current value function. The time aggregator $F(\cdot, \cdot)$ is defined as

$$V_t = F[c_t, \mu_t(V_{t+1})] \quad (365)$$

whereby F is a **concave function** that is **homogenous of degree 1**, c_t is current consumption, and $\mu_t(V_{t+1})$ is the **certainty equivalent operator**.

The time aggregator takes in current consumption and the certainty equivalent operator. The time aggregator that characterizes preferences in the absence of uncertainty (recall that even though V_{t+1} was unknown, applying $\mu(\cdot)$ gives a known value!).

13.3.5 Epstein Zin Preferences

We can now state the long awaited Epstein-Zin preferences.

Definition 302 (Epstein-Zin Preferences)

The Epstein-Zin (EZ) preferences are off the form:

$$V_t = \left[(1-\beta)c_t^{1-\rho} + \beta \left(\mathbb{E}_t[V_{t+1}^{1-\alpha}] \right)^{\frac{1-\rho}{1-\alpha}} \right]^{\frac{1}{1-\rho}} \quad (366)$$

Let us break this down. Mapping this to our notation, we have that the **time aggregator** $F(\cdot, \cdot)$ used is:

$$F(x, y) = \left((1-\beta)x^{1-\rho} + \beta y^{1-\rho} \right)^{\frac{1}{1-\rho}} \quad (367)$$

This is the **Constant Elasticity of Substitution** aggregate! (Similar to the CES utility functions from microeconomics).

Now, the **certainty equivalent operator** $\mu(\cdot)$ or the **risk aggregator** is (for $\alpha > 0$)

$$\mu(V_{t+1}) = G^{-1}\left(\mathbb{E}_t[G(V_{t+1})]\right) = \left[\mathbb{E}_t[V_{t+1}^{1-\alpha}]\right]^{\frac{1}{1-\alpha}} \quad (368)$$

whereby

$$G(x) = x^{1-\alpha} \quad \text{and} \quad G^{-1}(x) = x^{\frac{1}{1-\alpha}}$$

If $\alpha = 0$, then $G(x) = x$ and the certainty-equivalent operator is simply:

$$\mu(V_{t+1}) = \mathbb{E}_t[V_{t+1}].$$

Now, our whole goal was to get to the point where we had an utility function that does not link elasticity of intertemporal substitution and relative risk aversion. We have actually managed to achieve that through the EZ preferences! Looking at Eq. (366), we have that

$$\begin{cases} \alpha : \text{coefficient of risk aversion} \\ \frac{1}{\rho} : \text{intertemporal elasticity of substitution} \end{cases}$$

Now, what is even more interesting is that V_{t+1} was our *unknown term* and the coefficient of risk aversion deals with uncertainty. We can see that the α term is being applied to our V_{t+1} term but not our c_t term is V_{t+1} is the term that is unknown!!!

We now state a few useful results.

Proposition 303 (Homogeneity of EZ Preferences)

The Epstein-Zin preferences are homogenous of degree 1.

Proof. Follows immediately by the fact that CES preferences used are homogenous of degree 1. □

We now wish to derive the stochastic discount factor.

Proposition 304 (Stochastic Discount Factor of EZ Preferences)

The stochastic discount factor of EZ preferences is given by

$$\mathcal{M}_{t,t+1} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\rho} \left[\frac{V_{t+1}}{\mu(V_{t+1})} \right]^{\rho-\alpha}. \quad (369)$$

Proof. The stochastic discount factor in this set up is given by:

$$\mathcal{M}_{t,t+1} = \frac{\partial V_t / \partial C_{t+1}}{\partial V_t / \partial C_t} \quad (370)$$

By the chain rule, we can rewrite the numerator as:

$$\mathcal{M}_{t,t+1} = \frac{\frac{\partial V_t}{\partial V_{t+1}} \cdot \frac{\partial V_{t+1}}{\partial C_{t+1}}}{\frac{\partial V_t}{\partial C_t}}.$$

Looking at our EZ preferences:

$$V_t = \left[(1 - \beta)c_t^{1-\rho} + \beta \left(\mathbb{E}_t[V_{t+1}^{1-\alpha}] \right)^{\frac{1-\rho}{1-\alpha}} \right]^{\frac{1}{1-\rho}}$$

we can immediately compute that:

$$\begin{aligned} \frac{\partial V_t}{\partial C_t} &= (1 - \beta)V_t^\rho C_t^{-\rho} \\ \frac{\partial V_{t+1}}{\partial C_{t+1}} &= (1 - \beta)V_{t+1}^\rho C_{t+1}^{-\rho} \end{aligned}$$

The final term we need is: $\frac{\partial V_t}{\partial V_{t+1}}$. First, we can rewrite our EZ preferences as:

$$V_t = \left[(1 - \beta)c_t^{1-\rho} + \beta \mu_t(V_{t+1})^{1-\rho} \right]^{\frac{1}{1-\rho}} \quad (371)$$

whereby the we introduced the certainty equivalent operator notation $\mu_t(V_{t+1}) = \left(\mathbb{E}_t[V_{t+1}^{1-\alpha}] \right)^{\frac{1}{1-\alpha}}$. Going back to $\frac{\partial V_t}{\partial V_{t+1}}$, we use the chain rule trick again whereby we write this as:

$$\frac{\partial V_t}{\partial V_{t+1}} = \frac{\partial V_t}{\partial \mu_t} \cdot \frac{\partial \mu_t}{\partial V_{t+1}}$$

We can get the terms we need by differentiating Eq. (371) and therefore:

$$\begin{aligned} \frac{\partial V_t}{\partial \mu_t} &= \beta V_t^\rho (\mu_t(V_{t+1}))^{-\rho} \\ \frac{\partial \mu_t}{\partial V_{t+1}} &= (\mu_t(V_{t+1}))^\alpha V_{t+1}^{-\alpha} \end{aligned}$$

which gives us

$$\frac{\partial V_t}{\partial V_{t+1}} = \beta V_t^\rho (\mu_t(V_{t+1}))^{\alpha-\rho} V_{t+1}^{-\alpha}.$$

We now collect all our terms to get the stochastic discount factor:

$$\begin{aligned} \mathcal{M}_{t,t+1} &= \frac{\frac{\partial V_t}{\partial V_{t+1}} \cdot \frac{\partial V_{t+1}}{\partial C_{t+1}}}{\frac{\partial V_t}{\partial C_t}} \\ &= \frac{\beta V_t^\rho (\mu_t(V_{t+1}))^{\alpha-\rho} V_{t+1}^{-\alpha} \cdot (1 - \beta)V_{t+1}^\rho C_{t+1}^{-\rho}}{(1 - \beta)V_t^\rho C_t^{-\rho}} \\ &= \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left[\frac{V_{t+1}}{\mu(V_{t+1})} \right]^{\rho-\alpha}. \end{aligned}$$

□

Looking at our stochastic discount factor, and comparing it to our SDF for CRRA preferences:

$$\begin{aligned} \mathcal{M}_{t,t+1}^{EZ} &= \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left[\frac{V_{t+1}}{\mu(V_{t+1})} \right]^{\rho-\alpha} \\ \mathcal{M}_{t,t+1}^{CRRA} &= \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \end{aligned}$$

we can see that our new stochastic discount factor has an extra term

$$\left[\frac{V_{t+1}}{\mu(V_{t+1})} \right]^{\rho-\alpha}$$

This has an interesting interpretation. This is the continuation value over the certainty equivalent. Suppose that $\rho = 1$ and $\alpha > 1$ (risk aversion is higher). Now, suppose that the value of the program is higher than the expected value of the program $V_{t+1} > \mu_t(V_{t+1})$. This means that the expression

$$\left[\frac{V_{t+1}}{\mu(V_{t+1})} \right]^{\rho-\alpha}$$

will fall and cause the marginal utility/stochastic discount factor to fall. This is in line with our intuition that good states will have lower marginal utility (we want assets that pay off in bad states!).

13.3.6 Empirical Performance of EZ Preferences

We know that our EZ preferences V_t is homogenous of degree 1, therefore, from Euler's theorem, we have that:

$$V_t = \frac{\partial V_t}{\partial C_t} C_t + \mathbb{E}_t \frac{\partial V_t}{\partial V_{t+1}} V_{t+1}$$

We can define the variables

$$MC_t := \frac{\partial V_t}{\partial C_t}$$

$$MV_{t+1} := \frac{\partial V_t}{\partial V_{t+1}}$$

We therefore have

$$V_t = MC_t C_t + \mathbb{E}_t MV_{t+1} V_{t+1}$$

We can divide through by MC_t on both sides and define **household wealth** as $W_t := V_t/MC_t$ (value of program divided by marginal value with respect to consumption) to get

$$\frac{V_t}{MC_t} = C_t + \mathbb{E}_t \left[\left(\frac{MV_{t+1} MC_{t+1}}{MC_t} \right) \frac{V_{t+1}}{MC_{t+1}} \right]$$

to arrive at

$$W_t = C_t + \mathbb{E}_t \left[\mathcal{M}_{t,t+1} W_{t+1} \right] \quad (372)$$

which says that wealth is consumption plus expected discounted future wealth. We also define the **return on wealth** as

$$R_{wt+1} := \frac{W_{t+1}}{W_t - C_t}.$$

Doing a lot of algebra, we can show that the **stochastic discount factor** is

$$\mathcal{M}_{t,t+1} = \beta^\theta R_{wt+1}^{\theta-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}} \quad (373)$$

where $\theta := (1 - \alpha)/(1 - \rho)$ and $\psi := 1/\rho$. This means that we can test the equity premium using:

$$\mathbb{E}[R_{t+1}^i - R_t^f] = - \frac{\text{Cov} \left[\beta^\theta R_{wt+1}^{\theta-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}}, R_{t+1}^i - R_t^f \right]}{\mathbb{E}[\mathcal{M}_{t,t+1}]}. \quad (374)$$

Epstein and Zin (JPE, 1991) found that EZ preferences were able to match the equity premium. Furthermore, the risk-free-rate puzzle does not arise. However, a major issue is that the estimated rate of time preference is often negative and significant.

14 Exotic Preferences

14.1 Canzoneri, Cumby, and Diba (2007) - Empirical Performance of Exotic Preferences

We now look at the influential paper by Canzoneri, Cumby, and Diba (JME, 2007) which evaluates the models we have explored so far and their ability to match the interest rates observed in the real world. We shall see that the risk-free rates implied by the Euler equations from the models we have derived so far do a really bad job of explaining real world risk-free rates. This is a big deal as the Euler equation provides a direct link between consumption demand and monetary policy, and if our model is not able to capture interest rates well, this has big implications for monetary policy.

First, recall what the fundamental asset pricing equation said for risk-free interest rates:

$$\frac{1}{(1 + r_t)} = \mathbb{E}_t \left[\mathcal{M}_{t,t+1} \right] \quad (375)$$

where $\mathcal{M}_{t,t+1}$ is the stochastic discount factor and $(1 + r_t)$ is the **real** risk-free rate. A major issue is that we do not typically observe real risk-free bonds, and those that do exist can be uncommon or illiquid. However, we have a lot of data on nominal risk-free bonds. Therefore, we can rewrite our fundamental asset-pricing equation for nominal risk-free rates as:

$$\frac{1}{(1 + i_t)} = \mathbb{E}_t \left[\mathcal{M}_{t,t+1} \frac{P_t}{P_{t+1}} \right] \quad (376)$$

whereby P_t is the price level at t and $(1 + i_t)$ is the **nominal** risk-free rate.

We now describe the estimation strategy. The issue is that using generalised method of moments is not informative as it essentially boils down to running a χ^2 -test which tells us whether is our model specification incorrect. We want something that can give us more information. The idea used in the paper is to use a **vector autoregression** (VAR). The first trick is to write the stochastic discount factor in terms of direct observable variables such as consumption, wealth returns etc. Then, we can include any other variable from the economy we want into the model by looking at the reduced-form VAR:

$$Y_t = A_0 + A_1 Y_{t-1} + v_t \quad (377)$$

whereby Y_t are economy variables such as disposable income, expenditure on non-durable goods, the Federal funds rate, etc.

We can then estimate the coefficients of our VAR model on the historical data. To then test our model, we construct **forecasts of future consumption and price levels**. Given our consumption and price forecasts (and a specified utility function), we can then construct the implied interest rate by using the Euler equation described in Eq. (376).

In the next few subsections, we describe the different utility functions the paper utilised and the result of the implied interest rates under these preferences.

14.1.1 Baseline Case: CRRA Utility

The paper used a CRRA utility preference as the benchmark model to compare other preferences against. Here, the CRRA utility is defined to be:

$$U(C_t) = \frac{C_t^{1-\alpha}}{1-\alpha} \quad (378)$$

whereby the risk-aversion factor $\alpha = 2$ and the discount factor $\beta = 0.993$.

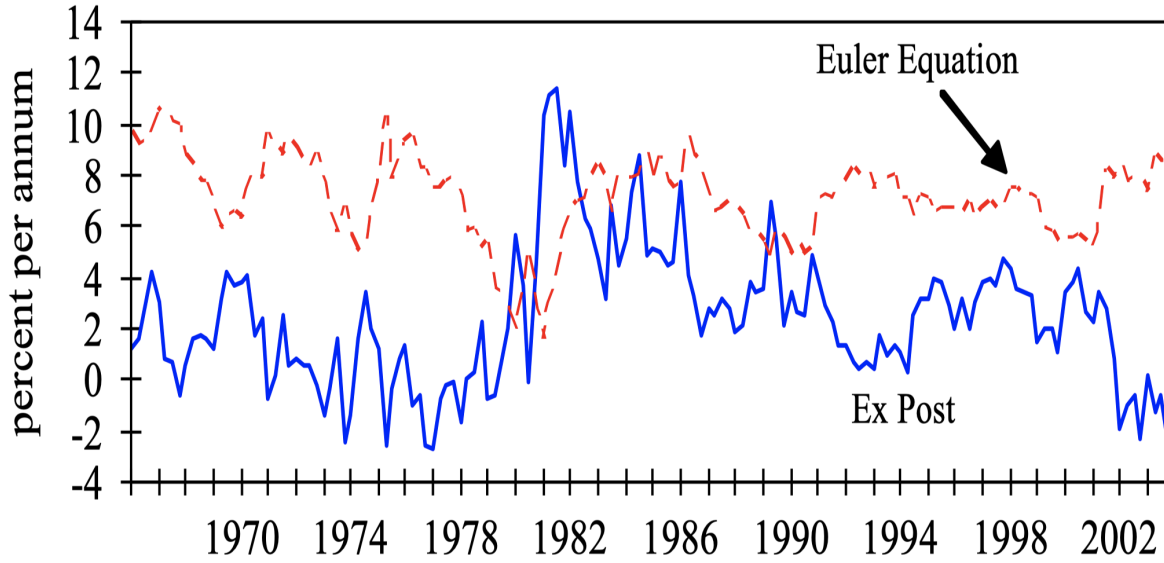


Fig. 1. Real interest rates: ex post and CRRA Euler equation.

Figure 32: Implied interest rates from Euler equation under CRRA utility.

Looking at Fig. 32, the red line is the interest rates implied by our Euler equation whilst the blue line is the real world observed nominal interest rates in the United States. We can see that the **Euler equation does a terrible job at predicting interest rates**. We see that both the **level** and **co-movement** of the implied interest rates is wrong. We see that the interest rates implied by the model have a negative covariance with the real world interest rates!!! This is bad. We will see that it does not get better with other exotic preferences.

14.1.2 Exotic Preference Case: Internal Habit

We now try a utility function that incorporates internal habits. We use the utility function of Boldrin, Christiano, and Fisher (2001), which is given by:

$$U(C_t) = \ln(C_t - bC_{t-1}) \quad (379)$$

and the degree of habit parameter is $b = 0.65$. We see that in Fig. 33, the volatility of the implied interest rate by the Euler equation is extremely high and off again.

14.1.3 Exotic Preference Case: External Habit

We now use a utility function which incorporates external habits. This is based off the work by Abel (1990). The idea is that our utility function depends how much we consume relative to the aggregate consumption. Define $\bar{C}_t$ to be aggregate consumption. Then, define current aggregate consumption to be:

$$Z_t = G^{\eta t} \bar{C}_{t-1}^{\gamma} \quad (380)$$

where G is trend growth over time. Then, the utility function at time t is given by:

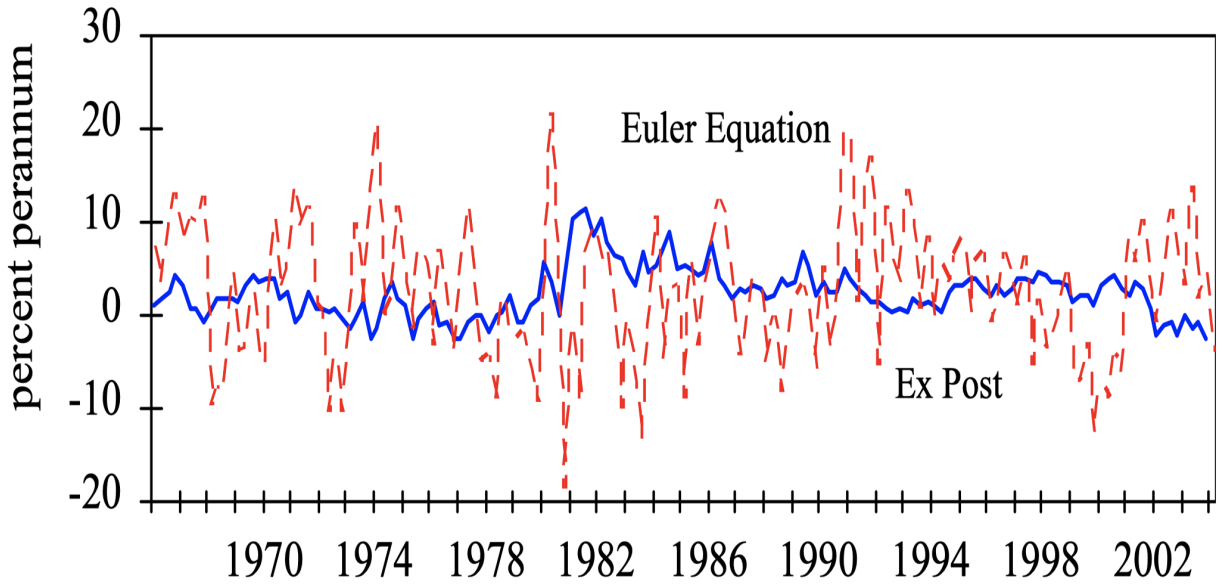


Fig. 3. Real interest rates: ex post and C–E–E Euler equation.

Figure 33: Implied interest rates from Euler equation under internal habits.

$$U(C_t) = \frac{(C_t/Z_t)^{1-\alpha}}{1-\alpha} \quad (381)$$

We can still choose several specifications for this utility function. The three panels in Fig. 34 are based on three further specifications.

1. **Log-normal consumption growth, parameters chosen to match the mean and variance of real money-market rates.** We see here that the implied interest rates are still too high and therefore fit the data poorly.
2. **Time-varying consumption growth.** We see here that this causes implied interest rates to be even higher and an even worse fit to the data.
3. **Parameters chosen to average interest rates and minimize standard deviations.** We now have that the average interest rate and volatility is in line with the data. However, our implied interest rates have a **negative covariance** with the actual interest rate which is a bad thing again.

14.1.4 Exotic Preference Case: Epstein-Zin Preferences

The final exotic preferences used were the Epstein–Zin preferences described by Tallarini (2000), with $IES = 1$, $RRA = \chi$, and $\beta = 0.993$:

$$V_t = \ln(C_t) + \frac{\beta \ln \left\{ \mathbb{E}_t \left[e^{(1-\beta)(1-\chi)V_{t+1}} \right] \right\}}{(1-\beta)(1-\chi)}. \quad (382)$$

Unfortunately, the paper does not include a figure of the interest rates implied by EZ preferences. It does state that both the mean and variance of the model-implied real interest rate match the data well. However, a major issue is that the model-implied and observed real interest rates are negatively correlated.

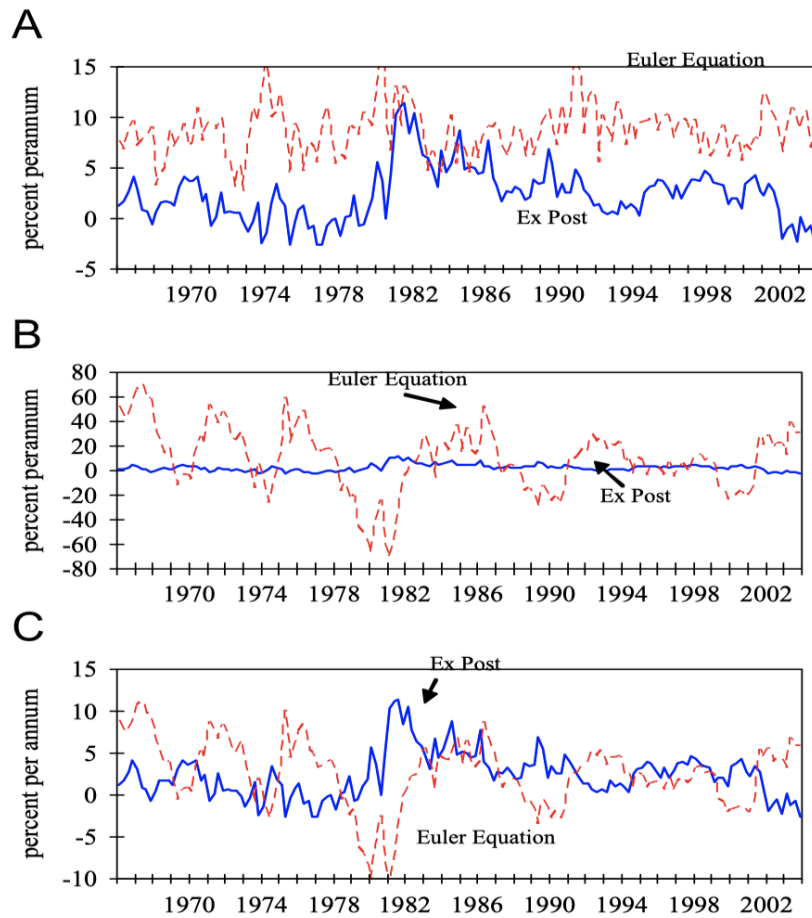


Fig. 4. (A) Real interest rates: ex post and Abel model (iid); (B) real interest rates: ex post and Abel model; and (C) real interest rates: ex post and Abel model (alternate).

Figure 34: Implied interest rates from Euler equation under external habits.

14.1.5 Summary

To summarise, we have no good model of real interest rates which is a big concern. We see that incorporating habits leads to **excess volatility of the risk-free rate**. We also see that recursive preferences lead to a **negative correlation between the model's implied risk-free rate and the empirical data**. The reason that this paper was influential is that it showed that the interest rates implied by the Euler equation do not match real world interest rates despite trying out many different utility preferences. This is a big issue for macroeconomics and finance theory!

15 Disaster Models

15.1 Motivation

We now look at an alternative approach to asset pricing.

In this section, we shall now construct models that contain a small probability that the economy will fall into disaster. In particular, we shall do this through the fundamental asset pricing equation for an arbitrary asset:

$$\mathbb{E}_t[\mathcal{M}_{t,t+1}r_{t+1}^i] = 1 \quad (383)$$

Part A: Twenty OECD Countries in Maddison [2003]				Part B: Eight Latin American and Seven Asian Countries in Maddison [2003]			
Event	Country	Years	% fall in real per capita GDP				
World War I	Austria	1913–1919	35	World War I	Argentina	1912–1917	29
	Belgium	1916–1918	30		Chile	1912–1915	16
	Denmark	1914–1918	16		Chile	1917–1919	23
	Finland	1913–1918	35		Uruguay	1912–1915	30
	France	1916–1918	31		Venezuela	1913–1916	17
	Germany	1913–1919	29	Great Depression	Argentina	1929–1932	19
	Netherlands	1913–1918	17		Chile	1929–1932	33
	Sweden	1913–1918	18		Mexico	1926–1932	31
Great Depression	Australia	1928–1931	20		Peru	1929–1932	29
	Austria	1929–1933	23		Uruguay	1930–1933	36
	Canada	1929–1933	33		Venezuela	1929–1932	24
	France	1929–1932	16		Malaysia	1929–1932	17
	Germany	1928–1932	18		Sri Lanka	1929–1932	15
	Netherlands	1929–1934	16	World War II	Peru	1941–1943	18
	New Zealand	1929–1932	18		Venezuela	1939–1942	22
	United States	1929–1933	31		Indonesia*	1941–1949	36
Spanish Civil War	Portugal	1934–1936	15		Malaysia**	1942–1947	36
	Spain	1935–1938	31		Philippines***	1940–1946	59
World War II	Austria	1944–1945	58		South Korea	1938–1945	59
	Belgium	1939–1943	24		Sri Lanka	1943–1946	21
	Denmark	1939–1941	24		Taiwan	1942–1945	51
	France	1939–1944	49	Post-WWII Depressions	Argentina	1979–1985	17
	Germany	1944–1946	64		Argentina	1998–2002	21
	Greece	1939–1945	64		Chile	1971–1975	24
	Italy	1940–1945	45		Chile	1981–1983	18
	Japan	1943–1945	52		Peru	1981–1983	17
	Netherlands	1939–1945	52		Peru	1987–1992	30
	Norway	1939–1944	20		Uruguay	1981–1984	17
	United States	1944–1947	28		Uruguay	1998–2002	20
Aftermaths of wars	Canada	1917–1921	30		Venezuela	1977–1985	24
	Italy	1918–1921	25		Indonesia	1997–1999	15
	United Kingdom	1918–1921	19		Philippines	1982–1985	18
	United Kingdom	1943–1947	15	Mean for 60 contractions (excluding 5 war aftermaths in part A)			29
	United States	1944–1947	28				

Source: Barro (2006)

Figure 35: 20th Century Economic Disasters. How much did real per capita GDP fall relative to pre-crisis trend. (Barro, 2006).

So far, we have focused on modifying the stochastic discount factor $\mathcal{M}_{t,t+1}$. When we incorporated habits, we introduced lagged consumption. When we introduced Epstein–Zin preferences, we introduced value functions and non-separable utility. We now focus on adding an exogenous component ω_{t+1}^i that serves as a "disaster" term for asset i .

$$\mathbb{E}_t[\mathcal{M}_{t,t+1}r_{t+1}^i\omega_{t+1}^i] = 1 \quad (384)$$

We cannot simply add an arbitrary term ω_{t+1}^i ; we need to impose discipline on it. In particular, we assume that $\omega_{t+1}^i = \omega_{t+1}$, so the disaster shock is independent of the asset i and affects all assets equally.

In Fig. 35, Barro (2006) maps out economic disasters that occurred in the twentieth century. The percentage fall in real per capita GDP is measured relative to the pre-crisis growth trend. We can see that such events have occurred

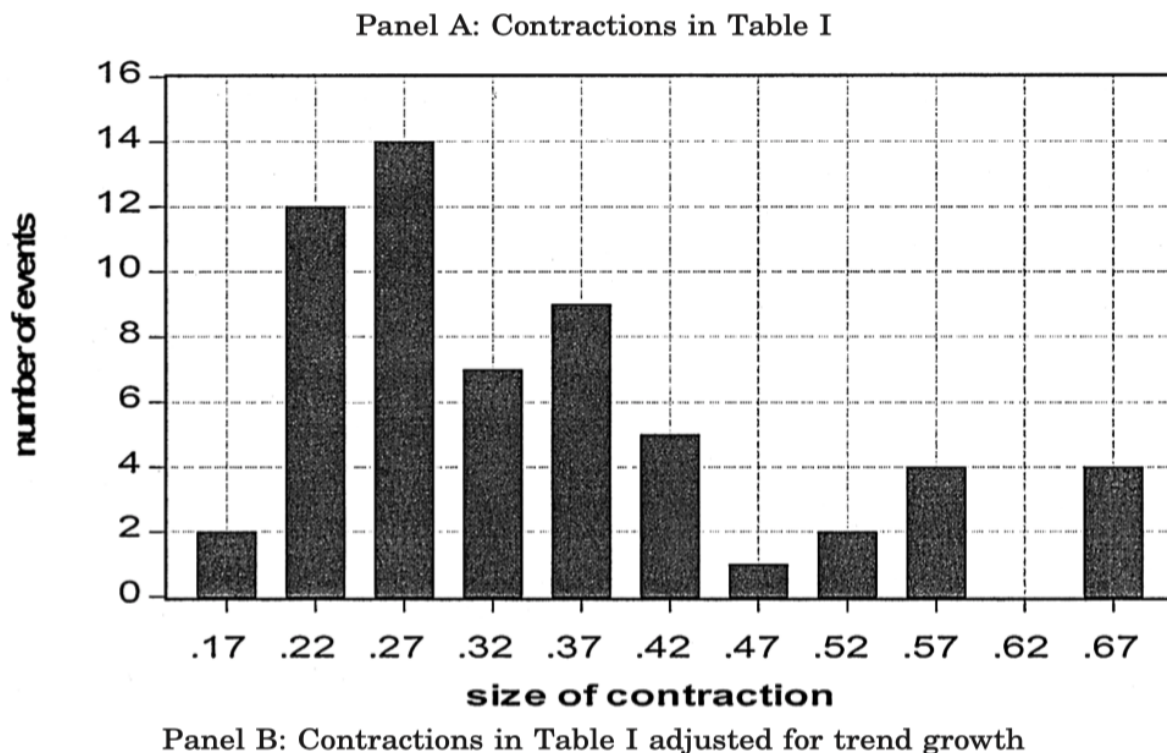


Figure 36: Histogram of declines in real per-capita GDP by 15% or more. A disaster is defined as being a fall in GDP of at least 15%.

repeatedly and have led to large contractions, suggesting that developed economies face some risk of a severe downturn. Figure 36 plots a histogram of these disasters, defining a disaster as a contraction of at least 15% relative to long-run GDP growth. Across 35 countries over 100 years, 60 such events occurred. Assuming homogeneous exposure to disaster risk, the probability of a fall in GDP of at least 15% is:

$$\mathbb{P}[\Delta y_{t+1} \leq -15\%] = 1.7\%. \quad (385)$$

Whilst so far we have looked at GDP, we are generally more interested in consumption when pricing assets. Output and consumption are strongly correlated. We see in Fig. 37 that GDP and consumption move closely together: an output disaster is accompanied by a consumption disaster. We now construct a model that takes such disaster risks into account.

15.2 Barro (2006)

We follow the production model setup based on Barro (2006) article. We define the following variables:

1. C_t : Consumption at time t ;
2. I_t : Investment at time t ;
3. A_t : The level of innovation at time t ;
4. K_{t+1} : The actual level of capital that goes into production at time $t+1$;
5. K_t^* : Level of capital chosen by the planner at time t (more on this later).

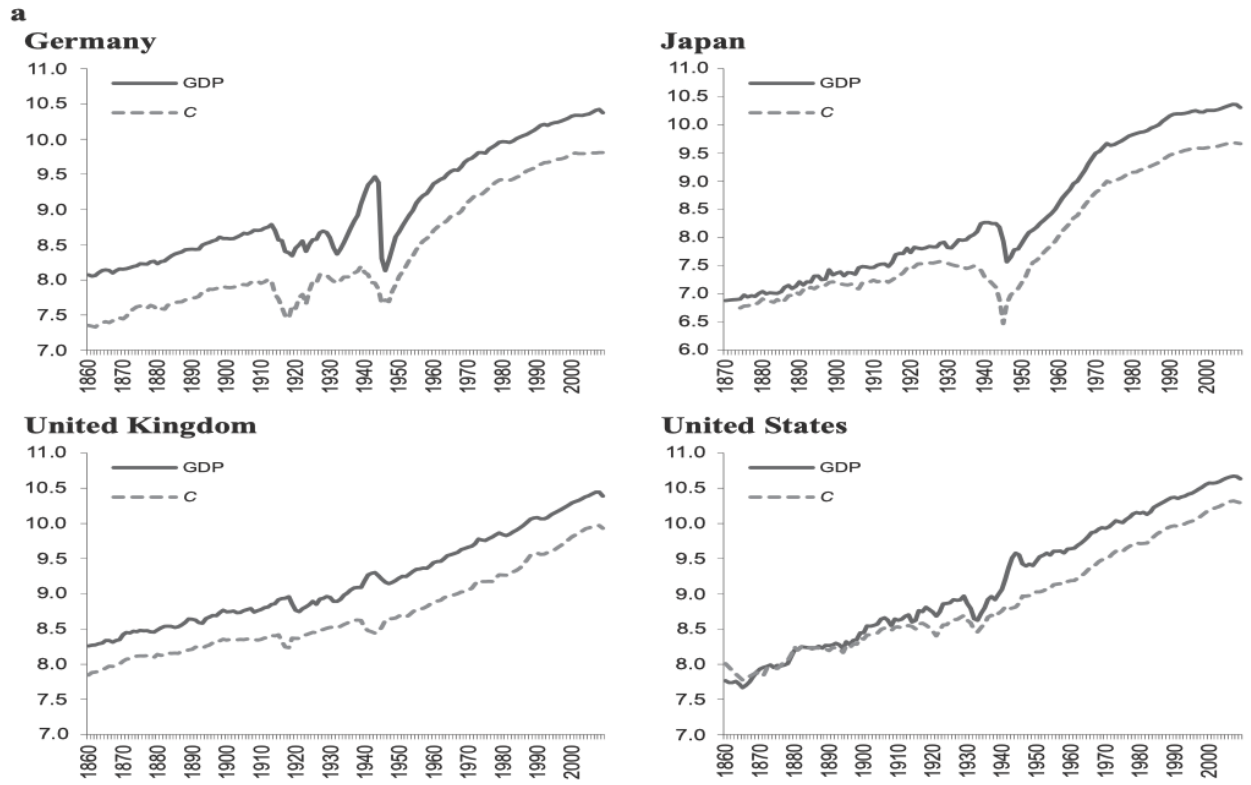


Figure 1
Data for real per-capita personal consumer expenditure (C) and real per-capita GDP for (a) four major countries and (b) four countries with newly available macroeconomic data, ending in 2009. Levels of series were anchored at 2005 values of series on purchasing-power-parity-adjusted, constant-dollar aggregates from *World Development Indicators*. Scales are in natural logs.

Figure 37: Fall in consumption and GDP as a result of disasters.

You can think of the innovation term A_t as the same thing that we saw in the AK model! One interesting setup that we have in this model is that there is a difference between K_t and K_t^* . The agent will be choosing an amount of capital they would like to have in the next period K_t^* but unfortunately random shocks will prevent that planned level of capital from being realised. That is, as we will introduce later, there will be shocks arising from disasters in this economy, which will cause our level of capital to fall and hence our realised level of capital K_{t+1} will differ to our planned level of capital K_t^* .

We now state the objective function of our AK disaster risk model.

Definition 305 (Objective Function of AK Disaster Risk Model)

We have a planner who chooses the sequence $\{C_t, K_t^*, I_t\}_{t=0}^{\infty}$ to maximize their lifetime utility. Therefore our optimisation problem is:

$$U(C_t) = \mathbb{E}_0 \sum_{t=0}^{\infty} \left[\beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \right] \quad (386)$$

Here, we are maximising our expected lifetime CRRA utility function. We now want to impose some constraints on this model. We will assume that the production function in this economy follows the AK production function:

$$Y_t = A_t K_t.$$

This allows us to state our first constraint.

Definition 306 (Resource Constraint)

At every period, the agent can choose to either consume or invest the amount of capital that is generated in that period

$$A_t K_t = C_t + I_t \quad (387)$$

whereby the production function is a linear AK production function.

Remark 307. *The right hand side is the agent's expenditure in that period. The left hand side is the amount of capital that has been produced.*

We now want to describe how the planned level of capital is determined. Here, we shall denote δ as the depreciation rate.

Definition 308 (Law of motion of capital)

At every period, the planned level of capital is determined by:

$$K_t^* = (1 - \delta)K_t + I_t \quad (388)$$

Remark 309. *The amount of capital we **plan** to have for next period K_t^* is the amount of capital from the previous period that has not been depreciated plus any new investment.*

We have been dancing around what is this disaster risk that causes our capital to fall. We shall now properly define it.

Definition 310 (Realisation of capital)

Define d_{t+1} to be a random variable such that

$$d_{t+1} = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } (1 - p) \end{cases}$$

Define θ_{t+1} to be the proportional loss of capital of K_t^* that will be destroyed. Then, we have that the amount of capital in the next period will be:

$$\ln(K_{t+1}) = \ln(K_t^*) - d_{t+1}\theta_{t+1} \quad (389)$$

Proposition 311 (Level realisation of capital)

The realisation of capital can be written as

$$K_{t+1} = K_t^* e^{-d_{t+1}\theta_{t+1}} \quad (390)$$

What is happening here is that d_{t+1} serves as the variable that indicates whether does a disaster occurs or not. Meanwhile, θ_{t+1} serves to indicate how severe the disaster is when it does occur. With all this, we can now specify our disaster risk model.

Definition 312 (AK Disaster Risk Model)

The AK disaster risk model is characterised by the optimisation problem:

$$\begin{aligned} \max_{\{C_t, K_t^*, I_t\}_{t=0}^{\infty}} \quad & \mathbb{E}_0 \sum_{t=0}^{\infty} \left[\beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \right] \\ \text{s.t.} \quad & A_t K_t = C_t + I_t \\ & K_t^* = (1 - \delta) K_t + I_t \\ & K_{t+1} = K_t^* e^{-d_{t+1} \theta_{t+1}} \end{aligned} \quad (391)$$

One thing we can do is to first combine our constraints so that it will be in terms of chosen level of capital.

Proposition 313 (Constraint in terms of chosen capital)

The resource constraint can be expressed in terms of the chosen level of capital:

$$C_t + K_t^* = [A_t + (1 - \delta)] K_{t-1}^* e^{-d_t \theta_t} \quad (392)$$

Proof. First plug in Eq. (390) into the constraints in Eq. (387) and Eq. (388) to get

$$\begin{aligned} A_t K_{t-1}^* e^{-d_t \theta_t} &= C_t + I_t \\ K_t^* &= (1 - \delta) K_{t-1}^* e^{-d_t \theta_t} + I_t \end{aligned}$$

We can then combine these two equations together through I_t to get:

$$C_t + K_t^* = [A_t + (1 - \delta)] K_{t-1}^* e^{-d_t \theta_t}$$

□

Let's pause and look at the equation we just derived

$$C_t + K_t^* = [A_t + (1 - \delta)] K_{t-1}^* e^{-d_t \theta_t}$$

This expression (as we are in a *real economy* setting) is expressing tomorrow's payout in terms of units of consumptions. If we give up unit of capital (in consumption terms), in the next period, we get $A_t K_{t-1}^*$ amount of capital in the next period in addition to the amount that is not depreciated. Alternatively, we can say that this is the marginal product of an unit of additional capital by giving up one of consumption today and getting back what the capital produces in the next period plus what is left over after depreciation. However, we need to take into account that a disaster may strike so we must discount tomorrow's consumption $e^{-d_t \theta_t}$.

We now want to start computing the return on capital in our model. In the paper, Barro makes a huge simplifying assumption.

Assumption 314 (Price of capital)

The price of capital in terms of units of consumption is 1.

We now derive the first-order conditions. We first set up the Lagrangian for our problem.

$$\mathcal{L} = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \left(\beta^t \frac{C_t^{1-\gamma}}{1-\gamma} + \lambda_t \left[(A_t + 1 - \delta) K_{t-1}^* e^{-d_t \theta_t} - C_t - K_t^* \right] \right) \right] \quad (393)$$

We first take first order conditions with respect to consumption:

$$\frac{\partial \mathcal{L}}{\partial C_t} : \beta^t C_t^{-\gamma} - \lambda_t = 0 \quad (394)$$

This gives us the relationships

$$\begin{aligned} \lambda_t &= \beta^t C_t^{-\gamma} \\ \lambda_{t+1} &= \beta^{t+1} C_{t+1}^{-\gamma} \end{aligned}$$

We now take first order conditions with respect to planned capital:

$$\frac{\partial \mathcal{L}}{\partial K_t^*} : \mathbb{E} \left[\lambda_{t+1} (A_{t+1} + (1 - \delta)) e^{-d_{t+1} \theta_{t+1}} - \lambda_t \right] = 0 \quad (395)$$

Re-arranging and plugging in our expression for λ_t and λ_{t+1} :

$$\mathbb{E} \left[\beta^{t+1} C_{t+1}^{-\gamma} (A_{t+1} + (1 - \delta)) e^{-d_{t+1} \theta_{t+1}} \right] = \beta^t C_t^{-\gamma}$$

We therefore arrive at our first useful equation.

Proposition 315 (Capital Pricing Equation)

The capital pricing equation is:

$$\beta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} (A_{t+1} + (1 - \delta)) e^{-d_{t+1} \theta_{t+1}} \right] = 1 \quad (396)$$

Looking at what we derived, recall the fundamental asset pricing equation

$$\mathbb{E} [\mathcal{M}_{t,t+1} R_{t+1}^e] = 1$$

whereby $\mathcal{M}_{t,t+1}$ is the stochastic discount factor. Looking at our capital pricing equation, we can see this is the same thing whereby our stochastic discount factor is

$$\mathcal{M}_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

and our return on equity (capital) is given by:

$$R_{t+1}^e = [A_{t+1} + (1 - \delta)] e^{-d_{t+1} \theta_{t+1}}$$

Definition 316 (Return on capital)

The return on capital is given by

$$R_{t+1}^e = [A_{t+1} + (1 - \delta)] e^{-d_{t+1} \theta_{t+1}} \quad (397)$$

Remark 317. *To interpret this, we are asking how many consumption units do I have to give up today in order to consume tomorrow. This can be thought of as what is the **real return** on capital.*

It is much easier to derive the bond pricing equation.

Proposition 318 (Bond pricing Equation)

The bond pricing equation is:

$$\beta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1}^f \right] = 1 \quad (398)$$

whereby R_{t+1}^f is the return on a (risk-free) bond.

Proof. Use the fundamental asset pricing equation

$$\mathbb{E}[\mathcal{M}_{t,t+1} R_{t+1}^f] = 1$$

We already know that the stochastic discount factor is

$$\mathcal{M}_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

and therefore we can just substitute it in. □

Since we have the capital pricing equation and the bond pricing equation, we can now compute the equity risk premium.

Theorem 319 (Excess-Return Pricing Condition)

Subtracting the two pricing equations gives

$$\mathbb{E}_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} (R_{t+1}^e - R_{t+1}^f) \right] = 0. \quad (399)$$

Proof. Subtract the capital pricing equation and bond pricing equation. □

We now want to examine what happens in this model. First, to make life easier, we can set technology to be fixed $A_t = A$ and for disaster to be fixed $\theta_t = \theta$. To proceed any further, we require a guess for our consumption control function C_t .

Proposition 320 (Policy function guess for consumption)

The policy function guess for consumption is

$$C_t = vK_t = vK_{t-1}^* e^{-d_t \theta} \quad (400)$$

Remark 321. *To interpret this guess, we are saying that the optimal level of consumption is proportional to the level of realised capital. Alternatively, it is equal to our planned amount of capital, discounted by the chance of a disaster risk.*

Proposition 322 (Optimal level of consumption)

The optimal level of consumption is

$$C_t = vK_t = vK_{t-1}^* e^{-d_t \theta} \quad (401)$$

whereby the proportion of capital v is given by

$$v = 1 - \delta + A - \left[\beta(1 - \delta + A) \mathbb{E}_t \left[e^{-d_{t+1} \theta (1-\gamma)} \right] \right]^{\frac{1}{\gamma}} \quad (402)$$

Sketch. The proof contains a lot of algebra. Using the relationship $C_t = \nu K_t = \nu K_{t-1}^* e^{-d_t \theta}$ and the resource constraint $A_t K_t = C_t + I_t$, first derive that

$$I_t = (A - \nu) K_t$$

From this, you can rewrite the budget constraint as

$$\begin{aligned} K_t^* &= (1 - \delta) K_t + I_t \\ &= (1 - \delta) K_t + (A - \nu) K_t \\ &= (1 - \delta) K_{t-1}^* e^{-d_t \theta} + (A - \nu) e^{-d_t \theta} K_{t-1}^* \\ &= (1 - \delta + A - \nu) K_{t-1}^* e^{-d_t \theta} \end{aligned}$$

We also need to use the fact that the budget constraint

$$C_t + K_t^* = [A_t + (1 - \delta)] K_{t-1}^* e^{-d_t \theta}$$

can give us

$$\frac{C_t + K_t^*}{K_{t-1}^*} = [A_t + (1 - \delta)] e^{-d_t \theta}$$

Now, look at our capital pricing equation

$$\beta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} (A_{t+1} + (1 - \delta)) e^{-d_{t+1} \theta_{t+1}} \right] = 1$$

substitute in our guess $C_t = \nu K_{t-1}^* e^{-d_t \theta}$ and the relationships we derived earlier. Then algebra your way to the solution! $\square$

Now that we know what the optimal level of consumption is, we can then derive the expected equity return (which actually does **not** involve what we just did).

Proposition 323 (Expected Equity Return Equation)

The expected equity return is given by

$$\mathbb{E}_t \left[R_{t+1}^e \right] = \mathbb{E}_t \left[\left(A + (1 - \delta) \right) e^{-d_{t+1} \theta} \right] \quad (403)$$

We can see that the expected equity return is

1. Decreasing in the probability of disaster p ;
2. Decreasing in the severity of the disaster θ .

Proof. Let us breakdown this equation. Recall that the probability p comes through in the d_{t+1} term. Therefore, as the probability p increases, d_{t+1} is more likely to become 1, which causes the entire expression to drop due to the negative exponential term. A similar line of reasoning holds for when we increase the severity of disaster loss θ . $\square$

Right now, we are actually at a big issue. We want to explain the equity premium puzzle (i.e. why it is so large). However, it seems that what we have done is to have created a model that has even **lower** expected equity returns. So what gives? As we shall see, whilst the expected equity return is lower, the risk-free rate will be even lower and this will cause an increase in the equity premium!

TABLE V
CALIBRATED MODEL FOR RATES OF RETURN

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
			Parameters				
	No disasters	Baseline	Low θ	High p	Low q	Low γ	Low ρ
θ (coeff. of relative risk aversion)	4	4	3	4	4	4	4
σ (s.d. of growth rate, no disasters)	0.02	0.02	0.02	0.02	0.02	0.02	0.02
ρ (rate of time preference)	0.03	0.03	0.03	0.03	0.03	0.03	0.02
γ (growth rate, deterministic part)	0.025	0.025	0.025	0.025	0.025	0.020	0.025
p (disaster probability)	0	0.017	0.017	0.025	0.017	0.017	0.017
q (bill default probability in disaster)	0	0.4	0.4	0.4	0.3	0.4	0.4
			Variables				
Expected equity rate	0.128	0.071	0.076	0.044	0.071	0.051	0.061
Expected bill rate	0.127	0.035	0.061–0.007		0.029	0.015	0.025
Equity premium	0.0016	0.036	0.016	0.052	0.042	0.036	0.036
Expected equity rate, conditional	0.128	0.076	0.081	0.052	0.076	0.056	0.066
Face bill rate	0.127	0.037	0.063–0.004		0.031	0.017	0.027
Equity premium, conditional	0.0016	0.039	0.019	0.056	0.045	0.039	0.039
Price-earnings ratio	9.7	19.6	17.8	37.0	19.6	27.8	24.4
Expected growth rate	0.025	0.020	0.020	0.018	0.020	0.015	0.020
Expected growth rate, conditional	0.025	0.025	0.025	0.025	0.025	0.020	0.025

Figure 38: Empirical performance of Barro (2006) model.

Proposition 324 (Bond Return Equation)

The (risk-free) bond return is given by

$$R_{t+1}^f = \frac{1}{\beta} \left\{ \mathbb{E}_t \left[\left((A + 1 - \delta - \nu) e^{-d_{t+1}\theta} \right)^{-\gamma} \right] \right\}^{-1} \quad (404)$$

Furthermore, we can see that the risk-free bond rate is

1. Decreasing in the probability of disaster p ;
2. Decreasing in the severity of the disaster θ .

However, this decrease will be larger for larger values of γ .

Sketch. Take the bond pricing equation

$$\beta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1}^f \right] = 1$$

and rewrite it as

$$R_{t+1}^f = \beta^{-1} \left(\mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right] \right)^{-1}$$

Use the fact that $C_{t+1} = \nu K_t^* e^{-d_{t+1}\theta}$ and our formula for ν to get

$$K_{t+1}^* = (1 - \delta) K_t^* e^{-d_{t+1}\theta} + I_{t+1} = (1 - \delta + A - \nu) e^{-d_{t+1}\theta} K_t^*$$

We then plug this into our bond pricing equation by using the guess $C_t = \nu K_t = \nu K_{t-1}^* e^{-d_t\theta}$. □

Remark 325. *Let us interpret the effect of increasing the probability of a disaster p or the severity of a disaster θ . This will cause $d_{t+1}\theta$ to increase, but due to negative in front of it, it will cause the whole expression to fall. But then it is raised to the power of $-\gamma$ which will make the expression bigger. But then there is one final inverse -1 which makes the expression smaller.*

This gives us a nice result. **We have a risk-free rate that is decreasing as we increase the probability of a disaster or the severity of a disaster. Furthermore, this decrease in the risk-free rate is larger, the larger the risk aversion parameter γ is.**

Using our two results, we have the following

Theorem 326 (Equity Risk Premium in AK Disaster Model)

The equity risk premium

$$\mathbb{E}_t \left[R_{t+1}^e \right] - R_{t+1}^f$$

increases as we increase the probability of a disaster p if the parameter γ is high enough.

Proof. We have shown that both the expected equity return $\mathbb{E}_t[R_{t+1}^e]$ and the risk-free bond rate R_{t+1}^f decrease as we increase the probability of a disaster p or the severity of a disaster θ . However, because the risk-aversion parameter γ appears as an exponent in the risk-free-return equation, there will be a **larger drop** in the risk-free rate than in the expected equity return. Therefore, this generates an equity premium. □

Barro shows that the equity premium in this AK production model is the same as in an endowment model. Now, we want to see how well does our model actually perform in matching the data. We can see how the model performs in Fig. 38. In column (1), this uses a CRRA utility function model with no disasters. If we look down the column, we

can see that the model produces an equity rate of 12.8% (0.128) and a safe rate of 12.7%. This leads to an extremely tiny equity premium of 0.1%, which is a terrible fit for our data.

Moving to column (2) in Fig. 38, we now have our baseline model, in which the probability of a disaster is $p = 0.017$. Barro also assumes that the safe asset (T-bills) can default, with probability $q = 0.4$. This means that the safe rate does not yield a guaranteed risk-free return. Nevertheless, as we will soon see, investors are willing to hold this asset because it provides protection during a disaster, and hence they are willing to hold it at a discount. In this model, the equity return is 7.1% and the risk-free rate is 3.5%. Both are much lower than in our CRRA model in column (1). This gives an equity premium of 3.6%, which is much closer to the data than the previous equity premium of 0.1%. Under the disaster model, both the equity and bill returns fall, but the bill return falls by more, thereby generating an equity premium. This is in line with the results we derived for the disaster model.

Moving to column (3) in Fig. 38, we lower the risk-aversion parameter from $\gamma = 4$ to $\gamma = 3$. The equity return is around 7.6%, close to the 7.1% obtained when $\gamma = 4$ in the baseline model. However, the risk-free rate is now 6.1%, compared with 3.5% in the baseline model. Thus, the risk-free rate falls by less when we **lower** risk aversion, in line with our earlier result. Consequently, the equity premium is also much smaller, at 1.6% rather than 3.6%.

So we now have a large equity risk premium! However, our risk-free rate is quite low. We move onto an extension of this model by Gourio to try and fix this.

15.3 Gourio (2012)

Gourio in his 2012 paper embeds these ideas of disasters into RBC models except with two new additional ideas:

1. **Capital adjustment costs:** There is a cost you need to pay for using your capital as investment.
2. **Time-varying disaster probabilities:** The probability of a disaster occurring is no longer a fixed constant p .

We now have that the agent can also decide how much labour they wish to supply N_t . However, recall that they value leisure so then that means $1 - N_t$ is what they enjoy.

Definition 327 (Agent's Preferences)

The agent's preferences are given by

$$V_t = \left\{ \left[C_t(1 - N_t)^\psi \right]^{1-\psi} + \beta \mathbb{E}_t \left[\left(V_{t+1}^{1-\gamma} \right)^{\frac{1-\psi}{1-\gamma}} \right] \right\}^{\frac{1}{1-\psi}} \quad (405)$$

whereby $\frac{1}{\psi}$ is the **intertemporal elasticity of substitution** and γ is the **coefficient of risk aversion**.

Remark 328. *This looks a lot like the Epstein–Zin preferences from before.*

We can also describe capital production by a Cobb–Douglas production.

Definition 329 (Production Function)

The production function is given by

$$Y_t = K_t^\alpha (z_t N_t)^{1-\alpha} \quad (406)$$

whereby N_t is labour and z_t is innovation.

Remark 330. *We will soon specify what form this innovation takes!*

Now, we shall define the **investment adjustment function**

$$\phi\left(\frac{I_t}{K_t}\right)$$

To interpret this, suppose we had instead:

$$\phi\left(\frac{I_t}{K_t}\right)K_t$$

This means that unlike before, if we invest one unit of capital, we do not get a 1-1 increase in the amount of capital we get back. It is sort of like decreasing returns to scale. With this in mind, we can now define what capital accumulation looks like in this model:

Definition 331 (Capital Accumulation)

Capital accumulation is given by

$$K_{t+1} = \left[(1 - \delta)K_t + \phi\left(\frac{I_t}{K_t}\right)K_t \right] e^{d_{t+1}\xi_{t+1}} \quad (407)$$

where the i.i.d. shocks satisfy:

$$\xi_{t+1} \sim \mathcal{N}(\mu_\xi, \sigma_\xi^2)$$

In this model, ξ_{t+1} is one form of our disaster shocks.

We can now specify the structure of our innovation terms in the production function z_t . We will see that it will have a permanent and transitory component of productivity.

Definition 332 (Productivity)

The innovation term z_t has two parts to it

$$\ln(z_t) = \ln(z_{p,t}) + \ln(z_{r,t}) \quad (408)$$

whereby the permanent component is

$$\ln(z_{p,t}) = \mu + \ln(z_{p,t-1}) + \epsilon_t + d_t\theta_t$$

and the transitory term is

$$\ln(z_{r,t}) = \rho_z \ln(z_{r,t-1}) + d_t(\psi_t - \theta_t)$$

where ϵ_t , θ_t , and ψ_t are distributed as $\mathcal{N}(0, \sigma_j^2)$ for $j \in \{\epsilon, \theta, \psi\}$.

Remark 333. Looking at the permanent component, μ is the long run level growth rate. Furthermore, $d_t\theta_t$ represents the permanent level effect of disaster on productivity.

In the transitory component, $d_t(\psi_t - \theta_t)$ captures the transitory effect of the disaster.

We can now incorporate the time-varying probabilities.

Definition 334 (Time-varying probabilities)

The probability of a disaster, conditioned of a disaster not currently happening is

$$\mathbb{P}\left(d_{t+1} = 1 | d_t = 0\right) = p_t$$

whereby p_t has an AR(1) structure

$$\ln(p_t) = (1 - \rho_p)\ln(\bar{p}) + \rho_p \ln(p_{t-1}) + \epsilon_{p,t}$$

whereby ρ_p are coefficients to be estimated.

Remark 335. Notice how we defined the probability of a disaster conditioning on a disaster not currently occurring. Gourio (2012) goes on to specify that if the economy is currently in a disaster, then

$$\mathbb{P}\left(d_{t+1} = 1 | d_t = 1\right) = \max\{q, p_t\}$$

whereby q is a number that approximately 91% according to the paper. In practice, $p_t \approx 5\%$ so then that means if we are in a disaster, then there is a very high chance that we will still remain in a disaster.

It is quite complex but one can compute the stochastic discount factor.

Definition 336 (Stochastic Discount Factor)

The stochastic discount factor is given by

$$\mathcal{M}_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\psi} \left(\frac{1 - N_{t+1}}{1 - N_t} \right)^{(1-\psi)\nu} \left[\frac{V_{t+1}}{(\mathbb{E}_t[V_{t+1}^{1-\gamma}])^{1/(1-\gamma)}} \right]^{\psi-\gamma} \quad (409)$$

The gross risk-free rate is given by

$$R_{t+1}^f = \frac{1}{\mathbb{E}_t[\mathcal{M}_{t,t+1}]}.$$

One final extension in Gourio (2012) accounts for the fact that governments often default on bonds during disasters. The government repays only a fraction $r < 1$ of the bond principal during a disaster.

Proposition 337 (T-bill Price)

The price of a one period government bond is given by

$$Q_{1,t} = \mathbb{E}_t \left[\mathcal{M}_{t,t+1} [1 - d_{t+1}(1 - r)] \right]$$

whereby $r < 1$ is the fraction that will be repaid.

From this, we can expand out and price bonds of longer maturities.

Proposition 338 (Term Structure)

$$Q_{n,t} = \mathbb{E}_t \left[\mathcal{M}_{t,t+1} [1 - d_{t+1}(1 - r)] Q_{n-1,t+1} \right]$$

with $Q_{0,t} = 1$ for all time periods t .

Parameters					
Capital share	α	.34	Mean permanent shock	μ_θ	-0.007
Depreciation rate	δ	.02	Std. dev. permanent shock	σ_θ	0.092
Leisure preference	v	2.33	Mean transitory shock	μ_φ	-0.055
Discount factor	β	.999	Std. dev. transitory shock	σ_φ	0.041
Adjustment cost	η	1.7	Persistence of productivity	ρ_z	0.71
Trend growth of TFP	μ	.005	Persistence of disaster state	q	0.914
Std. dev. of TFP shock	σ_ε	.01	Persistence of $\log(p)$	ρ_p	0.90
Elasticity of substitution	$1/\hat{\psi}$	2	Std. dev. of $\log(p)$	$\frac{\sigma_p}{\sqrt{1-\rho_p^2}}$	2.80
Risk aversion	γ	3.8	Average prob. of disaster		0.72

TABLE 1—PARAMETER VALUES FOR THE BENCHMARK MODEL. NOTE: THE DISTRIBUTION OF TRANSITORY SHOCKS IS TRUNCATED TO NEGATIVE VALUES, AS IN BARRO ET AL. (2011).

Figure 39: Parameter calibration of Gourio (2012) model.

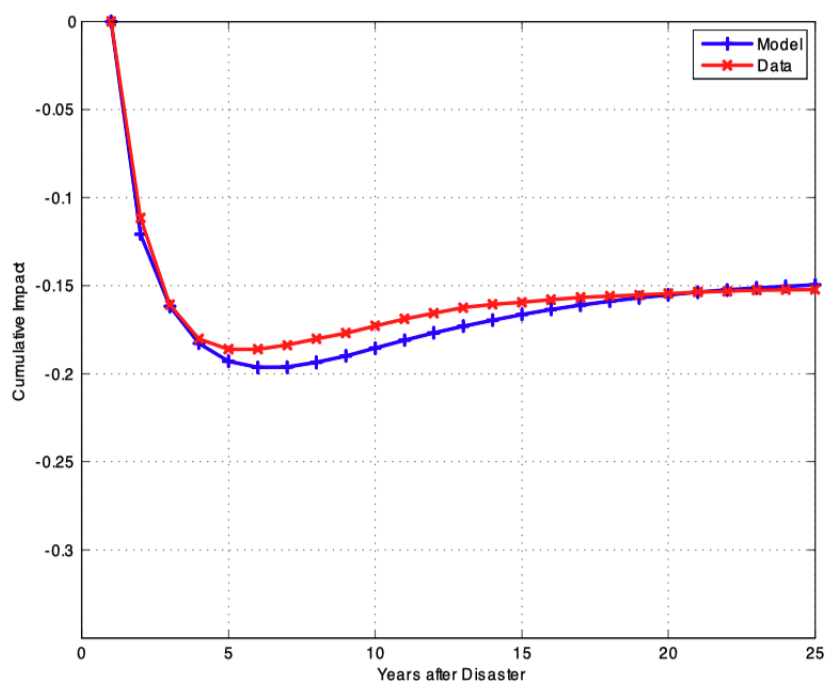


FIGURE 1. IMPULSE RESPONSE FUNCTION OF CONSUMPTION TO A DISASTER REALIZATION: MODEL VS. DATA (ESTIMATE OF BARRO ET AL. (2011)).

Figure 40: Impulse Response Function of Gourio (2012) model.

Remark 339. This is a recursive bond pricing formula whereby the price of a bond at time t with a maturity in n periods $Q_{n,t}$ depends on the price of a bond tomorrow with a maturity of $n - 1$ periods $Q_{n-1,t+1}$.

In Fig. 39, Gourio specifies the model's parameters. In particular, he sets the persistence of a disaster state to approximately $q = 0.914$. This means that once a disaster occurs, it is likely to persist for a long time.

We can use the impulse response functions in Fig. 40 to see how consumption in the model responds relative to the data. The model provides a reasonable fit.

		$\sigma(Y)$	$\sigma(C)$	$\sigma(I)$	$\sigma(N)$	$\rho_{C,Y}$	$\rho_{I,Y}$	$\rho_{N,Y}$
1	Data	0.99 (0.06)	0.54 (0.03)	2.75 (0.17)	0.93 (0.06)	0.48 (0.06)	0.61 (0.05)	0.74 (0.04)
2	No disasters	0.82 (0.04)	0.48 (0.03)	1.49 (0.08)	0.25 (0.01)	0.99 (0.00)	1.00 (0.00)	0.99 (0.01)
3	Constant p	0.82 (0.04)	0.48 (0.03)	1.56 (0.08)	0.25 (0.01)	0.99 (0.00)	1.00 (0.00)	0.99 (0.01)
4	Benchmark	0.93 (0.07)	0.69 (0.10)	2.96 (0.64)	0.69 (0.17)	0.32 (0.24)	0.87 (0.03)	0.75 (0.04)

TABLE 2—BUSINESS CYCLE STATISTICS FOR CONSUMPTION, INVESTMENT, OUTPUT AND EMPLOYMENT. NOTE: ALL SERIES ARE IN GROWTH RATE. STANDARD ERRORS IN PARENTHESIS. DATA 1947Q1-2010Q4.

Figure 41: Macro Variables Moments of Gourio (2012) model.

We now want to see more granular results from the model. We can see the moments computed for the macroeconomic variables in Fig. 41. Here, the standard deviation of output (Y), consumption (C), investment (I), and number of hours worked N is computed. Additionally, the correlation between these macro variables are computed. The first row shows what the empirical data say that these values are. In row 2, we compute the simulations from a model that does not incorporate any disasters. We see that the standard deviation of investment $\sigma(I) = 1.49$ and hours worked $\sigma(N) = 0.25$ is significantly lower than that observed in the data. In row 3, we now have a model with a probability of the risk of disaster being p , but this probability **does not vary** over time. We see that the standard deviation of investment $\sigma(I) = 1.56$ and hours worked $\sigma(N) = 0.25$ is still too low. This shows why it is important to have the risk of disaster vary over time. The fourth and final row shows the model we have developed with time varying disaster risk probabilities. We see that the standard deviation of investment $\sigma(I) = 2.96$ and hours worked $\sigma(N) = 0.69$. This much closer to what we observed in the data! This is quite a big improvement.

We also can look at the moments computed for the financial variables in Fig. 42. The first column is the risk-free rate. However, there is no such thing as a truly risk-free asset in the real world and hence why there is no value for this variable in the data row (the first row). The second column is the return on government bonds, which is usually a proxy for the risk-free rate. The third column is the expected equity risk premium. The subsequent columns are the standard deviations of the aforementioned financial variables. We see that in column 2 when we employ EZ preferences and have no disasters, there is no equity premium and there is little volatility of the excess returns. This is a bad fit of the data. In the third row, where we now incorporate non-varying disaster risk probabilities, we see that

		$E(R_f)$	$E(R_b)$	$E(R_e - R_b)$	$\sigma(R_f)$	$\sigma(R_b)$	$\sigma(R_e - R_b)$
1	Data	—	0.19 (0.05)	1.84 (0.52)	—	0.82 (0.06)	8.20 (0.44)
2	No disasters	0.56 (0.03)	0.56 (0.03)	0.01 (0.07)	0.05 (0.01)	0.05 (0.01)	1.03 (0.06)
3	Constant p	-0.17 (0.03)	0.04 (0.03)	2.30 (0.02)	0.05 (0.01)	0.05 (0.01)	0.61 (0.03)
4	Benchmark	0.16 (0.30)	0.28 (0.21)	2.24 (0.42)	1.00 (0.56)	0.74 (0.41)	7.67 (1.85)

TABLE 3—FINANCIAL STATISTICS. NOTE: MEAN AND STANDARD DEVIATION OF THE RISK-FREE, T-BILL, AND EQUITY EXCESS RETURN. STANDARD ERRORS IN PARENTHESIS. DATA 1947Q1-2010Q4.

Figure 42: Financial Variables Moments of Gourio (2012) model.

we now have a higher equity premium of 2.30% but the volatility $\sigma(R_e - R_b) = 0.61$ is way too low. Finally, in the fourth row with our benchmark model, which incorporates a time varying disaster probability, we see that we still have the equity premium but now our volatility is much higher $\sigma(R_e - R_b) = 7.67$ and hence a much better fit to the data!

We can also simulate the model after a disaster shock. The results appear in Fig. 43, where the disaster lasts for 10 periods. During this time, output and consumption keep falling because of the large decline in productivity and capital. After the disaster, productivity and economic activity remain at a **permanently** lower level because of the shock to the long-run productivity trend. Furthermore, real interest rates fall and remain low until the disaster has passed. T-bill rates fall below 1%, as investors are willing to hold them as insurance. Finally, stock prices fall sharply and then recover quickly.

One issue is that the model implies that disasters always destroy capital and permanently reduce productivity, neither of which is true for every disaster.

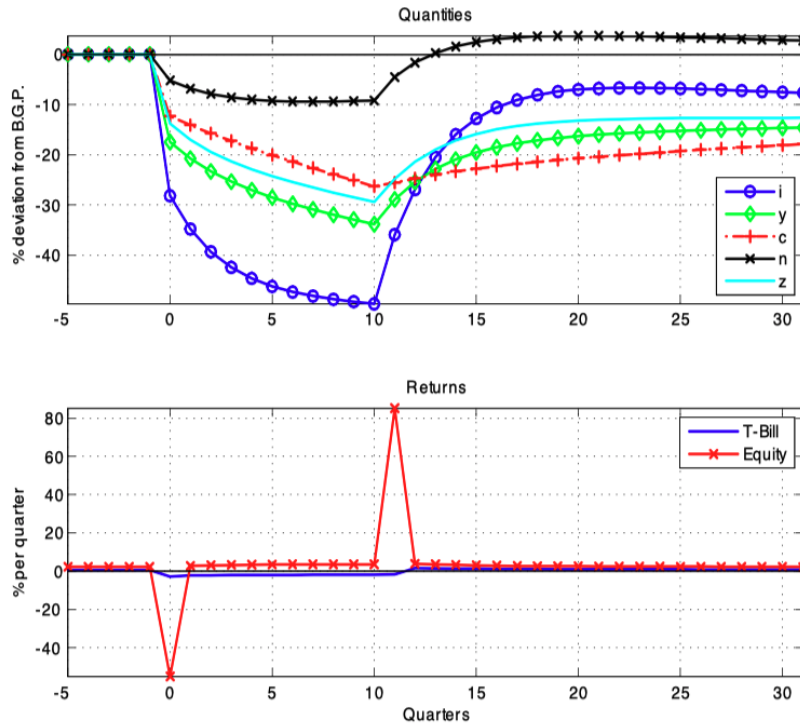


FIGURE 2. RESPONSE OF MACROECONOMIC QUANTITIES AND ASSET RETURNS TO A TYPICAL DISASTER IN THE MODEL.

Figure 43: Disaster Realization of Gourio (2012) model.

16 Government Debt

16.1 Government Debt

Government bond yields, both nominal and real, have been falling over the last 25 years and are at historically low levels. With yields below GDP growth rates ($r^f < g$), many have argued that the government can safely roll over its debt, so fiscal capacity may be unlimited. In particular, Olivier Blanchard argues that since the nominal T-bill rate is lower than GDP growth, the government can roll over the entire public debt every three months at this T-bill rate and, as a result, the debt-to-GDP ratio will drift towards zero over time. This led Blanchard to argue that there is no fiscal cost. The two papers that we examine argue that this reasoning is valid only in a deterministic model. From an asset-pricing perspective, **economic growth is risky**, and risks such as shocks to the permanent growth rate (for example, secular stagnation) should be captured by a risk-premium term RP . Therefore, they argue that the relevant comparison is:

$$r^f + RP > g \quad (410)$$

even if $r^f < g$.

16.1.1 The U.S. Public Debt Valuation Puzzle - Jiang et al. (2021a)

The main idea of this paper is to look at pricing the **entire** government bond portfolio (1-year, 2-year, 10-year bond etc) rather than individual bond securities. Then, in the absence of bubbles, the market value of outstanding debt should

equal the present discounted value of current and future primary surpluses (taxes less government spending without taking interest payments into account). By extension, the expected return on this debt portfolio should depend on the risk profile of the primary surpluses in question. Furthermore, they argue that the risk-free yields are **not** the right discount rate for valuing the entire government bond's portfolio and should instead take into account the short-run and long-run risks associated to primary surpluses.

We make a short run and long run assumption.

Assumption (Short Run Assumption)

Government spending is counter-cyclical and tax revenues are pro-cyclical.

Assumption (Long Run Assumption)

Tax revenues and government spending are cointegrated with GDP.

We can now define the one period government budget constraint.

Definition 340 (One Period Government Budget Constraint)

Let G_t denote nominal government spending, T_t as nominal government tax revenue, Q_t^h as the outstanding face value at time t of the government bond payments due at time $t + h$ where h is the maturity and where Q_{t-1}^1 is the total amount of debt payments that is due today. Let P_t^h be the price at time t of a nominal zero-coupon bond that pays \$1 at time $t + h$.

The one period government budget constraint is:

$$G_t + Q_{t-1}^1 = \sum_{h=1}^H \left(Q_t^h - Q_{t-1}^{h+1} \right) P_t^h + T_t \quad (411)$$

where expenditure is on the left-hand side and income is on the right-hand side.

We have many different types of bonds with respective prices floating around. We therefore need to impose a no arbitrage condition which intuitively says that a 5-year bond today is the present discounted value of a 4 year bond at the end of next year.

Assumption (No Arbitrage Bond Pricing Condition)

The no arbitrage condition says that:

$$P_t^h = \mathbb{E}_t \left[\mathcal{M}_{t,t+1} P_{t+1}^{h-1} \right] \quad (412)$$

for all $h \leq H$.

Now, we can take the one period budget constraint and iterate forward.

Proposition 341 (Value of Government Outstanding Debt)

Let D_t denote the nominal market value of the outstanding government debt portfolio. Then, the entire value of government's outstanding debt is given by:

$$D_t = \sum_{h=0}^H Q_{t-1}^{h+1} P_t^h = \mathbb{E}_t \left[\sum_{j=0}^T \mathcal{M}_{t,t+j} (T_{t+j} - G_{t+j}) \right] + \mathbb{E}_t [\mathcal{M}_{t,t+T} D_{t+T}]. \quad (413)$$

This is the value of outstanding government debt. It must equal the discounted primary surpluses (where "primary" means excluding interest payments), represented by the $(T - G)$ term, plus the discounted value of outstanding debt T periods from now. The stochastic discount factor $\mathcal{M}_{t,t+j}$ prices risky assets in the economy, including these future surpluses. We need to impose a no-bubble condition on the terminal-debt term.

Assumption (Transversality (No-Bubble) Condition)

The no-bubble condition is:

$$\lim_{T \rightarrow \infty} \mathbb{E}_t [\mathcal{M}_{t,t+T} D_{t+T}] = 0. \quad (414)$$

Theorem 342 (Value Equivalence)

In the absence of arbitrage opportunities and subject to a transversality condition, the market value of the outstanding government debt portfolio equals the expected risk-adjusted present discounted value of current and future primary surpluses:

$$D_t = \sum_{h=0}^H Q_{t-1}^{h+1} P_t^h = \mathbb{E}_t \left[\sum_{j=0}^{\infty} \mathcal{M}_{t,t+j} (T_{t+j} - G_{t+j}) \right] \equiv P_t^T - P_t^G. \quad (415)$$

where we have the PDV of the tax claim P_t^T and the spending claim P_t^G given by:

$$P_t^T \equiv \mathbb{E}_t \left[\sum_{j=0}^{\infty} \mathcal{M}_{t,t+j} T_{t+j} \right] \quad (416)$$

$$P_t^G \equiv \mathbb{E}_t \left[\sum_{j=0}^{\infty} \mathcal{M}_{t,t+j} G_{t+j} \right] \quad (417)$$

This result must hold ex ante in nominal and real terms. However, ex post, surprise inflation can erode the real value of debt. If we think about what this looks like on the government's balance sheet, here, P_t^T is tax revenue on the asset side of treasury balance sheet whilst P_t^G is spending commitments and value of debt D_t which both are on the liability side.

We can give another interpretation to the value equivalence equation:

$$\underbrace{D_t = \sum_{h=0}^H Q_{t-1}^{h+1} P_t^h}_{\text{The market value of government debt}} = \underbrace{\mathbb{E}_t \left[\sum_{j=0}^{\infty} \mathcal{M}_{t,t+j} (T_{t+j} - G_{t+j}) \right]}_{\text{Expected risk-adjusted PDV of future primary surpluses}}$$

Now, let us rewrite the product of the expectation using covariances:

$$D_t = \mathbb{E}_t \left[\sum_{j=0}^{\infty} \mathcal{M}_{t,t+j} (T_{t+j} - G_{t+j}) \right] = \sum_{j=0}^{\infty} \mathbb{E}_t [\mathcal{M}_{t,t+j}] \mathbb{E}_t [T_{t+j} - G_{t+j}] + \sum_{j=0}^{\infty} \text{Cov}_t [\mathcal{M}_{t,t+j}, T_{t+j}] - \sum_{j=0}^{\infty} \text{Cov}_t [\mathcal{M}_{t,t+j}, G_{t+j}]$$

This gives us our next result.

Theorem 343 (Fiscal Capacity Of Government)

The fiscal capacity of the government is:

$$D_t = \sum_{j=0}^{\infty} \mathbb{E}_t \left[\mathcal{M}_{t,t+j} \right] \mathbb{E}_t \left[T_{t+j} - G_{t+j} \right] + \sum_{j=0}^{\infty} \text{Cov}_t \left[\mathcal{M}_{t,t+j}, T_{t+j} \right] - \sum_{j=0}^{\infty} \text{Cov}_t \left[\mathcal{M}_{t,t+j}, G_{t+j} \right] \quad (418)$$

The first term on the right-hand side is the present discounted value of all expected future surpluses, using the term structure of risk-free bond prices. It is the expected PDV for a risk-neutral investor. If the SDF is deterministic, the covariance terms are zero, so this is the only term on the right-hand side. Fiscal capacity is then constrained by the government's ability to generate current and future surpluses, and lower interest rates increase fiscal capacity. The second and third terms encode the riskiness of the government debt portfolio and arise when discount rates are stochastic. If tax revenues tend to be high in good times, when $\mathcal{M}_{t,t+j}$ is low, then the tax covariance is negative. If government spending tends to be high in bad times, when $\mathcal{M}_{t,t+j}$ is high, then the spending covariance is positive but enters the expression with a minus sign. If both are true, the covariance terms lower the government's fiscal capacity. Put differently, the risk-neutral present value of future surpluses must be higher by an amount equal to the absolute value of the covariance adjustment to support a given positive amount of government debt D_t .

Now, from our assumptions, we have that:

$$D_t = \sum_{j=0}^{\infty} \mathbb{E}_t \left[\mathcal{M}_{t,t+j} \right] \underbrace{\mathbb{E}_t \left[T_{t+j} - G_{t+j} \right]}_{\approx 0} + \underbrace{\sum_{j=0}^{\infty} \text{Cov}_t \left[\mathcal{M}_{t,t+j}, T_{t+j} \right]}_{<0} - \underbrace{\sum_{j=0}^{\infty} \text{Cov}_t \left[\mathcal{M}_{t,t+j}, G_{t+j} \right]}_{>0} \quad (419)$$

The covariance terms lower fiscal capacity, and the paper also argues that these terms are large. This implies a large negative risk adjustment to fiscal capacity. Intuitively, if the government wants to support debt equal to 100% of GDP, standard risk-neutral logic would require an equally large present value of surpluses. The covariance terms imply that an even larger present value of surpluses is needed once aggregate risk is taken into account.

There are two key asset pricing insights related to the above equation Eq. (419).

(1) There are business-cycle risks associated with the cash flows from future primary surpluses. Imagine a representative investor who buys all government debt issuance and receives all redemptions, thereby acquiring a claim to future primary surpluses $\{S_{t+j}\}_{j \geq 1}$, where $S_{t+j} = T_{t+j} - G_{t+j}$. These surpluses are the cash flows from the investment strategy. They are **highly procyclical** because tax revenues T_{t+j} are procyclical (governments cut taxes in bad times and raise them in good times) and government spending G_{t+j} is countercyclical (governments spend more in recessions). The cash flow therefore has the wrong kind of business-cycle risk: it pays well in good times, when marginal utility is low, and poorly in bad times, when marginal utility is high. Consequently, the government debt portfolio carries a **business-cycle risk premium**, much like the equity risk premium associated with procyclical dividends.

(2) There is also long-run risk. The authors argue that government spending G and taxes T must be **cointegrated** with GDP. Since surpluses S are the difference between taxes and spending, they are also cointegrated with GDP. Therefore, the government debt portfolio inherits the long-run risks in GDP and carries a long-run risk premium. This is similar to the long-run risk in the disaster model.

They then tried to estimate the PDV of surplus claim and compare it to the total amount of debt issued. What they found was a strange result.

Definition 344 (Government Debt Valuation Puzzle)

The government bond portfolio is more valuable than the surplus claim when measured in the data:

$$D_t > 0 > \mathbb{E}_t \left[\sum_{j=0}^{\infty} \mathcal{M}_{t,t+j} (T_{t+j} - G_{t+j}) \right]. \quad (420)$$

Remark 345. *There is much more government debt issued compared to what the government should be able to fiscally support through their primary surpluses!*

Equivalently, the interest rate that the government pays on its debt is much lower than the risk-adjusted rate on the surplus claim where this risk-adjusted rate contains both the business cycle risk premium and the long-run risk premium we discussed earlier.

To summarise the paper, the authors argue that government bonds are overpriced as they do not take into account sufficient business cycle risk and long run risk. Furthermore, they argued that the right discount rate to use $r^f + RP$ where RP is the risk premium term is much higher than the growth rate g . This is because the correct discount rate takes into account that economic growth is very risky.

16.1.2 Manufacturing Risk-free Government Debt - Jiang et al. (2021b)

This paper looks more carefully at government debt. Here, governments face a trade-off between insuring bondholders and insuring taxpayers against output risk. If they insure bondholders by manufacturing risk-free zero-beta (beta measures covariance of payout from the debt with the broader economy) debt, then their capacity to insure taxpayers by lowering tax rates or raising government spending during recessions is limited.

Before proceeding, let us review corporate finance. Debt and equity are factored in when assessing a company's risk profile. Beta is the slope of the coefficient for a stock regressed against a benchmark market index. A key determinant of beta is leverage, which measures the level of a company's debt to its equity. Levered beta measures the risk of a firm with debt and equity in its capital structure to the volatility of the market. Unlevered beta (or asset beta) measures the market risk of the company without the impact of debt. *Unlevering* a beta removes the financial effects of leverage thus isolating the risk due solely to company assets. In other words, how much did the company's equity contribute to its risk profile.

Theorem 346 (Modigliani-Miller For Corporate Finance)

The Modigliani-Miller equation is:

$$\underbrace{\frac{D+E}{D}\beta^A}_{\text{Risk Of Unlevered asset}} = \underbrace{\beta^D}_{\text{Risk Of Unlevered asset}} + \underbrace{\frac{E}{D}\beta^E}_{\text{Risk Of Corporate Equity}} \quad (421)$$

where D is the amount of debt held by the company and E is the amount of equity held by the company.

The Modigliani-Miller equation relates the beta of a levered firm (a firm financed by both debt and equity) to that of its unlevered (i.e., a firm which has no debt) counterpart.

We now go back to the paper. The government bond portfolio is backed by a long position in a claim to tax revenue and a short position in a claim to government spending. We want to measure the riskiness to the claim on taxes (long position on primary surpluses). Conversely, you are said to take a short position on primary surpluses via the spending claim. The riskiness of these two claims determines the risk-premium on the treasury portfolio **regardless of the**

maturity choices that the government makes. The Modigliani-Miller theorem says that ultimately, the risk-premium or the riskiness of the entire portfolio is entirely pinned down by the risk characteristics of the tax claim and the spending claim.

The Treasury's long position in the tax claim exceeds the short position in the spending claim by the value of outstanding debt. To ensure risk-free debt, defined as a zero beta on government debt $\beta^D = 0$ in the language of asset pricing, the claim to tax revenues needs to be less risky than the spending claim, i.e., it must have a lower beta: $\beta^T < \beta^G$. Recast in the language of Modigliani-Miller, the claim to tax revenue can be regarded as the government's unlevered asset, which is divided into government debt and the claim to government spending. To manufacture risk-free debt, the spending claim has to be a levered version of the tax claim. Therefore, just as the equity of a firm has to be riskier than its assets in order to generate risk-free debt, the government's spending beta has to be higher than its tax beta to ensure zero-beta debt. The tax beta has to be low.

The tax claim has a low beta if the present discounted value (PDV) of future tax revenues increases in bad times, times in which the investor's marginal utility is high. Since the taxpayers pay the taxes, they have a short position on the tax revenue claim. From their perspective, a low-beta tax claim is a risky tax liability. The larger the amount of outstanding debt, the more levered the government becomes, and the larger the gap between the tax beta and the spending beta needs to be to keep the debt risk-free. As the debt grows, the beta of the tax claim has to go to zero, holding fixed the spending beta. The trade-off between insuring taxpayers and bondholders steepens.

In the paper, we can derive betas associated to the PDV government spending and taxes.

Proposition 347 (Betas of Taxes and Government Spending)

The betas associated to taxes and government spending is:

$$\beta_t^T = - \frac{\text{Cov}(\mathcal{M}_{t,t+1}, \frac{P_{t+1}^T}{(P_t^T - T_t)})}{\text{Var}[\mathcal{M}_{t,t+1}]} \quad (422)$$

$$\beta_t^G = - \frac{\text{Cov}(\mathcal{M}_{t,t+1}, \frac{P_{t+1}^G}{(P_t^G - G_t)})}{\text{Var}[\mathcal{M}_{t,t+1}]} \quad (423)$$

The paper then sets up a benchmark model where we assume risk-free treasury portfolio. They then set the excess return to 0 and from it, they can solve to get:

$$\beta_t^T < \beta_t^G.$$

This implies that the claim to future revenues from tax has to be safer than the government spending claim in order to keep the debt risk-free Furthermore, they found that the more debt you have $B = D_t - S_t$, the safer the tax claim has to be.

We can finally arrive at a Modigliani-Miller equation for government finance.

Theorem 348 (Modigliani-Miller For Government Finance)

The Modigliani-Miller equation is:

$$\underbrace{\frac{D_t + P_t^G}{D_t} \beta^T}_{\text{Risk Of Tax Revenue}} = \underbrace{\beta^D}_{\text{Risk Of Govt Debt}} + \underbrace{\frac{P_t^G}{D_t} \beta^G}_{\text{Risk Of Govt Spending}} \quad (424)$$

where D_t is the total market value of government debt issued.

The beta on the tax claim is a weighted average of the beta of the spending claim and the beta of the debt. The tax revenue claim is the unlevered version of the spending claim, or, equivalently, the spending claim is the levered

version of the tax claim. This result is analogous to the Miller-Modigliani relation between the unlevered return on equity (the return on the tax claim) and the levered return on equity (the return on the spending claim).

First, consider the case in which the spending claim has a positive beta ($\beta_t^G > 0$). Then, the government engineers risk-free debt by lowering the beta of the tax claim relative to that of the spending claim: $\beta_t^T < \beta_t^G$. A low beta for the tax claim means that tax revenue must fall by less than GDP in a recession. Tax rates must rise in recessions. The more debt there is outstanding (higher D_t), the lower the beta of the tax claim needs to be relative to that of the spending claim. With more debt, the trade-off between insuring bondholders and taxpayers becomes steeper. The restriction on the betas holds true regardless of the specific dynamics of the tax and spending process. In the next sections, we will derive restrictions on the underlying cash flows by committing to particular processes for debt/output and spending/output. The only way the government can provide insurance to debt holders, while keeping the debt risk-free, is by saving—choosing $D_t < 0$. In other words, the government can only insure taxpayers at the expense of bondholders.

Second, consider the case in which the spending claim has a negative beta ($\beta_t^G < 0$). To ensure risk-free debt, the tax claim must also have a negative beta when $D_t > 0$ ($\beta_t^T < 0$). Tax payments increase during recessions. Taxpayers are insuring the bondholders.

17 Carbon Pricing

We are interested in pricing the social cost of carbon. Before we proceed, we revisit some concepts from consumption under uncertainty.

17.1 Precautionary Savings and Prudence

First, suppose an agent must decide on consumption c_t and bond holdings (savings) b_{t+1} over an infinite horizon. The bond pays a gross return $(1 + r)$, and the agent receives a **deterministic endowment** every period.

Definition 349 (Infinite Horizon Deterministic Household Problem)

The problem facing the household is:

$$\begin{aligned} \max_{\{c_t, b_{t+1}\}_{t=1}^{\infty}} \quad & \sum_{t=1}^{\infty} \beta^t u(c_t) \\ \text{s.t.} \quad & c_t + b_{t+1} = (1 + r)b_t + y_t. \end{aligned} \tag{425}$$

We make an assumption on the utility function.

Assumption (CRRA Preferences)

We assume that the utility function is:

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma} \tag{426}$$

where γ is the relative risk-aversion parameter.

Theorem 350 (Euler Equation for Consumption)

The Euler equation for consumption is:

$$c_{t+1} = \left(\frac{1+r}{1+\delta} \right)^{\frac{1}{\gamma}} c_t \tag{427}$$

where δ is the discount rate in $\beta = \frac{1}{1+\delta}$.

Proof. We can set up the usual Lagrangian:

$$\mathcal{L} = \sum_{t=1}^{\infty} \beta^t \left[u(c_t) + \lambda_t \left((1+r)b_t + y_t - c_t - b_{t+1} \right) \right].$$

The FOCs are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t} : u'(c_t) &= \lambda_t \\ \frac{\partial \mathcal{L}}{\partial b_{t+1}} : \beta(1+r)\lambda_{t+1} &= \lambda_t \end{aligned}$$

We combine these to get:

$$\beta(1+r)u'(c_{t+1}) = u'(c_t).$$

We can compute the marginal utility of consumption for CRRA utility:

$$u'(c_t) = c_t^{-\gamma}.$$

Plug this in:

$$\beta(1+r)c_{t+1}^{-\gamma} = c_t^{-\gamma}.$$

Then substitute in $\beta = \frac{1}{1+\delta}$ and re-arrange:

$$\left(\frac{1+r}{1+\delta}\right) = \left(\frac{c_{t+1}}{c_t}\right)^{\gamma}$$

and then we are done:

$$\frac{c_{t+1}}{c_t} = \left(\frac{1+r}{1+\delta}\right)^{1/\gamma}.$$

□

The Euler equation for consumption has a nice interpretation:

$$c_{t+1} = \left(\frac{1+r}{1+\delta}\right)^{\frac{1}{\gamma}} c_t.$$

We can think of the term in front of c_t as the *growth rate* of consumption. It can then be shown that the growth rate is:

$$g \equiv \left(\frac{1+r}{1+\delta}\right)^{\frac{1}{\gamma}} - 1 \approx \frac{r-\delta}{\gamma}. \quad (428)$$

Definition 351 (Growth Rate of Consumption)

Under CRRA preferences, the growth rate of consumption is:

$$g \approx \frac{r-\delta}{\gamma} \quad (429)$$

where r is the interest rate, δ is the discount rate, and γ is the coefficient of relative risk aversion.

We get some nice results from this growth rate of consumption.

Proposition 352 (Increasing Consumption Growth)

The growth rate of consumption is increasing if the interest rate on savings is greater than the discount rate:

$$r > \delta. \quad (430)$$

Proposition 353 (Smoothing of Consumption Growth)

An agent only perfectly smooths consumption if the rate of interest equals the discount rate:

$$r = \delta. \quad (431)$$

Furthermore, as the relative risk aversion γ of the agent increases, then the growth rate of consumption falls.

Previously, we had three different forces regarding the agent's decision to save. First, due to decreasing marginal utility of consumption, the agent is incentivised to smooth consumption over time. Second, due to the impatience of

agents, they want to consume earlier. Finally, the interest rate on savings encourages the agent to save and consume later. On this last point, this is equivalent to saying that if $r > \delta$, then the agent is more incentivised to save and consume later and hence their consumption is growing over time.

We now restrict our attention to a 2-period model $t = 1, 2$. Furthermore, suppose that the endowment y_t is known but $\tilde{y}_{t+1}$ is unknown and therefore a random variable. This means that in the second period, the agent's consumption is also a random variable and hence unknown. Therefore, we need to take an expectation.

Definition 354 (2-Period Household Problem with Uncertainty)

The problem facing the household is:

$$\begin{aligned} \max_{c_t, c_{t+1}, b_t} \quad & u(c_t) + \beta \mathbb{E}[u(c_{t+1})] \\ \text{s.t.} \quad & c_t + b_t \leq y_t \\ & c_{t+1} \leq \tilde{y}_{t+1} + (1+r)b_t \end{aligned} \tag{432}$$

The result of now introducing this random endowment term is that the Euler equation now changes.

Proposition 355 (Euler Equation for Consumption)

The Euler equation for consumption is:

$$u'(y_t - b_t) = (1+r)\beta \mathbb{E}[u'((1+r)b_t + \tilde{y}_{t+1})]. \tag{433}$$

Proof. First substitute in the budget constraints into the objective function:

$$\max_{b_t} u(y_t - b_t) + \beta \mathbb{E}[u(\tilde{y}_{t+1} + (1+r)b_t)]$$

Now take first order conditions and we are done:

$$-u'(y_t - b_t) + (1+r)\beta \mathbb{E}[u'((1+r)b_t + \tilde{y}_{t+1})] = 0$$

□

Notice that the Euler equation now contains the **expected marginal utility of consumption**. Uncertainty about future income gives the household an additional motive to save: accumulating wealth helps protect against future risk. This is the precautionary-saving motive, and it relies on prudent behaviour.

Suppose that the optimal amount of savings (bonds) that the consumer chose in the above 2-period problem is b_t^* . We are now interested in seeing how does this choice of savings compared to when we no longer have uncertain future income $\tilde{y}_{t+1}$, so we can replace it by its expectation $\mathbb{E}_t[\tilde{y}_{t+1}]$. Finally, to simplify the analysis, we set $\beta = 1$. Then, our optimisation problem is now:

$$\max_{b_t} u(y_t - b_t) + u\left(\mathbb{E}_t[\tilde{y}_{t+1}] + (1+r)b_t\right)$$

where now notice that the expectation is pushed inside the utility function and only applies to the future income. Denote the optimal savings b_t to this problem with certain future income as $\hat{b}_t$.

We want to know whether optimal saving under uncertainty, b_t^* , is greater than optimal saving under certainty, $\hat{b}_t$.

To think about this, define the value function for the problem with no uncertainty as:

$$\hat{V}(\hat{b}_t) = \max_{b_t} u(y_t - b_t) + u\left(\mathbb{E}_t[\tilde{y}_{t+1}] + (1+r)b_t\right).$$

It can be verified that $\hat{V}(b_t)$ is concave in b_t , so moving away from its optimum $\hat{b}_t$ lowers $\hat{V}$. Hence, if $b_t^* \geq \hat{b}_t$, then reducing saving marginally from b_t^* raises lifetime utility under certainty. Because $\hat{V}$ is concave, this is equivalent to its first derivative being non-positive at b_t^* :

$$\begin{aligned}\hat{V}'(b_t^*) &= -u'(y_t - b_t^*) + (1+r)u'\left(\mathbb{E}_t[\tilde{y}_{t+1}] + (1+r)b_t^*\right) \\ &= -(1+r)\mathbb{E}\left[u'((1+r)b_t^* + \tilde{y}_{t+1})\right] + (1+r)u'\left((1+r)b_t^* + \mathbb{E}[\tilde{y}_{t+1}]\right) \leq 0\end{aligned}$$

where the red and blue terms come from the Euler equation for consumption we derived earlier. We can then see that from this last inequality, the level of precautionary saving is positive if and only if:

$$\mathbb{E}[u'((1+r)b_t^* + \tilde{y}_{t+1})] \geq u'((1+r)b_t^* + \mathbb{E}[\tilde{y}_{t+1}]) \quad (434)$$

This is an application of **Jensen's inequality**. The displayed inequality holds for all distributions when marginal utility $u'(\cdot)$ is a convex function. Convexity of $u'(\cdot)$ requires its second derivative, $u''(\cdot)$, to be non-negative. This gives us our next definition.

Definition 356 (Prudence)

The prudence of the utility function is defined as:

$$P(x) = -\frac{u'''(x)}{u''(x)} \quad (435)$$

as the degree of absolute prudence of the agent with utility u .

The third derivative of the utility function $u'''(\cdot)$ measures the **prudence** of the agent.

Thus, prudence is necessary if precautionary saving is to be positive for all possible distributions of future risk. A consumer with concave marginal utility would instead reduce saving in response to future risk and would exhibit "imprudent behaviour". Prudence corresponds to a positive third derivative of utility, just as risk aversion corresponds to a negative second derivative. Risk aversion and prudence are distinct: an agent can be risk-averse but imprudent, or prudent but risk-loving.

17.2 Pricing Carbon in Discrete Time

We now want to derive the social cost of carbon in **discrete time**. To get started, we need to use the fundamental asset pricing equation:

$$P_t = \mathbb{E}_t\left[\mathcal{M}_{t,t+1}\left(D_{t+1} + P_{t+1}\right)\right]$$

We now expand out

$$\begin{aligned}
P_t &= \mathbb{E}_t \left[\mathcal{M}_{t,t+1} \left(D_{t+1} + P_{t+1} \right) \right] \\
&= \mathbb{E}_t \left[\mathcal{M}_{t,t+1} D_{t+1} + \mathcal{M}_{t,t+1} P_{t+1} \right]
\end{aligned}$$

We can substitute in our assumption $D_{t+1} = \chi \theta c_{t+1}$ and also use the fact that the marginal utility is $u'(c_t) = c_t^{-\gamma}$, the stochastic discount factor will be:

$$\mathcal{M}_{t,t+1} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma}$$

We can then continue on:

$$\begin{aligned}
&= \mathbb{E}_t \left[\mathcal{M}_{t,t+1} \chi \theta c_{t+1} + \mathcal{M}_{t,t+1} P_{t+1} \right] \\
&= \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \chi \theta c_{t+1} + \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} P_{t+1} \right] \\
&= \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \chi \theta c_{t+1} \right] + \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} P_{t+1} \right]
\end{aligned}$$

We then multiply and divide by c_t in the first term:

$$\begin{aligned}
&= \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \chi \theta c_{t+1} \frac{c_t}{c_t} \right] + \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} P_{t+1} \right] \\
&= \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{1-\gamma} \chi \theta c_t \right] + \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} P_{t+1} \right]
\end{aligned}$$

To summarise what we have so far for the price, we have that:

Theorem 357 (Price for carbon)

The price for carbon is:

$$P_t = \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{1-\gamma} \chi \theta c_t \right] + \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} P_{t+1} \right]$$

We can now guess that the price is proportional to consumption, $P_t = \pi c_t$, and substitute this into the pricing equation:

$$\begin{aligned}
\pi c_t &= \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{1-\gamma} \chi \theta c_t \right] + \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \pi c_{t+1} \right] \\
&= \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{1-\gamma} \chi \theta c_t \right] + \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \pi c_{t+1} \frac{c_t}{c_t} \right] \\
&= \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{1-\gamma} \chi \theta c_t \right] + \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{1-\gamma} \pi c_t \right] \\
&= \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{1-\gamma} \right] \beta c_t (\chi \theta + \pi)
\end{aligned}$$

Therefore, we have that

$$\Rightarrow \pi c_t = \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{1-\gamma} \right] \beta c_t (\chi \theta + \pi)$$

Now, we can cancel out c_t on both sides

$$\pi = \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{1-\gamma} \right] \beta (\chi \theta + \pi)$$

By assumption, we know that the logarithm of consumption growth is log-normal. The expectation of a log-normal distribution is given by:

$$\mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{1-\gamma} \right] = \exp \left\{ (1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2 \right\}$$

So we can now apply this to what we have so far:

$$\begin{aligned} \pi &= \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{1-\gamma} \right] \beta (\chi \theta + \pi) \\ &\Leftrightarrow \pi = \exp \left\{ (1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2 \right\} \beta (\chi \theta + \pi) \\ &= \exp \left\{ (1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2 \right\} \beta \chi \theta + \exp \left\{ (1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2 \right\} \beta \pi \end{aligned}$$

Therefore, we have that:

$$\pi = \exp \left\{ (1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2 \right\} \beta \chi \theta + \exp \left\{ (1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2 \right\} \beta \pi$$

We group the π terms on both sides and solve for π :

$$\begin{aligned} \pi - \exp \left\{ (1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2 \right\} \beta \pi &= \exp \left\{ (1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2 \right\} \beta \chi \theta \\ \Rightarrow \pi &= \frac{\beta \exp \left\{ (1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2 \right\} \chi \theta}{1 - \beta \exp \left\{ (1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2 \right\}} \end{aligned}$$

We now substitute back in $P_t = \pi c_t \Leftrightarrow \pi = \frac{P_t}{c_t}$ and multiply the c_t out.

Theorem 358 (The price of social cost of carbon)

The social cost of carbon is:

$$P_t = \frac{\beta \exp \left\{ (1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2 \right\} \chi \theta c_t}{1 - \beta \exp \left\{ (1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2 \right\}} \quad (436)$$

Recall that:

$$\beta = \frac{1}{1+\delta}$$

and therefore we can rewrite our social cost of carbon as:

$$P_t = \frac{\exp\left\{(1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2\right\}\chi\theta c_t}{(1+\delta) - \exp\left\{(1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2\right\}} \quad (437)$$

We want to move the numerator into the denominator. To do so, we use the property that:

$$\frac{1}{\exp(x)} = \exp(-x)$$

So then for the numerator, we have that

$$\exp\left\{(1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2\right\} = \frac{1}{\exp\left\{(\gamma-1)\mu - \frac{1}{2}(1-\gamma)^2\sigma^2\right\}}$$

So we can apply this inverse property to Eq. (437):

$$\begin{aligned} P_t &= \frac{\exp\left\{(1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2\right\}\chi\theta c_t}{(1+\delta) - \exp\left\{(1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2\right\}} \\ &= \frac{\chi\theta c_t}{\left[(1+\delta) - \exp\left\{(1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2\right\}\right] \exp\left\{(\gamma-1)\mu - \frac{1}{2}(1-\gamma)^2\sigma^2\right\}} \\ &= \frac{\chi\theta c_t}{(1+\delta)\exp\left\{(\gamma-1)\mu - \frac{1}{2}(1-\gamma)^2\sigma^2\right\} - \exp\left\{(1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2\right\}\exp\left\{(\gamma-1)\mu - \frac{1}{2}(1-\gamma)^2\sigma^2\right\}} \end{aligned}$$

We now look at the second term in our denominator and use the exponential property:

$$\exp(A)\exp(B) = \exp(A+B)$$

So then for our second term in the denominator, we can simplify it as:

$$\begin{aligned} &\exp\left\{(1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2\right\}\exp\left\{(\gamma-1)\mu - \frac{1}{2}(1-\gamma)^2\sigma^2\right\} \\ &= \exp\left\{(1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2 + (\gamma-1)\mu - \frac{1}{2}(1-\gamma)^2\sigma^2\right\} \\ &= \exp(0) = 1 \end{aligned}$$

Therefore, we have that:

$$\begin{aligned} &= \frac{\chi\theta c_t}{(1+\delta)\exp\left\{(\gamma-1)\mu - \frac{1}{2}(1-\gamma)^2\sigma^2\right\} - \exp\left\{(1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2\right\}\exp\left\{(\gamma-1)\mu - \frac{1}{2}(1-\gamma)^2\sigma^2\right\}} \\ &= \frac{\chi\theta c_t}{(1+\delta)\exp\left\{(\gamma-1)\mu - \frac{1}{2}(1-\gamma)^2\sigma^2\right\} - 1} \end{aligned}$$

We now add and subtract a $\frac{1}{2}(\gamma-1)\sigma^2$ term inside the exponential in the denominator.

$$= \frac{\chi^{\theta c_t}}{(1 + \delta) \exp \left\{ (\gamma - 1)\mu + \frac{1}{2}(\gamma - 1)\sigma^2 - \frac{1}{2}(\gamma - 1)\sigma^2 - \frac{1}{2}(1 - \gamma)^2\sigma^2 \right\} - 1}$$

To proceed further, recall that given to us, we know that as the logarithm of consumption growth is a normal distribution, we have that

$$\mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \right] = \exp \left\{ -\gamma\mu + \frac{1}{2}\gamma^2\sigma^2 \right\}$$

and it was defined for us in class that:

$$g = \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right) - 1 \right] = \exp \left\{ \mu + \frac{1}{2}\sigma^2 \right\} - 1$$

Combining these two results together, we can derive an expression for the growth rate of logarithmic consumption.

Definition 359 (Growth rate)

The growth rate is given by:

$$(1 + g)^{\gamma-1} = \left[\exp \left\{ \mu + \frac{1}{2}\sigma^2 \right\} \right]^{\gamma-1} = \exp \left\{ (\mu + \frac{1}{2}\sigma^2)(\gamma - 1) \right\} \quad (438)$$

So going back to our term, we want to incorporate this growth rate. To do so, we need to add and subtract $\frac{1}{2}(\gamma - 1)\sigma^2$ into the exponential:

$$\begin{aligned} & \frac{\chi^{\theta c_t}}{(1 + \delta) \exp \left\{ (\gamma - 1)\mu + \frac{1}{2}(\gamma - 1)\sigma^2 - \frac{1}{2}(\gamma - 1)\sigma^2 - \frac{1}{2}(1 - \gamma)^2\sigma^2 \right\} - 1} \\ &= \frac{\chi^{\theta c_t}}{(1 + \delta) \exp \left\{ (\gamma - 1)\mu + \frac{1}{2}(\gamma - 1)\sigma^2 \right\} \exp \left\{ -\frac{1}{2}(\gamma - 1)\sigma^2 - \frac{1}{2}(1 - \gamma)^2\sigma^2 \right\} - 1} \end{aligned}$$

Looking at the exponential in the denominator, and grouping terms, we have that:

$$\exp \left\{ (\gamma - 1)\mu + \frac{1}{2}(\gamma - 1)\sigma^2 \right\} = \exp \left\{ \left(\mu + \frac{1}{2}\sigma^2 \right) (\gamma - 1) \right\} = (1 + g)^{\gamma-1}$$

We therefore have the growth rate now in our cost of carbon and we can then go back to our equation and continue to simplify:

$$\begin{aligned}
&= \frac{\chi^{\theta_{C_t}}}{(1+\delta)\exp\left\{(\gamma-1)\mu + \frac{1}{2}(\gamma-1)\sigma^2\right\}\exp\left\{-\frac{1}{2}(\gamma-1)\sigma^2 - \frac{1}{2}(1-\gamma)^2\sigma^2\right\} - 1} \\
&= \frac{\chi^{\theta_{C_t}}}{(1+\delta)(1+g)^{\gamma-1}\exp\left\{-\frac{1}{2}(\gamma-1)\sigma^2 - \frac{1}{2}(1-\gamma)^2\sigma^2\right\} - 1} \\
&= \frac{\chi^{\theta_{C_t}}}{(1+\delta)(1+g)^{\gamma-1}\exp\left\{-\frac{1}{2}\sigma^2\left((\gamma-1) + (1-\gamma)^2\right)\right\} - 1} \\
&= \frac{\chi^{\theta_{C_t}}}{(1+\delta)(1+g)^{\gamma-1}\exp\left\{-\frac{1}{2}\gamma(\gamma-1)\sigma^2\right\} - 1}
\end{aligned}$$

We have therefore arrived at the social cost of carbon.

Theorem 360 (Risk-adjusted Social Cost of Carbon)

$$P_t = \frac{\chi^{\theta_{C_t}}}{(1+\delta)(1+g)^{\gamma-1}\exp\left\{-\frac{1}{2}\gamma(\gamma-1)\sigma^2\right\} - 1} \quad (439)$$

17.3 The Risk-Adjusted Carbon Price - Bremer and Van der Ploeg (2021)

AER

We now go through the big ideas in how to price carbon in continuous time.

Proposition 361 (Social Cost of Carbon)

The social cost of carbon $P(t)$ is the expected present discounted value of all future damages associated with global warming that results from emitting one ton of carbon today:

$$P(t) \equiv \mathbb{E}_t \left[\int_t^\infty \left(U'(C(s)) \left\{ \frac{\partial \Pi(s)}{\partial T(s)} \frac{\partial T(s)}{\partial S(s)} \right\} \frac{\partial S(s)}{\partial F(t)} \right) e^{-\rho(s-t)} ds \right] \frac{1}{U'(C(t))} \quad (440)$$

where Π is aggregate output, S is the carbon stock, T is temperature, F is fossil fuel use, D as damages to output, and C is consumption.

We now break down all the terms one at a time. First, look at the chain rule in the middle:

$$\left\{ \frac{\partial \Pi(s)}{\partial T(s)} \frac{\partial T(s)}{\partial S(s)} \right\} \frac{\partial S(s)}{\partial F(t)}.$$

We can read this from right to left. First, what is the change on the carbon stock at time s , $S(s)$, as we increase the amount of fossil fuel we use now at time t , $F(t)$. Note that $s \geq t$ so:

$$\frac{\partial S(s)}{\partial F(t)}$$

captures how does changing the amount of fossil fuel we use today at time t affect the amount of carbon stock at

time $s \geq t$. Now, by altering the carbon stock at time s , this then affects the temperature at time s , which is capture by:

$$\frac{\partial T(s)}{\partial S(s)}.$$

Finally, by changing the temperature, this can induce global warming damages to output, and hence this is capture in the term:

$$\frac{\partial \Pi(s)}{\partial T(s)}.$$

Hence, we can chain all the effects together and that's what the following expression said:

$$\left\{ \frac{\partial \Pi(s)}{\partial T(s)} \frac{\partial T(s)}{\partial S(s)} \right\} \frac{\partial S(s)}{\partial F(t)}$$

which tells us what is damages to output at time s if we increase fossil fuel use today at time t .

Now, we multiply the damages to output at time s by the marginal utility of consumption at time s , which converts the damages into utility units. We then discount them back to time t :

$$\int_t^\infty \left(U'(C(s)) \left\{ \frac{\partial \Pi(s)}{\partial T(s)} \frac{\partial T(s)}{\partial S(s)} \right\} \frac{\partial S(s)}{\partial F(t)} \right) e^{-\rho(s-t)} ds.$$

This is just like what we did in the Ramsey model when discounting with integrals!

Finally, recall from asset pricing, the fundamental asset pricing equation:

$$P_t = \mathbb{E}_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1} \right]$$

where x_{t+1} is the payoff from the asset at time $t + 1$ and $\beta \frac{u'(c_{t+1})}{u'(c_t)}$ is the stochastic discount factor. Hence, you can sorta match this to our expression for the social cost of carbon to see that this is the fundamental asset pricing equation applied in continuous time:

$$P(t) \equiv \mathbb{E}_t \left[\int_t^\infty \left(U'(C(s)) \left\{ \frac{\partial \Pi(s)}{\partial T(s)} \frac{\partial T(s)}{\partial S(s)} \right\} \frac{\partial S(s)}{\partial F(t)} \right) e^{-\rho(s-t)} ds \right] \frac{1}{U'(C(t))}.$$

There are a few other things going on in the background that drives this model that we briefly mention:

1. The utility function follows Epstein-Zin recursive preferences. This means that the coefficient of relative risk aversion is no longer coupled with the intertemporal elasticity of substitution.
2. We assume that consumption is evolving like a Brownian motion process with a drift term (technically its a geometric Brownian motion but don't worry about that).
3. Production occurs through an AK model (as we saw with Martin).

Now, once we embed all these different things into our model, we actually end up with some nice closed form expressions!

First, we consider the price with economic uncertainty but without climate-related uncertainty.

Proposition 362 (Risk-Adjusted Social Cost of Carbon without Climate Uncertainty)

The risk-adjusted social cost of carbon with economic uncertainty but without climate uncertainty is given by:

$$P(t) = \frac{\mu \Theta C(t)}{r^* + \phi} \quad (441)$$

where μ is the fraction of emissions entering the atmospheric carbon reservoir, Θ is the normalised marginal-damage-to-output ratio, $C(t)$ is consumption, ϕ is the decay rate of atmospheric carbon, and r^* is the risk-adjusted discount rate:

$$r^* = \rho + (\gamma - 1) \left(g - \frac{1}{2} \eta \sigma_K^2 \right). \quad (442)$$

Here, ρ is the utility discount rate, γ is the coefficient of relative intergenerational inequality aversion (so the intertemporal elasticity of substitution is $1/\gamma$), η is relative risk aversion, g is economic growth, and σ_K is the volatility of the capital stock.

To break this social cost of carbon down, we multiply the damages to output (ratio) by consumption as a measure of how much we will be consumption. We then discount this by the discount rate r^* :

$$r^* = \rho + (\gamma - 1) \left(g - \frac{1}{2} \eta \sigma_K^2 \right).$$

As generations become more patient (ρ falls), the discount rate r^* falls and the risk-adjusted social cost of carbon $P(t)$ rises. The term involving growth captures rising **affluence**: if future generations will be richer, the marginal cost of shifting climate damages to them is lower. Growth also raises the future scale of damages, which pushes the cost of carbon in the opposite direction. Finally, the variance term captures **prudence**, or the precautionary response to uncertainty about the future.

We now allow for all four sources of uncertainty in the model.

Assumption (Uncertainties in Model)

There are 4 sources of uncertainties in the model:

1. Economic growth: We don't know how economic growth will occur in the future.
2. Carbon cycle: We don't know how long CO₂ stays in the atmosphere.
3. Temperature response: We don't know how temperature responds to greenhouse gases.
4. Climate Damage: How does increased temperature damage the economy.

Now, once we take into account these 4 types of uncertainties, we get a different social cost for carbon.

Proposition 363 (Risk-Adjusted Social Cost of Carbon under uncertainty)

The risk-adjusted social cost of carbon under uncertainty is given by:

$$P(t) = \frac{\mu \Theta C(t)}{r^* + \phi} \left[1 + \Delta_x + \Delta_\lambda + \Delta_{CK} + \Delta_{CC} \right] \quad (443)$$

where each Δ term captures the sensitivity to each of the 4 uncertainties we mentioned earlier, Θ is normalised marginal damage to output ratio, $C(t)$ is consumption, ϕ is the decay of atmospheric carbon and r^* is the risk-adjusted discount rate under uncertainty:

$$r^* = \rho + \gamma g - g - \frac{1}{2}(1 + \gamma)\eta\sigma_K^2 + \eta\sigma_K^2 \quad (444)$$

where ρ is the utility discount rate, γ is relative intergenerational inequality aversion (equal to the inverse of the intertemporal elasticity of substitution), g is economic growth, $1 + \gamma$ is relative prudence, and η is relative risk aversion.

We now want to interpret this new risk-adjusted discount rate:

$$r^* = \underbrace{\rho}_{\text{Impatience}} + \underbrace{\gamma g}_{\text{Rising Affluence}} - \underbrace{g}_{\text{Growing Damages}} - \underbrace{\frac{1}{2}(1 + \gamma)\eta\sigma_K^2}_{\text{Prudence}} + \underbrace{\eta\sigma_K^2}_{\text{Insurance}}$$

The discount rate captures how impatient policymakers are. The term γg captures rising affluence: if future generations are richer, current generations may place less weight on transferring resources to them. The term $-g$ captures growing damages as output expands. The prudence term captures the fact that greater future volatility increases precautionary saving, pushes down the interest rate, and raises the carbon price.

The insurance term captures the following intuition. Suppose that there is a future state of nature whereby economic activity is high, so consumption is high but then marginal utility of consumption is low. However, during such times when consumption is high, damages to the environment is also high. Hence in good times, damages to the environment is high, so your rate of interest goes up and the social cost of carbon falls. That is, when there is high damages to climate, this occurs during good times when output is high and consumption is high.

Finally, ϕ captures that if carbon decays from the atmosphere (e.g. go back into the oceans), then the higher the discount rate, the lower the price of carbon, and hence the less action you need to take.

Hence, this paper looked at how can we go about pricing carbon and the different factors that go into it. The following paragraph is one of the closing remarks of the paper.

We confirm earlier results that the optimal SCC decreases with increasing intergenerational inequality aversion if economic growth adjusted for its uncertainty is positive, but increases in risk aversion if economic growth (or asset returns) are volatile provided the elasticity of intertemporal substitution is less than one. If damages are proportional to GDP, there is an insurance effect which curbs the optimal SCC. If the elasticity of damages with respect to GDP is below one, there is less insurance potential, which increases the SCC, but damages grow less rapidly, which reduces the SCC (cf. Dietz, Gollier, and Kessler 2018). In our simulations, the first effect dominates if economic volatility is derived from asset returns, but the second effect dominates if volatility is derived from GDP growth.

18 Credit and Business Cycles

18.1 Introduction

We now look at the role of credit in our macroeconomic models. Let us recall some things we observed in the RBC model. **Amplification** refers to a model's ability to make output react by substantially more than the exogenous shock, that is, to take "small" shocks and produce "large" fluctuations. This happens when we apply a technology shock to the RBC model and observe the model's reaction to it. Recall that the only amplification mechanism in the RBC model is labour supply, through which workers substitute between work and leisure as they make an intertemporal consumption–leisure trade-off. This mechanism was fairly weak and not what we desire.

The **propagation** of a model refers to a model's ability to make shocks have persistent effects. The only propagation mechanism in the RBC model was capital accumulation, but that was weak as well. Hence, the time-series properties of output in the RBC model end up looking essentially like those of technology: output is about as variable and persistent as the technology shock.

Therefore, the RBC model is currently unable to explain internal propagation well and has little amplification. Furthermore, the RBC model does not explain prices well, since wages are quite volatile and asset prices are quite smooth. By incorporating financial frictions, we aim to introduce amplification and propagation into these models.

18.2 The Role of Leverage

Before we proceed, it is important to get an intuitive idea of leverage. One way to think about leverage is the use of borrowed capital to increase the potential return of an investment. By borrowing money, one can multiply the potential returns that one could have otherwise received. When a company is described as highly leveraged, it means that the company has a high amount of debt compared to equity.

Let's do an example. Suppose you have \$100 and you are interested in buying stock in a potato company (we shall call it Potato Co.). Each share of Potato Co. is trading at \$100. Suppose you spend all your money and buy \$100 worth of Potato Co. stock (one share). Now suppose that the stock price rises by \$5 (equivalently, 5%). This means that you make \$5 and your total return is 5%. Conversely, if the stock falls by \$5, you lose \$5 and your return is –5%.

Leverage amplifies these effects. Let us reset our scenario and say that you take out an interest-free loan for another \$100 and purchase \$200 worth of Potato Co. stock (two shares). In our good scenario, in which the share price rises by \$5 (equivalently, 5%), we make \$10 because both shares have risen from \$100 to \$105. When we compute returns, we compare the gain with the initial amount we had before borrowing because we need to repay what we borrowed. We have two shares worth \$210 in total and need to repay the \$100 loan, leaving us with \$110. We have therefore made a \$10 profit on our initial capital of \$100, so our return is now 10%. We have amplified our return through leverage from 5% to 10%. This is the amplification effect of leverage.

However, the converse also holds: leverage can amplify our losses. Suppose that each share falls from \$100 to \$95. The total value of our shares is now \$190. After repaying the \$100 loan, we are left with \$90. We began with \$100 but now have \$90, so we have made a 10% loss. This loss is larger than it would have been without leverage.

The punchline of all this is that leverage amplifies gains and losses.

We will look at three different types of Kiyotaki-Moore models. The key features of each of the models are:

1. We have a borrowing constraint with future output as collateral.
2. We have a borrowing constraint with a fixed asset as collateral.

3. We instead have a resaleability constraint.

18.3 Kiyotaki-Moore I

The key insight of the Kiyotaki–Moore model is that borrowers can abscond with funds. We will work with real variables, so we are in a setting similar to the closed-economy RBC model with flexible prices.

Now suppose that we are in a world with two types of agents. One type is productive and the other is unproductive. Here, “unproductive” simply means less productive than the productive type. Both types have access to a technology, but the technology available to the productive agent pays a higher return. Agents can borrow up to a credit limit determined by the fraction of future output that can be committed. Because borrowers may abscond with the funds, we limit how much they can borrow.

As a result of this setup, the model admits a new transmission mechanism for shocks. Net worth matters to lenders: if a borrower has higher net worth, lenders are more willing to provide funds. We can then introduce a one-period negative productivity shock that causes net worth to fall. This disproportionately affects productive agents because of their leveraged positions and causes their net worth to fall sharply. Consequently, productive agents cut investment by more than the decline in savings.

We now go into the model. We first require one of many key assumptions.

Assumption 364 (Limited Commitment Constraint)

The **limited commitment constraint** is given by:

$$b_{t+1} \leq \theta y_{t+1} \quad (445)$$

where b_{t+1} is the amount an agent promises to repay and y_{t+1} is the borrower’s output next period. Only a fraction θ of this output can be pledged to lenders. This is known as the **Kiyotaki–Moore constraint**.

Remark 365. *The amount that lenders are willing to lend is bounded by the exogenous parameter θ . We will later see that, in equilibrium, the agent has no incentive to abscond.*

Here, a lender provides funds against promised repayment b_{t+1} , while the borrower produces y_{t+1} . If the borrower absconds, the lender can recover only θy_{t+1} .

Example 366

(Numerical Example of Limited Commitment Constraint). Suppose that a loan contract promises repayment $b_{t+1} = 100$ and the production function yields $y_{t+1} = 200$. If $\theta = 0.5$, the lender can recover 100 units. If instead the recovery rate is $\theta = 0.1$, the lender can recover only 20 units of the good, so a promised repayment of 100 is not credible.

In this model, lenders restrict credit so that borrowers prefer repayment to absconding.

We can now state the household problem.

Definition 367 (Household Problem)

There is a continuum of infinitely-lived agents with identical preferences given by:

$$\mathbb{E}_0 \left[\sum_{\tau=0}^{\infty} \beta^{\tau} \log(c_{\tau}) \right] \quad (446)$$

In each period t , the market for one-period credit is competitive, so one unit of the good at time t returns r_t units at time $t + 1$.

Next, we see another departure from the RBC model: rather than having a firm, each agent has access to their own production technology. We now describe the agents' production functions.

Definition 368 (Production functions for agents)

A **productive agent** has the production function:

$$y_{t+1} = a_h x_t \quad (447)$$

where a_h is the productivity term for productive agents and x_t is the input.

An **unproductive agent** has the production function:

$$y_{t+1} = a_l x_t \quad (448)$$

where $a_l > 1$ is the productivity term for the unproductive agent.

We impose some assumptions on the production function for the agents and the distribution of the types.

Assumption (Production Function and Distribution of Agent Assumptions) 1. Both agents have constant returns to scale production functions with productivity parameters:

$$1 < a_l < a_h.$$

2. Agent types evolve according to an independent Markov process:

$$h \rightarrow l \quad \text{with probability } \delta$$

$$l \rightarrow h \quad \text{with probability } n\delta$$

3. The ratio of productive to unproductive agents is fixed at a constant ratio $n : 1$.

Remark 369. The first assumption states that high-type agents are more productive than low-type agents. Furthermore, even low-type agents generate a return greater than 1.

The second assumption states that a fraction δ of productive agents become unproductive, while a fraction $n\delta$ of unproductive agents become productive. This keeps the distribution of agent types stable.

18.3.1 Unconstrained Model

In a world with no constraints, the aggregate production function will be the productive agents' production function. Here, heterogeneity does not matter because all funds can be allocated to productive agents in each period. Credit

markets allow funds to flow from unproductive to productive agents. Hence, it is the borrowing constraint that generates the financial friction in this model.

In the **unconstrained model**, the unproductive agents should never invest. They should simply always lend to the productive agents. In each period, the productive agents will do all the investment.

Proposition 370 (Interest Rates in Unconstrained Model)

In the **unconstrained** model, the interest rate r_t is:

$$r_t = a_h \quad (449)$$

where a_h is the high type agent's productivity.

Proof. Suppose that $r_t < a_h$. Then there would be unlimited demand for funds, so this cannot be an equilibrium. Now suppose that $r_t > a_h$. The interest rate is then too high, so there is no demand for funds. $\square$

Remark 371. *There is a special case in which we lend to a productive agent who becomes unproductive next period with probability δ . In that event, the economy receives a_l rather than a_h .*

We now continue to build up the model. First, we need to impose a few more assumptions.

First of all, we don't want the probability δ (probability of going from productive to unproductive) to be too high or else we don't want to invest with the productive worker as there is a chance of them becoming unproductive in the next period. In fact, if δ is too high, we may actually want to make loans to unproductive workers as there is a high enough probability that they will become productive in the next period.

Assumption 372 (Agents do not switch types too easily)

Let n be the ratio parameter and δ the probability that a high-type agent becomes a low-type agent. We require that:

$$n\delta + \delta < 1 \quad (450)$$

Remark 373. *We do not want it to be too easy for agents to switch between being productive and unproductive. We want there to be correlation with future productivity as this will encourage the productive agent to borrow.*

We also impose a condition ensuring that leverage remains finite.

Assumption 374 (Finite Leverage)

To ensure a positive down-payment requirement, assume that:

$$\theta a_h < a_l \quad (451)$$

Remark 375. *This condition keeps $1 - \theta a_h / a_l$ positive and prevents productive agents from taking unbounded leverage. It implies that θ cannot be too high.*

We now state the full household problem.

Definition 376 (Household Problem)

Each individual chooses a sequence of consumption, investment, output, and debt $\{c_t, x_t, y_{t+1}, b_{t+1}\}_{t=0}^{\infty}$:

$$\begin{aligned} \max_{\{c_t, x_t, y_{t+1}, b_{t+1}\}_{t=0}^{\infty}} \quad & \mathbb{E}_0 \left[\sum_{\tau=0}^{\infty} \beta^{\tau} \log(c_{\tau}) \right] \\ \text{s.t.} \quad & y_{t+1} = ax_t \\ & b_{t+1} \leq \theta y_{t+1} \\ & c_t + x_t = y_t - b_t + \frac{b_{t+1}}{r_t} \end{aligned} \tag{452}$$

We want to quickly interpret the timing on the term:

$$\frac{b_{t+1}}{r_t}$$

For example, if the gross interest rate is $r_t = 1.1$, borrowing 10 units today requires a repayment of 11 units next period. At time t , the agent chooses the promised repayment b_{t+1} and receives b_{t+1}/r_t units now.

Now, with this setup, we can derive the level of consumption in this model.

Proposition 377 (Individual Consumption in Unconstrained Model)

Under the unconstrained model, the individual agent consumption is given by:

$$c_t = (1 - \beta)W_t$$

where $W_t \equiv (y_t - b_t)$ is net worth and savings is given by:

$$s_t = \beta W_t.$$

Proof. Since we are in the unconstrained model, we do not have a borrowing constraint, so the optimisation problem is:

$$\begin{aligned} \max_{\{c_t, x_t, y_{t+1}, b_{t+1}\}_{t=0}^{\infty}} \quad & \mathbb{E}_0 \left[\sum_{\tau=0}^{\infty} \beta^{\tau} \log(c_{\tau}) \right] \\ \text{s.t.} \quad & y_{t+1} = ax_t \\ & c_t + x_t = y_t - b_t + \frac{b_{t+1}}{r_t} \end{aligned}$$

We then restrict attention to high-type agents, since they are the only agents that invest. Combining the two constraints gives:

$$c_t + \frac{y_{t+1}}{a_h} = y_t - b_t + \frac{b_{t+1}}{r_t}.$$

Now, we have shown that the interest rate will be equal to the high type agent's productivity:

$$r_t = a_h$$

so we can plug this in to get:

$$c_t + \frac{y_{t+1}}{r_t} = y_t - b_t + \frac{b_{t+1}}{r_t}.$$

Now, re-arrange to get:

$$y_{t+1} - b_{t+1} = r_t [y_t - b_t - c_t].$$

Then use the definition of net worth $W_t \equiv y_t - b_t$:

$$W_{t+1} = r_t [W_t - c_t].$$

This expression for W_{t+1} is the **evolution of wealth** in the system.

We also can set up the Lagrangian:

$$\mathcal{L} = \mathbb{E}_0 \left[\sum_{\tau=0}^{\infty} \beta^\tau \left\{ \log(c_\tau) + \lambda_\tau [r_\tau (W_\tau - c_\tau) - W_{\tau+1}] \right\} \right]$$

and derive the Euler equation for consumption to be:

$$\frac{1}{c_t} = \frac{\beta r_t}{c_{t+1}}$$

and re-arrange this to get:

$$c_{t+1} = \beta r_t c_t$$

which we will need later.

We now shall combine this Euler equation with our earlier expression for the evolution of wealth:

$$W_{t+1} = r_t [W_t - c_t]$$

to get the result.

Since it is an infinite horizon problem, we need to guess our policy function for c_t as a function of the state variable W_t . We guess that it is a linear solution of the form:

$$c_t = A W_t$$

where A is an unknown coefficient we need to solve for. First, rewrite this as:

$$W_t = \frac{c_t}{A}.$$

Now, plug this into the evolution of wealth:

$$\begin{aligned} W_{t+1} &= r_t [W_t - c_t] \\ \Leftrightarrow W_{t+1} &= r_t \left[\frac{c_t}{A} - c_t \right] \\ \Leftrightarrow \frac{c_{t+1}}{A} &= r_t \left[\frac{c_t}{A} - c_t \right] \\ \Leftrightarrow \frac{\beta r_t c_t}{A} &= r_t \left[\frac{c_t}{A} - c_t \right] \\ \Leftrightarrow \frac{\beta r_t c_t}{A} &= r_t c_t \left[\frac{1}{A} - 1 \right] \\ \Leftrightarrow \frac{\beta}{A} &= \left[\frac{1}{A} - 1 \right] \\ \Leftrightarrow \beta &= 1 - A \\ \Rightarrow A &= 1 - \beta. \end{aligned}$$

Hence, our coefficient is:

$$A = 1 - \beta$$

and therefore our policy function is:

$$c_t = (1 - \beta)W_t.$$

Consumption plus savings must equal total wealth, so:

$$\begin{aligned} c_t + s_t &= W_t \\ \Leftrightarrow (1 - \beta)W_t + s_t &= W_t \\ \Rightarrow s_t &= \beta W_t. \end{aligned}$$

□

Hence, consumption c_t is always a fraction of net worth. Since we assumed log utility, this policy is of the Gorman polar form, so we can aggregate individual consumption. We next find aggregate investment in the economy.

Proposition 378 (Aggregate Investment)

The aggregate investment X_t in the **unconstrained** model is:

$$X_t = \beta a_h X_{t-1}. \quad (453)$$

Proof. First, we use the fact that investment must equal savings and the fact that in a closed economy, the net supply of bonds is 0 $B_t = 0$:

$$\begin{aligned} X_t &= \beta W_t \\ &= \beta(Y_t - B_t) \\ &= \beta Y_t \\ &=_{(1)} \beta a_h X_{t-1} \end{aligned}$$

where $=_{(1)}$ follows from the fact that only productive agents operate the technology that produces Y_t . □

Hence, investment X_t is simply the amount of savings that is invested into the production by the productive agents.

Now that we know how much is invested in the economy, we can determine its growth in this unconstrained setting.

Proposition 379 (Aggregate Growth of Wealth)

Aggregate growth G_t of wealth in the **unconstrained model** is constant and independent of the wealth distribution:

$$G_{t+1} = \beta a_h \quad (454)$$

where $G_{t+1} \equiv \frac{W_{t+1}}{W_t}$.

Proof. Plug in the definitions:

$$\begin{aligned}
\frac{W_{t+1}}{W_t} &= \frac{(Y_{t+1} - B_{t+1})}{(Y_t - B_t)} \\
&= \frac{Y_{t+1}}{Y_t} \\
&= \frac{a_h X_t}{a_h X_{t-1}} \\
&= \frac{X_t}{X_{t-1}} \\
&=_{(1)} \beta a_h
\end{aligned}$$

where $=_{(1)}$ uses the aggregate-investment result in Eq. (453), $X_t = \beta a_h X_{t-1}$. □

To summarise, aggregate consumption C_t obeys the same proportional rule as individual consumption because preferences are of the Gorman polar form:

$$C_t = (1 - \beta)W_t.$$

The aggregate investment in this economy is dependent on the productivity of only the productive agent:

$$X_t = \beta a_h X_{t-1}.$$

Finally, the aggregate growth of wealth in the economy only depends on the productive agent:

$$G_{t+1} = \beta a_h.$$

18.3.2 Constrained Model

We now impose restrictions on the Kiyotaki–Moore parameter θ . Previously, borrowing was unconstrained, so unproductive agents could lend all their funds to productive agents. We will no longer allow this. To do so, we require θ to be sufficiently low that unproductive agents are the marginal investors.

Assumption 380 (Kiyotaki Moore Parameter)

We ensure that borrowing parameter θ is a low value by assuming that:

$$\delta > \theta \left(\frac{a_h - a_l}{a_l} \right) \left(\frac{a_l - \theta a_h}{a_l - \theta a_h - n\theta a_l} \right) \quad (455)$$

Remark 381. *To interpret this, the first term on the right-hand side says that the productivity gap between the two types of agents cannot be too large. The second term depends on the ratio n of productive to unproductive agents.*

A low θ restricts how much productive agents can pledge and therefore how much they can borrow. Once the borrowing constraint binds, unproductive agents cannot lend all their savings to productive agents and instead invest the remaining funds in their own technology. The constrained model therefore features investment by both productive and unproductive agents.

From this, we can figure out what the interest rate is in this model.

Proposition 382 (Interest Rate under Constrained Model)

The interest rate under the constrained model is equal to the productivity of the **unproductive agent**:

$$r_t = a_l. \quad (456)$$

Proof. The unproductive agent is now the marginal lender, that is, they want to keep lending but are unable to do so due to the borrowing constraint. Hence, they will be indifferent between lending if the interest they earn on lending is $r_t = a_l$, which is the productivity of the technology if they invested in it themselves. $\square$

Remark 383. Notice that the interest rate is now different to the unconstrained model! Previously, in the unconstrained model, $r_t = a_h$.

Now, under this setting, we can figure out how much is borrowed.

Proposition 384 (Borrowing Constraint Binds Under Constrained Model)

The amount that the productive agent borrows is at the binding borrowing constraint:

$$b_{t+1} = \theta a_h x_t. \quad (457)$$

Proof. We have that:

$$b_{t+1} = \theta y_{t+1} = \theta a_h x_t.$$

$\square$

Now, we know what the interest rate is and how much the productive agent borrows. From this, we can figure out how much investment is made.

Proposition 385 (Investment by productive agents)

The investment made by productive agents x_t is given by:

$$x_t = \frac{y_t - b_t - c_t}{1 - \left(\frac{\theta a_h}{r_t}\right)} \quad (458)$$

where the numerator is **unconsumed** net worth (internal funds) and the denominator is the down payment required for a unit of investment.

Proof. Since $r_t = a_l$ as the marginal investor is the unproductive type, then the productive agents will borrow up to the borrowing limit:

$$b_{t+1} = \theta y_{t+1} = \theta a_h x_t.$$

Then, we have that the resource constraint:

$$c_t + x_t = y_t + \frac{b_{t+1}}{r_t} - b_t$$

becomes:

$$x_t = y_t + \frac{\theta a_h x_t}{r_t} - b_t - c_t$$

This then gives us the desired result:

$$x_t = \frac{y_t - b_t - c_t}{1 - \left(\frac{\theta a_h}{r_t}\right)}.$$

$\square$

Remark 386. *The numerator is unconsumed net worth, which is what remains after consumption, while the denominator is the required down payment for a unit of investment. This down payment depends on the productivity of the high-type agent relative to that of the low-type agent.*

If we have \$100 and the down payment is 10%, then we can lever up to \$1,000. The amount of leverage depends on θ . As $\theta \rightarrow 0$, the down payment approaches one: none of the investment can be financed by borrowing because lenders recover nothing if the borrower absconds.

This down-payment term is new. In the RBC model, we have complete markets in which agents can simply borrow if they do not have any funds. In this model, an agent can borrow only if they have enough to make the down payment. This gives the borrower skin in the game. Hence, this is a source of financial friction: agents can no longer borrow freely without sufficient internal funds.

We can see that the investment made by productive agents, x_t , depends on three terms:

1. Productive agents' investment x_t is increasing in their net worth.
2. Productive agents' investment x_t is decreasing in the interest rate r_t , as it costs more to borrow. Hence, the productivity gap between unproductive and productive agents matters.
3. Productive agents' investment x_t is increasing in their productivity a_h .

We can now write a difference equation for how net wealth evolves.

Proposition 387 (Evolution of Net Wealth)

Net wealth $y_t - b_t$ evolves over time as:

$$y_{t+1} - b_{t+1} = \frac{(1 - \theta)a_h}{1 - \frac{\theta a_h}{a_l}} \beta(y_t - b_t). \quad (459)$$

Proof.

$$\begin{aligned} y_{t+1} - b_{t+1} &= y_{t+1} - \theta y_{t+1} \\ &= (1 - \theta)y_{t+1} \\ &= (1 - \theta)a_h x_t \\ &= (1 - \theta)a_h \left(\frac{\beta(y_t - b_t)}{1 - \frac{\theta a_h}{a_l}} \right) \\ &= \left(\frac{(1 - \theta)a_h}{1 - \frac{\theta a_h}{a_l}} \right) \beta(y_t - b_t) \end{aligned}$$

□

Reading off the evolution of net wealth:

$$y_{t+1} - b_{t+1} = \frac{(1 - \theta)a_h}{1 - \frac{\theta a_h}{a_l}} \beta(y_t - b_t)$$

we can interpret γ , defined below, as the leveraged rate of return on productive agents' savings.

Definition 388 (Leveraged Rate of Return)

The leveraged rate of return under the constrained model is:

$$\gamma \equiv \frac{(1 - \theta)a_h}{1 - \frac{\theta a_h}{a_l}}. \quad (460)$$

Importantly, $\gamma > a_h$. Although this may initially seem strange, the productive agent can borrow at the low rate a_l and earn the margin $a_h - a_l$. The agent then magnifies this margin through leverage, so productive agents earn a higher return on their own net worth in the constrained economy.

The net worth of a productive agent grows faster in the constrained economy than in the unconstrained economy.

We now look at the unproductive agents. Here, as the unproductive agent is the marginal investor, they are indifferent between lending and investing in their own unproductive technology. As a result, the unproductive agent will also save a fraction of their net worth $\beta(y_t - b_t)$. Now, under this constrained model, we have the net worth for both the productive and unproductive agents, which differ from one another. Hence, we need to sum over the net wealth for both the productive and unproductive agents.

We now define the aggregate net worth of only **productive agents**. We will later see why we do this.

Definition 389 (Aggregate Net Worth of Productive Agents)

The aggregate net worth of productive agents is given by:

$$A_t \equiv \sum_{i \in H} (y_t^i - b_t^i). \quad (461)$$

We now want to determine how wealth grows in this economy.

Definition 390 (Growth Rate of Aggregate Wealth)

Let W_t be **aggregate wealth**. The constrained-economy growth factor $\tilde{G}_{t+1}$ is defined as:

$$\tilde{G}_{t+1} \equiv \frac{W_{t+1}}{W_t}. \quad (462)$$

We can derive how aggregate wealth evolves over time in the constrained model.

Proposition 391 (Growth Rate of Wealth)

The growth rate of **aggregate wealth** is given by:

$$\tilde{G}_{t+1} \equiv \frac{W_{t+1}}{W_t} = \beta \left(a_l + (a_h - a_l) \frac{1}{1 - \frac{\theta a_h}{a_l}} S_t \right) \quad (463)$$

where $S_t = \frac{A_t}{W_t}$ is the share of aggregate net worth held by productive agents.

Proof. We aggregate across net wealth for productive and unproductive agents:

$$\begin{aligned}
W_{t+1} &=_{(1)} Y_{t+1}^h + Y_{t+1}^l \\
&=_{(2)} a_h X_t^h + a_l X_t^l \\
&=_{(3)} a_h \frac{\beta A_t}{1 - \frac{\theta a_h}{a_l}} + a_l \left(\beta W_t - \frac{\beta A_t}{1 - \frac{\theta a_h}{a_l}} \right) \\
&=_{(4)} a_l \beta W_t + (a_h - a_l) \frac{1}{1 - \frac{\theta a_h}{a_l}} \beta \frac{A_t}{W_t} W_t
\end{aligned}$$

Here, $=_{(1)}$ aggregates net wealth across high- and low-type agents; aggregate debt cancels by market clearing. The equality $=_{(2)}$ uses the two linear production technologies. The equality $=_{(3)}$ uses the expressions for each type's investment. We have already derived productive agents' investment, while unproductive agents invest total savings βW_t less the amount invested by productive agents. Finally, $=_{(4)}$ follows from algebra.

We then divide through by W_t on both sides to get our desired result. $\square$

Aggregate net wealth W_{t+1} depends on the productive agents' net-worth share S_t . To interpret S_t , note that A_t is the net worth of productive agents, while W_t is total net worth.

Definition 392 (Productive Agents' Net-Worth Share)

The productive agents' net-worth share S_t is:

$$S_t = \frac{A_t}{W_t} \quad (464)$$

where A_t is the net wealth of productive agents and W_t is the aggregate net wealth of all agents.

Remark 393. Because S_t involves A_t , the distribution of net worth now matters for the evolution of aggregate wealth.

Importantly, the aggregate-wealth growth factor in the constrained economy, $\tilde{G}_{t+1}$, is never as high as the unconstrained growth factor βa_h .

Thus, the evolution of aggregate net wealth depends on the productive agents' net-worth share. **The growth rate of wealth depends on how much wealth productive agents hold. Loans are made to high types to invest in their more productive technology, but the borrowing constraint limits how much can be invested there.**

We can also write an expression for the evolution of the productive agents' net-worth share.

Proposition 394 (Evolution of Productive Agents' Net-Worth Share)

The evolution of the productive agents' net-worth share is given by the difference equation:

$$S_{t+1} \equiv \frac{A_{t+1}}{W_{t+1}} = \frac{(1 - \delta)\gamma S_t + n\delta a_l(1 - S_t)}{\gamma S_t + a_l(1 - S_t)} \quad (465)$$

Proof. First, we write out the productive agents' aggregate net worth:

$$\begin{aligned}
A_{t+1} &= (1 - \delta)(Y_{t+1}^h - B_{t+1}) + n\delta(Y_{t+1}^l + B_{t+1}) \\
&= (1 - \delta)[\gamma\beta(Y_t^h - B_t)] + n\delta[\beta a_l(W_t - A_t)] \\
&= (1 - \delta)\gamma\beta A_t + n\delta\beta a_l(W_t - A_t)
\end{aligned}$$

This then allows us to write:

$$S_{t+1} \equiv \frac{A_{t+1}}{W_{t+1}} = \frac{\beta W_t}{W_{t+1}} [(1 - \delta)\gamma S_t + n\delta a_l(1 - S_t)]$$

and we can re-arrange this to get our desired result. □

This is a difference equation for the productive agents' net-worth share, so we can use a phase diagram to study how it evolves.

A few comments on the phase diagrams in lectures.

1. There is a steady state to which the net-worth share S_{t+1} converges. If a shock moves the economy away from equilibrium, the share converges back to its steady state.
2. Now suppose that a productivity shock lowers the productivity of **all agents**. In this case, all agents are temporarily unproductive. The productive agents therefore produce a_l rather than the expected a_h . Because they produce less than expected while holding highly leveraged positions, their net worth is sharply reduced. It then takes time for their net worth to reaccumulate. Since the net-worth share S_t depends on productive agents' net worth A_t , the share falls and takes time to recover.

After a productivity shock, the model generates persistence in the productive agents' net-worth share because their net worth declines. This differs from the unconstrained model, in which the allocation can rebound immediately after the shock. We have therefore incorporated persistence into the model.

18.4 Kiyotaki-Moore II

We now want to add amplification to our model. We can do so by using a fixed asset as collateral. Introducing a fixed asset can add substantial amplification because the asset serves as both collateral and a factor of production. When productivity falls, the borrowing constraint induces persistence. Since the return to the fixed asset is lower, forward-looking agents value land less. Because land also serves as collateral, this price decline weakens the balance sheets of leveraged agents, especially those with high leverage. The fall in net worth then further reduces borrowing capacity.

Definition 395 (Production Function)

The production function is given by:

$$y_{t+1}^j = a_j \left(\frac{k_t}{\sigma} \right)^\sigma \left(\frac{x_t}{1 - \sigma} \right)^{1-\sigma} \quad (466)$$

where $0 < a_l < a_h$ and $\sigma \in (0, 1)$.

We now impose a new borrowing constraint under which only land is recoverable.

Definition 396 (Borrowing Constraint)

The borrowing constraint now assumes that only land is recoverable:

$$b_{t+1} \leq q_{t+1} k_t. \quad (467)$$

Contrast this to what we had earlier where $b_{t+1} \leq \theta y_{t+1}$. Now, k_t is used in both the production function and borrowing constraint. Here, the leverage varies with the land price.

We can now describe the per-period budget constraint. Firms buy land as a fixed asset rather than renting it.

Definition 397 (Flow of Funds)

The flow of funds is given by:

$$c_t + x_t + q_t(k_t - k_{t-1}) = y_t + \frac{b_{t+1}}{r_t} - b_t. \quad (468)$$

where firms buy land as a fixed asset rather than renting it.

We want to be able to aggregate the land that both the productive and unproductive agents buy.

Assumption (Land is fixed)

Land is fixed in aggregate across productive and unproductive agents:

$$K_t + K'_t = 1$$

where K'_t is the land owned by unproductive agents.

We now describe the results when we have this setup.

Suppose we have a negative productivity shock. Then the following happens.

1. Aggregate net worth decreases, but proportionately more for productive agents because of the leverage effect of debt. Recall that they took on more leverage because they earned a higher return.
2. The share of net worth of productive agents is lower and hence it will take time for this share and aggregate wealth to recover.
3. Since the economy will be weaker in coming periods, the user cost of land falls. Forward-looking agents anticipate this, so the price of land falls today. The demand for land reflects the state of the economy, and as its price falls, lenders are willing to provide even less to productive types.
4. The fall in land prices therefore reduces aggregate wealth even further and even disproportionately affects the productive agents even further.

Introducing such a fixed asset generates substantial amplification in the model.

18.5 Kiyotaki-Moore III

We now want to incorporate money and, more specifically, liquidity into our model. We retain the borrowing constraint from before and add a constraint involving financial variables. So far, we have focused on models of real variables, but we now want to consider monetary variables. In particular, the paper asks why people hold money and argues that it solves an intertemporal double coincidence of wants. In this setting, an agent who wants to borrow must sell equity, while an agent who wants to lend buys equity. We then impose constraints on what agents can do with the equity they hold in a given period.

We assume a linear production function.

Definition 398 (CRS Production)

The firms produce with a CRS production function:

$$y_t = r_t k_t. \quad (469)$$

The return they get on their investment is now a function of the aggregate economy.

Definition 399 (Rate of interest)

The rate of interest is given as a decreasing function of aggregate capital:

$$r_t = a_t K_t^{\alpha-1}. \quad (470)$$

We also assume that capital depreciates in production at rate λ .

Here is the key part of the model. With probability π , an entrepreneur has an investment opportunity to make new capital with a linear technology using output goods. The issue is that the entrepreneur needs to raise funds quickly enough in order to invest and take advantage of this situation.

An entrepreneur can issue claims on future returns to raise funds for a new project. This is equivalent to selling equity. The equity pays r_{t+1} at time $t + 1$, λr_{t+2} at time $t + 2$, and so on. We now impose constraints on the equity that agents can buy and sell.

Assumption (Borrowing and Re-saleability Constraints)

The **borrowing constraint** says that entrepreneurs who invest x_t can issue at most θx_t in equity.

The **resaleability constraint** says that entrepreneurs can resell at most a fraction ϕ_t of outside and inside equity at time t .

The first constraint says that entrepreneurs who invest an amount x_t can raise at most θx_t by issuing equity. This limits how much an entrepreneur can raise. The second constraint says that entrepreneurs can sell at most a fraction ϕ_t of the equity they bought from other firms. It therefore limits how much an entrepreneur can liquidate at a given point in time.

Under these assumptions, agents trade equity. If there is a productivity shock, demand for equity holdings is procyclical. Furthermore, liquidity shocks—changes in how much equity can be resold—propagate to the rest of the economy. Hence, the model offers a way to think about why individuals demand liquid assets, represented here by resaleable equity.

19 Financial Frictions

19.1 Introduction

Before the global financial crisis of 2007–08, interest-rate spreads were at their normal levels and the financial system functioned smoothly, in the sense that first-best allocations were supported in equilibrium. Then, with the collapse in bankers' net worth, interest-rate spreads jumped and financial markets became dysfunctional: the volume of intermediation and investment fell below their first-best levels. This section aims to explain this phenomenon by incorporating financial frictions.

We look at two main models of financial frictions. The first is a moral-hazard model in which banks can potentially abscond with funds, as in Gertler and Karadi (2009) and Gertler and Kiyotaki (2011). According to the model, banks respond to a decline in their own net worth by restricting the amount of deposits that they issue. They do so because, if they tried to maintain deposits after a decline in net worth, depositors would lose confidence and take their money elsewhere. Depositors would correctly anticipate that higher bank leverage gives bankers a stronger incentive to abscond with bank assets.

The second model we look at is the model with asymmetric information and monitoring costs by Bernanke, Gertler, and Gilchrist (1999). This model focuses on asymmetric information and costly monitoring as the source of financial frictions. In the model, bankers combine their own net worth with loans to acquire the securities of firms with projects that are subject to idiosyncratic risk. We assume that a bank can purchase the securities of at most one firm, so that the asset side of bank balance sheets is risky (banks cannot diversify by purchasing securities of multiple firms). There are no financial frictions between a bank and the firm whose securities it purchases. The realization of uncertainty in a firm's project is observed by its bank, but can be observed by bank creditors only by their paying a monitoring cost. We assume that creditors offer banks a *standard debt contract*. The contract specifies a loan amount and an interest rate. The bank repays the loan with interest, if it can. If the securities of a bank are bad because the issuing firm has an adverse idiosyncratic shock then the bank declares bankruptcy, is monitored by its creditor, and loses everything.

19.2 Moral Hazard - Gertler and Karadi (2009)

We construct a two-period model. In the first period, households make deposits in banks. Bankers combine these deposits with their own net worth and provide funds to firms. In the second period, households purchase the goods produced by firms by using income generated by bank profits and interest payments on bank deposits. The source of moral hazard is that bankers have an option to default by absconding with an exogenously fixed fraction of their total assets, leaving the rest to depositors. When a sufficiently large fraction of a bank's assets is purchased with bankers' own net worth, then a bank simply hurts itself by defaulting, and it chooses not to do so.

We show that when banks' net worth is sufficiently large that the option to default is irrelevant, the equilibrium allocations correspond to the first-best efficient allocations. We refer to this scenario as a "normal time." When banks' net worth is sufficiently low, banks restrict their demand for deposits. They do so because they know that, if they planned a higher level of deposits, depositors would rationally lose confidence and take their funds elsewhere: bankers have little net worth and hence little skin in the game. With banks' demand for deposits reduced and household supply unchanged, the market-clearing interest rate on deposits is low. Because the return on bank assets is fixed by assumption, banks' interest-rate spreads increase. We call this situation, in which bank net worth is so low that the banking system conducts too little intermediation, a *crisis time*. Government policy can push the economy out of crisis and back to normal by addressing the underlying cause. One way to do so is to purchase bank assets.

19.2.1 Household

We look at a two-period model in which households maximise utility over the two periods. In the first period, they receive an endowment of y goods. They then choose between consumption c_1 and a bank deposit d . In the second period, the household earns a gross return R on the deposit. Furthermore, it receives profits brought home by bankers, π (every household contains a mix of workers and bankers). We can now state the household's optimisation problem.

Definition 400 (Household Optimisation Problem)

The optimisation problem facing the household is:

$$\begin{aligned} \max_{c_1, c_2, d} \quad & u(c_1) + \beta u(c_2) \\ \text{s.t.} \quad & c_1 + d \leq y \\ & c_2 \leq Rd + \pi \end{aligned} \tag{471}$$

We can rewrite the budget constraint.

Proposition 401 (Household's Intertemporal Budget Constraint)

The household's intertemporal budget constraint is:

$$c_1 + \frac{c_2}{R} = y + \frac{\pi}{R}. \tag{472}$$

Proof. First, re-arrange the first constraint:

$$d = y - c_1$$

and substitute it into second one:

$$c_2 = R[y - c_1] + \pi.$$

Now re-arrange to get the combined budget constraint:

$$c_1 + \frac{c_2}{R} = y + \frac{\pi}{R}.$$

□

Before we proceed, we impose an assumption on the utility function.

Assumption (CRRA Utility)

The household has CRRA preferences:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \tag{473}$$

where $0 < \gamma < 1$ is the coefficient of risk aversion.

Remark 402. By our choice of $0 < \gamma < 1$, substitution effects dominate income effects. Therefore, if the interest rate R rises, the household will increase deposits rather than current consumption.

We can now derive the household's Euler equation.

Proposition 403 (Euler Equation for Consumption)

The household's Euler equation for consumption is:

$$c_2 = c_1(R\beta)^{\frac{1}{\gamma}}. \quad (474)$$

Proof. First construct the Lagrangian:

$$\mathcal{L} = u(c_1) + \beta u(c_2) + \lambda \left[y + \frac{\pi}{R} - c_1 - \frac{c_2}{R} \right].$$

Take first order conditions:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_1} : u'(c_1) &= \lambda \\ \frac{\partial \mathcal{L}}{\partial c_2} : \beta u'(c_2) &= \frac{\lambda}{R}. \end{aligned}$$

If we plug the first FOC into the second, we get:

$$\beta u'(c_2) = \frac{u'(c_1)}{R}$$

which can be re-arranged to get:

$$R\beta u'(c_2) = u'(c_1).$$

Now, if our utility function $u(c)$ is CRRA, then its derivative is:

$$u'(c) = c^{-\gamma}.$$

We plug this in to get our expression:

$$\begin{aligned} R\beta u'(c_2) &= u'(c_1) \\ \Leftrightarrow R\beta c_2^{-\gamma} &= c_1^{-\gamma}. \end{aligned}$$

Then re-arrange:

$$c_2 = c_1(R\beta)^{\frac{1}{\gamma}}.$$

□

Now that we have derived the Euler equation for consumption, we can derive consumption in periods 1 and 2, c_1 and c_2 , and the household's deposit.

Proposition 404 (Household's Consumption and Deposits)

The household's consumption in the two periods is given by:

$$c_1 = \frac{y + \frac{\pi}{R}}{1 + \frac{(\beta R)^{\frac{1}{\gamma}}}{R}} \quad (475)$$

$$c_2 = Rd + \pi. \quad (476)$$

The household's deposit is:

$$d = y - c_1. \quad (477)$$

Proof. First, use the household's Euler equation:

$$c_2 = c_1(R\beta)^{\frac{1}{\gamma}}$$

and plug this into the household's intertemporal budget constraint:

$$c_1 + \frac{c_2}{R} = y + \frac{\pi}{R}$$

to get:

$$\begin{aligned} \Leftrightarrow c_1 + \frac{c_1(R\beta)^{1/\gamma}}{R} &= y + \frac{\pi}{R} \\ \Leftrightarrow c_1 \left[1 + \frac{(R\beta)^{1/\gamma}}{R} \right] &= y + \frac{\pi}{R} \end{aligned}$$

Now re-arrange for c_1 and you get the level of consumption in the first period:

$$c_1 = \frac{y + \frac{\pi}{R}}{1 + \frac{(R\beta)^{1/\gamma}}{R}}.$$

The household's second period consumption immediately follows from the second period budget constraint binding:

$$c_2 = Rd + \pi.$$

The household's deposit immediately follows from the first period budget constraint binding:

$$c_1 + d = y.$$

□

The level of interest rates on deposits, R , and the profits earned by bankers, π , are endogenous and will be determined in equilibrium.

19.2.2 Firms

We now look at the role of firms. We impose an assumption on the market structure.

Assumption (Perfect Competition)

Firms are perfectly competitive and make no profits.

Firms issue securities in order to finance the capital they use to produce consumption goods in period 2. Intermediation is crucial in this economy. If firms receive no resources from banks in period 1, then there can be no production, and therefore no consumption, in period 2.

The firm sells securities s to bankers in period 1 (where s is the value of those securities). Firms use the proceeds from s to purchase an equal quantity of period-1 goods that they turn into capital. They then produce sR^k goods in period 2, where R^k is the gross return on the technology.

Assumption (Securities Are Paid Back with Certainty)

The securities that firms sell to bankers are paid back in the second period with certainty at an interest rate of R^k .

The interest rate R^k is **exogenous**.

This is also the return they must pay to the bankers. Hence, firms make no profits.

Definition 405 (Firm's Optimisation Problem)

The optimisation problem facing the firm is:

$$\max_s \quad sR^k - sR^k \quad (478)$$

19.2.3 Bankers

We assume that bankers are endowed with N goods in the first period.

Definition 406 (Net Worth)

The bankers' endowment of N goods in the first period is referred to as the bankers' **net worth**.

The bankers accept deposits d from households to help them smooth consumption. They also purchase securities s from firms to finance firms' first-period investment in capital. Bankers purchase these securities using the deposits they receive plus their net worth N . In the second period, they earn a gross return R^k on the securities and repay households Rd .

Because repayment on firms' securities is guaranteed by assumption, bankers purchase as many securities as their funding permits. Firms therefore issue securities equal in value to banks' deposits plus net worth, so that:

$$s = N + d.$$

Therefore, the banker's only decision is to decide how much deposits to accept in the first period.

Definition 407 (Banker Optimisation Problem)

The optimisation problem facing the banker is:

$$\max_d \quad \left[(N + d)R^k - Rd \right] \quad (479)$$

19.2.4 Benchmark Equilibrium - No Financial Frictions

We first examine the equilibrium where we have no financial frictions.

Definition 408 (Equilibrium With No Financial Frictions)

The equilibrium is characterised by an interest rate on deposits R and decisions $\{c_1, c_2, d\}$ such that:

1. Households solve their utility maximisation problem.
2. Bankers solve their profit-maximisation problem.
3. Markets for goods and deposits clear.
4. $d, c_1, c_2 > 0$ so that we are in an interior solution.

We have previously solved for c_1 , c_2 , and d . We now determine the equilibrium interest rate that clears the markets.

Proposition 409 (Equilibrium Rate of Interest)

The equilibrium level of interest rate is:

$$R = R^k. \quad (480)$$

Proof. The proof follows from an arbitrage argument. If $R > R^k$, then the bank pays more on deposits than it earns from firms' securities. Therefore, the bank would take no deposits, so $d = 0$. This violates the interior-solution condition. If $R < R^k$, then the bank earns more on securities than it pays on deposits. It would therefore demand an unbounded amount of deposits, which is not feasible. $\square$

Thus, without financial frictions, the interest rate R faced by households coincides with the return on capital R^k . The first-best allocations are therefore achieved: the allocations in the efficient benchmark equilibrium coincide with those that solve the social planner's problem.

We define an important object that we will use repeatedly.

Definition 410 (Interest Rate Spread)

The interest rate spread is defined as the difference between the return on capital and interest rate faced by households.

$$R^k - R. \quad (481)$$

We can derive the interest rate spread under this setting.

Proposition 411 (Interest Rate Spread In Baseline Model)

The interest rate spread under no financial frictions is:

$$R^k - R = 0. \quad (482)$$

Proof. Hence, we have that the interest rate spread is 0 in this efficient benchmark as:

$$R^k - R = R^k - R^k = 0.$$

$\square$

Deposits d are also at their socially optimal level.

Proposition 412 (Deposits Are at Their First-Best Level)

The amount of deposits d in an equilibrium without financial frictions equals the socially efficient amount under the social planner's problem.

Finally, in this setting, the Barro-Wallace irrelevance proposition holds and hence tax-financed government purchases of private assets have no impact on consumption or total intermediation.

19.2.5 Model Equilibrium - With Financial Frictions

We have established the frictionless baseline, in which deposits are at their socially efficient level. We now introduce a financial friction by allowing bankers to default in the second period.

Assumption (Bankers Can Default)

In period 2, a bank has two options:

1. Default.
2. Not default.

If a bank defaults, its bankers keep a fraction of the assets financed by first-period deposits d and bank net worth N .

Assumption (Bankers Abscond With Fraction of Assets)

If a bank defaults, they keep an exogenously fixed fraction θ of their assets.

We now compute the payoffs the bankers earn from the options of defaulting or not.

Proposition 413 (Payoff From Defaulting)

The payoff from defaulting for banks is:

$$\theta R^k (N + d). \quad (483)$$

Proof. In period 2, the bankers earn a return R^k on the securities they bought from the firm. The principal invested is the sum of the deposits d raised in the first period and their net worth N . $\square$

Proposition 414 (Payoff From Not Defaulting)

The payoff from **not** defaulting for banks is:

$$(N + d)R^k - Rd. \quad (484)$$

Proof. The banker earns $(N + d)R^k$ on the principal amount they invested in buying securities from the firm at a rate of return R^k . They then must pay back Rd to the households in the second period since the households withdraw their deposits. $\square$

Using these two payoffs, we can derive the incentive-compatibility constraint under which bankers choose **not** to default.

Proposition 415 (No-Default Condition)

The no-default condition is:

$$(N + d)R^k - Rd \geq \theta R^k (N + d). \quad (485)$$

Let us look more carefully at this no-default condition:

$$(N + d)R^k - Rd \geq \theta R^k (N + d).$$

We can compare the coefficients on net worth N on the two sides:

$$R^k N > \theta R^k N$$

as $0 \leq \theta < 1$. Therefore, an increase in net worth N raises the left-hand side of the inequality by more than the right-hand side, so a banker is less likely to default. This is because the banker loses some of their net worth upon default: the more of their own assets they commit to the project, the less willing they are to default.

We now determine the equilibrium when the no-default condition must be taken into account.

Each bank takes the interest rate on deposits as given and sets its own level of deposits d . Banks must post their intended values of d at the start of the period so that households can assess whether a bank will default. We consider symmetric equilibria in which no bank defaults and the value of d posted by each bank satisfies the no-default condition. The banker's problem is similar to the baseline case, except that it now includes a no-default condition.

Definition 416 (Banker Optimisation Problem Under Incentive Compatibility)

The optimisation problem facing the banker is:

$$\begin{aligned} \max_d \quad & \left[(N + d)R^k - Rd \right] \\ \text{s.t.} \quad & (N + d)R^k - Rd \geq \theta R^k(N + d) \end{aligned} \quad (486)$$

where the constraint ensures that the banker does not default.

Now, we can define the general equilibrium under this model.

Definition 417 (Equilibrium With Financial Frictions)

The equilibrium is characterised by an interest rate on deposits R and decisions $\{c_1, c_2, d\}$ such that:

1. Households solve their utility maximisation problem.
2. Bankers solve their profit-maximisation problem **under incentive compatibility**.
3. Markets for goods and deposits clear.
4. $d, c_1, c_2 > 0$ so that we are in an interior solution.

We are interested in deriving the interest-rate spread in this model.

Proposition 418 (Interest Rate Spread Under Financial Frictions)

The interest rate spread is given by:

$$R^k - R = \frac{\lambda}{1 + \lambda} \theta R^k \quad (487)$$

where λ is the Lagrange multiplier on the default constraint.

Proof. Take the banker's optimisation problem under incentive compatibility.

$$\begin{aligned} \max_d \quad & \left[(N + d)R^k - Rd \right] \\ \text{s.t.} \quad & (N + d)R^k - Rd \geq \theta R^k(N + d) \end{aligned}$$

Form the Lagrangian:

$$\mathcal{L} = (N + d)R^k - Rd + \lambda[(N + d)R^k - Rd - \theta R^k(N + d)]$$

Now take first order conditions with respect to d :

$$\frac{\partial \mathcal{L}}{\partial d} : R^k - R + \lambda R^k - \lambda R - \lambda \theta R^k = 0.$$

Re-arrange:

$$R^k - R + \lambda R^k - \lambda R = \lambda \theta R^k.$$

Then factorise the left-hand side:

$$(1 + \lambda)(R^k - R) = \lambda \theta R^k.$$

Then divide through to conclude:

$$R^k - R = \frac{\lambda}{1 + \lambda} \theta R^k.$$

□

The big takeaway is that **interest-rate spreads are no longer necessarily zero when we have financial frictions**. There are two possible cases. First, suppose that the no-default condition does **not** bind:

$$(N + d)R^k - Rd > \theta R^k(N + d).$$

This implies that the Lagrange multiplier $\lambda = 0$. This means that the interest rate spread is now 0:

$$R^k - R = \frac{\lambda}{1 + \lambda} \theta R^k = 0$$

so then the rate of interest on deposits R is equal to return on investment:

$$R = R^k.$$

This is the same setting as in the baseline model without financial frictions. It also implies that deposits are at their socially optimal, first-best level.

The second case is the interesting one. Suppose that the no-default condition **does** bind:

$$(N + d)R^k - Rd = \theta R^k(N + d).$$

This implies that the Lagrange multiplier $\lambda > 0$. This means that the interest rate spread is now **not** 0:

$$R^k - R = \frac{\lambda}{1 + \lambda} \theta R^k > 0.$$

We now have an interest-rate spread $R^k - R > 0$. This means that the endogenous return on deposits is below the exogenous return on investment, $R < R^k$. As a result, R is below the socially optimal level in the frictionless baseline. Furthermore, because the deposit rate R is lower, deposits d are also below their socially optimal level.

Proposition 419 (Suboptimal Interest Rate and Deposits)

Under a binding no-default condition, the interest rate on deposits R and the amount of deposits d are below their first-best, socially efficient levels without financial frictions.

It is not households' supply of deposits that is directly affected by the financial friction, but banks' demand for deposits. The key takeaway is that **frictions in financial markets impose real costs on the economy and reduce social welfare**.

19.2.6 Policy Implications

We see that the cause of the financial frictions is the no-default condition:

$$(N + d)R^k - Rd \geq \theta R^k(N + d)$$

which creates an interest-rate spread. A natural policy response is to increase bankers' net worth N . Higher net worth makes default less attractive. In the Lagrangian, the no-default constraint then binds less tightly, so its multiplier λ falls and the deposit rate R moves closer to the return on investment R^k . The model's focus on bankers' net worth helps explain why central banks pay close attention to bank balance sheets and devoted substantial resources to repairing them after the financial crisis.

19.3 Asymmetric Information and Monitoring Costs - Bernanke, Gertler and Gilchrist (1999)

This model focuses on the broad credit channel and closely follows the Kiyotaki–Moore model. Its key idea is costly monitoring of bankers by mutual funds, so frameworks of this kind are often known as costly-state-verification models. Financial markets do not work perfectly because asymmetric information between a banker and a mutual fund is costly to resolve. When an entrepreneur borrows from a bank, the project is subject to a shock. An incentive-compatibility constraint governs the borrower's report of whether the productivity shock was good or bad, and an optimal contract determines the terms under which the entrepreneur can borrow. As in the Kiyotaki–Moore model, the borrower's net worth matters to lenders. The greater the entrepreneur's net worth, the lower the external-finance premium. Hence, if borrowers' balance sheets move procyclically, the external-finance premium moves countercyclically. This creates feedback between the financial and real sectors: when the real economy performs well, borrowing becomes easier and leverage rises; when a shock occurs, leverage amplifies its effects.

There are two periods. A large number of workers and bankers with identical utility functions live in a representative, competitive household. Workers have an exogenous endowment y of income in the first period, and bankers possess N units of net worth. All variables are expressed in household per capita units. Households allocate y to first period household consumption and to deposits in identical, competitive mutual funds. Each banker combines his or her net worth with a loan from a mutual fund to acquire the securities of one firm. The firm operates a technology perturbed by an idiosyncratic technology shock to build capital in period 1 and uses the capital in period 2 to produce goods. Because we assume markets are competitive and because firms contribute nothing themselves to building capital and selling goods, they enjoy zero earnings after paying the return on the security sold to their bank. Because a bank can invest in the securities of at most one firm, the asset side of a bank's balance sheet is risky. As in the previous section, we simplify the exposition by pretending that a firm's activity building capital and selling goods is undertaken directly by its bank.

A financial friction arises because the bank observes the realization of the idiosyncratic technology shock on the asset side of its balance sheet, whereas its mutual fund can observe the technology shock only by paying a monitoring cost. The loan received by banks from their mutual fund comes in the form of a standard debt contract. The contract specifies a loan amount and an interest rate. Banks with a sufficiently low realization of their idiosyncratic productivity shock are not able to repay their loan, and these banks must transfer whatever assets they have to their mutual fund after being monitored. The measure of microeconomic uncertainty referred to in the opening paragraph pertains to the variance across banks in their idiosyncratic technology shock. We consider a characterization of the financial crisis that involves just a drop in bank net worth as one that also involves an increase in the cross-sectional variance of the technology shock. Household consumption in the second period is financed out of income on period 1 deposits as well as bank profits.

The model implies that a fall in bank net worth causes interest rate spreads, the cost of funds to banks minus the risk-free rate, to rise and investment to fall. We consider various policy responses. We find that a policy that subsidizes the cost of funds to banks is welfare improving in both normal and crisis times. Moreover, the optimal value

of the subsidy is greater in crisis times, so that the model suggests a more aggressive policy stance then. We also find that, absent government intervention, bank leverage is too low. In this sense, the model does not include the kind of features that rationalize the current interest in imposing leverage restrictions on economic agents. The reason for our “underborrowing” result is that the marginal return on loans by mutual funds to banks is higher than the average return, whereas the economics of the model implies that bank creditors focus on the average return.

20 Neutrality of Money

20.1 Introduction

Previously, we saw that technology shocks were the main drivers of economic fluctuations in RBC models. These real disturbances affected the model's real variables. However, this approach leaves no role for either money or monetary policy.

There are two ways to address this. The first is to disregard the RBC agenda and return to the old Keynesian approach. The problem is the **Lucas critique**: such models contain relationships that are not **structural**, so their policy recommendations need not remain valid under regimes not observed in the data.

The second option, which we shall follow, is to keep some elements of the RBC agenda and now introduce money and monetary policy into the model. With this new model, we shall run into the **classical dichotomy**, which says that monetary policy only affects nominal variables (such as inflation) and not real variables, which is inconsistent with the empirical evidence. This gives rise to the unintuitive result that there is **no** inflation-output tradeoff and hence central banks should focus on inflation. However, as mentioned earlier, there are actually *real effects* of monetary policy and hence such a tradeoff actually does exist.

20.2 RBC + Money

20.2.1 Household

We embed money in the RBC model. As before, the representative household values consumption C_t and dislikes working N_t . In addition, the household values real money balances, so M_t/P_t enters the utility function. The parameter χ describes the importance of holding money relative to consumption.

The household spends on consumption $P_t C_t$, investment $P_t I_t$, government bonds B_t , and money M_t . Its resources consist of the gross payoff on bonds purchased last period, $(1 + i_{t-1})B_{t-1}$; rental income from capital, $R_t^k K_{t-1}$; previously accumulated money balances, M_{t-1} ; and labour income, $W_t N_t$, less taxes T_t .

As the household chooses how much to work and wages are determined in equilibrium, the wage rate creates a substitution effect on households' labour supply.

Definition 420 (Frisch Elasticity Of Labour Supply)

The **Frisch elasticity of labour supply** $1/\psi$ captures the elasticity of hours worked with respect to the wage rate, holding the marginal utility of wealth constant. The higher the Frisch elasticity, the more willing people are to work when the wage rises.

Finally, there is a law of motion for how capital K_t develops over time as a function of depreciation, investment, and the previous period level of capital K_{t-1} . With all this, we can now state the representative household optimisation problem.

Definition 421 (Household Optimisation Problem)

The optimisation problem facing the household is:

$$\begin{aligned} \max_{C_t, N_t, M_t, B_t, I_t, K_t} \quad & \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\ln(C_t) + \chi \ln \left(\frac{M_t}{P_t} \right) - \frac{N_t^{1+\psi}}{1+\psi} \right) \right] \\ \text{s.t.} \quad & P_t C_t + P_t I_t + B_t + M_t = (1 + i_{t-1}) B_{t-1} + M_{t-1} + R_t^k K_{t-1} + W_t N_t - T_t \\ & K_t = (1 - \delta) K_{t-1} + I_t \end{aligned} \quad (488)$$

where χ captures the importance of money to the household, $1/\psi$ is the Frisch elasticity of labour supply, I_t is investment, K_t is capital, B_t denotes bonds, T_t denotes lump-sum taxes, and M_t/P_t denotes real money balances.

As $\chi \rightarrow 0$, bonds dominate money because they pay interest, so households hold no money in equilibrium. Including money in utility therefore generates a demand for money. One interpretation is that money provides a **liquidity service** by facilitating exchange.

We can take the budget constraint and rewrite it in real terms by dividing through with the price level P_t :

$$\begin{aligned} \Rightarrow P_t C_t + P_t I_t + B_t + M_t &= (1 + i_{t-1}) B_{t-1} + M_{t-1} + R_t^k K_{t-1} + W_t N_t - T_t \\ \Leftrightarrow C_t + I_t + \frac{B_t}{P_t} + \frac{M_t}{P_t} &= (1 + i_{t-1}) \frac{B_{t-1}}{P_t} + \frac{M_{t-1}}{P_t} + \frac{R_t^k}{P_t} K_{t-1} + \frac{W_t}{P_t} N_t - \frac{T_t}{P_t} \\ \Leftrightarrow C_t + I_t + \frac{B_t}{P_t} + \frac{M_t}{P_t} &= \frac{(1 + i_{t-1})}{P_t} \frac{P_{t-1} B_{t-1}}{P_{t-1}} + \frac{P_{t-1}}{P_t} \frac{M_{t-1}}{P_{t-1}} + \frac{R_t^k}{P_t} K_{t-1} + \frac{W_t}{P_t} N_t - \frac{T_t}{P_t} \\ \Leftrightarrow C_t + I_t + \frac{B_t}{P_t} + \frac{M_t}{P_t} &= \frac{(1 + i_{t-1})}{\Pi_t} \frac{B_{t-1}}{P_{t-1}} + \frac{1}{\Pi_t} \frac{M_{t-1}}{P_{t-1}} + r_t^k K_{t-1} + w_t N_t - \tau_t \end{aligned}$$

where we defined $\Pi_t \equiv P_t/P_{t-1}$, $r_t^k \equiv R_t^k/P_t$, $w_t \equiv W_t/P_t$, and $\tau_t \equiv T_t/P_t$. Eliminating investment reduces the six choice variables to five: C_t , B_t/P_t , M_t/P_t , K_t , and N_t .

We can take the law of motion of capital:

$$I_t = K_t - (1 - \delta) K_{t-1}$$

and substitute it into our budget constraint that is in real terms to eliminate the investment choice variable:

$$C_t + K_t + \frac{B_t}{P_t} + \frac{M_t}{P_t} = \frac{(1 + i_{t-1})}{\Pi_t} \frac{B_{t-1}}{P_{t-1}} + \frac{1}{\Pi_t} \frac{M_{t-1}}{P_{t-1}} + (r_t^k + (1 - \delta)) K_{t-1} + w_t N_t - \tau_t$$

The **current-value Lagrangian** is given by:

$$\mathcal{L}_0 = \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[\left(\ln(C_t) + \chi \ln \left(\frac{M_t}{P_t} \right) - \frac{N_t^{1+\psi}}{1+\psi} \right) + \lambda_t \left(\frac{(1 + i_{t-1})}{\Pi_t} \frac{B_{t-1}}{P_{t-1}} + \frac{1}{\Pi_t} \frac{M_{t-1}}{P_{t-1}} + (r_t^k + (1 - \delta)) K_{t-1} + w_t N_t - \tau_t - C_t - K_t - \frac{B_t}{P_t} - \frac{M_t}{P_t} \right) \right] \right\} \quad (489)$$

We can then solve for the first order conditions:

Theorem 422 (Household's First Order Conditions)

The household's first order conditions are given by:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial C_t} : \frac{1}{C_t} &= \lambda_t && \text{(Consumption FOC)} \\ \frac{\partial \mathcal{L}}{\partial B_t/P_t} : \beta \mathbb{E}_t \left[\lambda_{t+1} \frac{1+i_t}{\Pi_{t+1}} \right] &= \lambda_t && \text{(Real Bond Holdings FOC)} \\ \frac{\partial \mathcal{L}}{\partial M_t/P_t} : \frac{\chi}{M_t/P_t} + \beta \mathbb{E}_t \left[\lambda_{t+1} \frac{1}{\Pi_{t+1}} \right] &= \lambda_t && \text{(Real Money Holdings FOC)} \\ \frac{\partial \mathcal{L}}{\partial K_t} : \beta \mathbb{E}_t \left[\lambda_{t+1} (r_{t+1}^K + 1 - \delta) \right] &= \lambda_t && \text{(Capital FOC)} \\ \frac{\partial \mathcal{L}}{\partial N_t} : N_t^\psi &= \lambda_t w_t && \text{(Hours Worked FOC)}\end{aligned}$$

Proof. Take the Lagrangian:

$$\begin{aligned}\mathcal{L}_0 = \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[\left(\ln(C_t) + \chi \ln \left(\frac{M_t}{P_t} \right) - \frac{N_t^{1+\psi}}{1+\psi} \right) \right. \right. \\ \left. \left. + \lambda_t \left(\frac{(1+i_{t-1})}{\Pi_t} \frac{B_{t-1}}{P_{t-1}} + \frac{1}{\Pi_t} \frac{M_{t-1}}{P_{t-1}} + (r_t^K + 1 - \delta) K_{t-1} + w_t N_t - \tau_t - C_t - K_t - \frac{B_t}{P_t} - \frac{M_t}{P_t} \right) \right] \right\}.\end{aligned}$$

Now work through the 5 first order conditions.

Differentiate with respect to consumption C_t :

$$\mathbb{E}_t \left[\frac{\beta^t}{C_t} - \beta^t \lambda_t \right] = 0$$

which then gives us:

$$\frac{1}{C_t} = \lambda_t$$

Differentiate with respect to real bond holdings B_t/P_t :

$$\mathbb{E}_t \left[\beta^{t+1} \frac{1+i_t}{\Pi_{t+1}} \lambda_{t+1} - \beta^t \lambda_t \right] = 0.$$

Re-arrange to get:

$$\mathbb{E}_t \left[\beta \frac{1+i_t}{\Pi_{t+1}} \lambda_{t+1} \right] = \lambda_t.$$

Differentiate with respect to M_t/P_t :

$$\mathbb{E}_t \left[\frac{\beta^t \chi}{M_t/P_t} + \beta^{t+1} \lambda_{t+1} \frac{1}{\Pi_{t+1}} - \beta^t \lambda_t \right] = 0.$$

Re-arrange to get:

$$\mathbb{E}_t \left[\frac{\chi}{M_t/P_t} + \beta \lambda_{t+1} \frac{1}{\Pi_{t+1}} \right] = \lambda_t.$$

Differentiate with respect to K_t :

$$\mathbb{E}_t \left[\beta^{t+1} \lambda_{t+1} (r_{t+1}^K + (1 - \delta)) - \beta^t \lambda_t \right] = 0.$$

Re-arrange to get:

$$\mathbb{E}_t \left[\beta \lambda_{t+1} [r_{t+1}^K + (1 - \delta)] \right] = \lambda_t.$$

Differentiate with respect to N_t :

$$\mathbb{E}_t \left[-\beta^t N_t^\psi + \beta^t \lambda_t w_t \right] = 0$$

Re-arrange to get:

$$N_t^\psi = \lambda_t w_t.$$

□

Eq. ([Consumption FOC](#)) says that the Lagrange multiplier equals the marginal utility of consumption. Eq. ([Real Bond Holdings FOC](#)) says that the marginal utility of consumption today (λ_t) equals the discounted expected marginal utility of consumption tomorrow times the return on bonds. Eq. ([Real Money Holdings FOC](#)) says that the marginal utility of consumption equals the utility from holding money (the χ term) plus the discounted expected marginal utility of consumption tomorrow times the return on money. Money pays no interest, so inflation erodes its real value. Eq. ([Capital FOC](#)) and Eq. ([Hours Worked FOC](#)) have the standard interpretations for capital and labour, respectively.

Theorem 423 (Household's Euler Equations for Bonds, Capital, and Money)

The household's Euler equations for bonds, capital, and money are given by:

$$\begin{aligned} \frac{1}{C_t} &= \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \frac{1+i_t}{\Pi_{t+1}} \right] && \text{(Bonds)} \\ \frac{1}{C_t} &= \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} (r_{t+1}^k + 1 - \delta) \right] && \text{(Capital)} \\ \frac{1}{C_t} &= \frac{\chi}{M_t/P_t} + \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \frac{1}{\Pi_{t+1}} \right] && \text{(Money)} \end{aligned}$$

Proof. To derive the Euler bond equation, we substitute Eq. ([Consumption FOC](#)) into Eq. ([Real Bond Holdings FOC](#)) to get the Euler bond equation:

$$\begin{aligned} \beta \mathbb{E}_t \left[\lambda_{t+1} \frac{1+i_t}{\Pi_{t+1}} \right] &= \lambda_t \\ \Leftrightarrow \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \frac{1+i_t}{\Pi_{t+1}} \right] &= \frac{1}{C_t} \end{aligned}$$

To derive Eq. ([Capital](#)), substitute in Eq. ([Consumption FOC](#)) into Eq. ([Capital FOC](#)).

Finally, to derive Eq. ([Money](#)), substitute Eq. ([Consumption FOC](#)) into Eq. ([Real Money Holdings FOC](#)).

□

The Euler equations describe the equalisation of the marginal utility of consumption λ_t to the returns of the three assets bonds, capital, and money.

If we look at Eq. ([Money](#)), we can see that the equation has a useful economic interpretation.

$$\underbrace{\frac{1}{C_t}}_{\text{Marginal Cost}} = \underbrace{\frac{\chi}{M_t/P_t} + \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \frac{1}{\Pi_{t+1}} \right]}_{\text{Marginal Benefit}}$$

The marginal cost represents the marginal utility of consumption that has been given up to buy money M_t . The first term on the marginal benefit side represents the liquidity service from holding money. We can see that as the relative value of holding money $\chi \rightarrow 0$, then this liquidity service goes to 0. The second term of the marginal benefit describes the discounted value of future consumption after adjusting for inflation of the money you hold.

We can in fact view money as a stock and follow similar arguments from the asset pricing lecture to find the *price* of money.

Before we proceed, first recall that from our first order condition for consumption:

$$\lambda_t = \frac{1}{C_t}.$$

Recall that λ_t measures the increase in utility from relaxing the real budget constraint by one unit, so it has units of utility per consumption good. Because P_t is the amount of currency per consumption good, dividing λ_t by P_t gives marginal utility per unit of currency:

$$\lambda_t \cdot \frac{1}{P_t} = \frac{\text{Util}}{\text{Real Terms}} \cdot \frac{\text{Real Terms}}{\text{Monetary Amount}} = \frac{\text{Util}}{\text{Monetary Amount}}.$$

This object lets us value a unit of money as an asset.

Theorem 424 (Value of Money as an Asset)

The value of money as an asset is:

$$\frac{1/C_t}{P_t} = \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i \chi(1/M_{t+i}) \right] \quad (490)$$

Proof. First, look at the Euler equation for money in Eq. (Money):

$$\frac{1}{C_t} = \frac{\chi}{M_t/P_t} + \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \frac{1}{\Pi_{t+1}} \right]$$

We can divide by P_t in this equation to get:

$$\frac{1/C_t}{P_t} = \frac{\chi}{M_t} + \beta \mathbb{E}_t \left[\underbrace{\frac{1/C_{t+1}}{P_{t+1}}}_{(*)} \right]$$

We then iterate this equation forward one period:

$$\frac{1/C_{t+1}}{P_{t+1}} = \frac{\chi}{M_{t+1}} + \beta \mathbb{E}_{t+1} \left[\frac{1/C_{t+2}}{P_{t+2}} \right]$$

We then plug this iterated equation into (*):

$$\begin{aligned} \frac{1/C_t}{P_t} &= \frac{\chi}{M_t} + \beta \mathbb{E}_t \left[\frac{\chi}{M_{t+1}} + \beta \mathbb{E}_{t+1} \left[\frac{1/C_{t+2}}{P_{t+2}} \right] \right] \\ \frac{1/C_t}{P_t} &= \frac{\chi}{M_t} + \beta \mathbb{E}_t \left[\frac{\chi}{M_{t+1}} \right] + \beta \mathbb{E}_t \left[\beta \mathbb{E}_{t+1} \left[\frac{1/C_{t+2}}{P_{t+2}} \right] \right] \\ \frac{1/C_t}{P_t} &= \frac{\chi}{M_t} + \beta \mathbb{E}_t \left[\frac{\chi}{M_{t+1}} \right] + \beta^2 \mathbb{E}_t \left[\frac{1/C_{t+2}}{P_{t+2}} \right] \end{aligned}$$

where in the last line we applied the law of iterated expectations: $\mathbb{E}_t[\mathbb{E}_{t+1}[X_{t+2}]] = \mathbb{E}_t[X_{t+2}]$.

We therefore have the expression:

$$\frac{1/C_t}{P_t} = \frac{\chi}{M_t} + \beta \mathbb{E}_t \left[\chi(1/M_{t+1}) \right] + \beta^2 \mathbb{E}_t \left[\frac{1/C_{t+2}}{P_{t+2}} \right]$$

By induction, we can see that if we do this for T substitutions, we get:

$$\frac{1/C_t}{P_t} = \mathbb{E}_t \left[\sum_{i=0}^{T-1} \beta^i \chi(1/M_{t+i}) \right] + \beta^T \mathbb{E}_t \left[\frac{1/C_{t+T}}{P_{t+T}} \right]$$

We are interested in taking limits $T \rightarrow \infty$. Before we do this, we need another assumption.

Assumption (No Bubble Condition)

The no bubble condition for the value of money is given by:

$$\lim_{T \rightarrow \infty} \beta^T \mathbb{E}_t \left[\frac{1/C_{t+T}}{P_{t+T}} \right] = 0 \quad (491)$$

Therefore, we can now take limits and impose the no bubble condition:

$$\lim_{T \rightarrow \infty} \frac{1/C_t}{P_t} = \lim_{T \rightarrow \infty} \mathbb{E}_t \left[\sum_{i=0}^{T-1} \beta^i \chi(1/M_{t+i}) \right] + \lim_{T \rightarrow \infty} \beta^T \mathbb{E}_t \left[\frac{1/C_{t+T}}{P_{t+T}} \right]$$

where we employed the **dominated convergence theorem** to interchange the expectation and limit. We therefore have the desired expression:

$$\Rightarrow \frac{1/C_t}{P_t} = \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i \chi(1/M_{t+i}) \right]$$

□

The value of money as an asset equation has the nice interpretation:

$$\underbrace{\frac{1/C_t}{P_t}}_{\text{Current Value of Money}} = \underbrace{\mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i \chi(1/M_{t+i}) \right]}_{\text{Present Discounted Value}}$$

We now want to derive an equation for the **demand for money**.

Theorem 425 (Demand for Money Balances)

The **demand for real money balances** is given by:

$$\frac{M_t}{P_t} = \chi \frac{1 + i_t}{i_t} C_t \quad (492)$$

Proof. Take the Euler equation for bonds Eq. (Bonds):

$$\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \frac{1 + i_t}{\Pi_{t+1}} \right]$$

and re-arrange this to get:

$$\frac{1}{1 + i_t} = \beta \mathbb{E}_t \left[\frac{C_t}{C_{t+1}} \frac{1}{\Pi_{t+1}} \right] \quad (493)$$

Now take the Euler equation for money in Eq. (Money):

$$\frac{1}{C_t} = \frac{\chi}{M_t/P_t} + \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \frac{1}{\Pi_{t+1}} \right]$$

and re-arrange this to get:

$$1 = \frac{\chi C_t}{M_t/P_t} + \beta \mathbb{E}_t \left[\frac{C_t}{C_{t+1}} \frac{1}{\Pi_{t+1}} \right] \quad (494)$$

We now combine Eq. (493) and Eq. (494):

$$\begin{aligned}
\frac{1}{1+i_t} &= \beta \mathbb{E}_t \left[\frac{C_t}{C_{t+1}} \frac{1}{\Pi_{t+1}} \right] \\
\leftrightarrow \frac{1}{1+i_t} &= \left[1 - \frac{\chi C_t}{M_t/P_t} \right] \\
\leftrightarrow \frac{\chi C_t}{M_t/P_t} &= 1 - \frac{1}{1+i_t} \\
\leftrightarrow \chi C_t &= \frac{M_t}{P_t} - \frac{M_t}{P_t} \frac{1}{1+i_t} \\
\leftrightarrow \chi C_t &= \frac{M_t}{P_t} \left(1 - \frac{1}{1+i_t} \right) \\
\leftrightarrow \frac{M_t}{P_t} &= \frac{\chi C_t}{\left(1 - \frac{1}{1+i_t} \right)} \\
\leftrightarrow \frac{M_t}{P_t} &= \chi \frac{1+i_t}{i_t} C_t
\end{aligned}$$

□

The equation for real money demand shows that demand for real balances:

1. Increases with consumption as the household needs more money for its liquidity services;
2. Decreases with the nominal interest rate. This is due to the opportunity cost of holding money.

20.2.2 Firm Problem

We now turn to the firm problem in this model. We assume that there is a representative firm that hires labour and rents capital in competitive markets. We also shall assume a Cobb-Douglas production function.

Definition 426 (Firm Optimisation Problem)

The profit maximisation problem facing the firm is given by:

$$\begin{aligned}
\max_{K_{t-1}, N_t} \quad & P_t Y_t - R_t^k K_{t-1} - W_t N_t \\
\text{s.t.} \quad & Y_t = A_t K_{t-1}^\alpha N_t^{1-\alpha}
\end{aligned} \tag{495}$$

where $\alpha \in (0, 1)$ and A_t is a stationary TFP process.

To solve this, we can construct the **Lagrangian**:

$$\mathcal{L} = P_t A_t K_{t-1}^\alpha N_t^{1-\alpha} - R_t^k K_{t-1} - W_t N_t. \tag{496}$$

From this, we can derive the first order conditions.

Theorem 427 (Firm's First Order Conditions)

The firm's first-order conditions are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial K_{t-1}} : R_t^k &= \alpha P_t A_t K_{t-1}^{\alpha-1} N_t^{1-\alpha} & (\text{Capital}) \\
\frac{\partial \mathcal{L}}{\partial N_t} : W_t &= (1-\alpha) P_t A_t K_{t-1}^\alpha N_t^{-\alpha} & (\text{Labour})
\end{aligned} \tag{497}$$

20.2.3 Government

We now turn to the government. Recall that it issues bonds B_{t-1} that pay interest i_{t-1} . Its resulting liability is $(1 + i_{t-1})B_{t-1}$. Furthermore, taxes T_t less government spending $P_t G_t$ give the **primary surplus**, $T_t - P_t G_t$. Finally, the government prints money and earns **seigniorage**, $M_t - M_{t-1}$. We can therefore write the amount of bonds B_t that the government issues this period.

Definition 428 (Government Budget Constraint)

The government budget constraint is given by:

$$B_t = (1 + i_{t-1})B_{t-1} - (T_t - P_t G_t) - (M_t - M_{t-1}) \quad (498)$$

where B_t denotes bonds issued, i_t is the interest rate, $T_t - P_t G_t$ is the primary government surplus (taxes less government spending), and $M_t - M_{t-1}$ is seigniorage.

Throughout the rest of this lecture, we impose an assumption on the government's fiscal policy.

Assumption (Exogenous Fiscal Policy)

We assume that government spending G_t is an exogenous stationary process.

Remark 429. In this model, bonds B_t and taxes T_t are also fiscal policy. Furthermore, as T_t is a lump sum tax, **Ricardian equivalence** holds and hence only government spending G_t matters for fiscal policy.

For **monetary policy**, the government can set either the quantity of money M_t or its opportunity cost, represented by the nominal interest rate i_t .

20.3 Steady State Equilibrium

We now collect the equations and variables that constitute an equilibrium in the RBC model with money.

Theorem 430 (Equilibrium of RBC with Money)

The equilibrium of the model is given by the equations:

$$\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \frac{1 + i_t}{\Pi_{t+1}} \right] \quad (\text{Bond Market})$$

$$\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} (\alpha A_{t+1} K_t^{\alpha-1} N_{t+1}^{1-\alpha} + 1 - \delta) \right] \quad (\text{Capital Market})$$

$$\frac{M_t}{P_t} = \chi \frac{1 + i_t}{i_t} C_t \quad (\text{Money Market})$$

$$(1 - \alpha) A_t K_{t-1}^{\alpha} N_t^{-\alpha} = N_t^{\psi} C_t \quad (\text{Labour Market})$$

$$A_t K_{t-1}^{\alpha} N_t^{1-\alpha} = C_t + K_t - (1 - \delta) K_{t-1} + G_t \quad (\text{Goods Market})$$

where A_t and G_t are exogenous processes. Given the predetermined capital stock, the endogenous sequences are $\{C_t, i_t, M_t/P_t, K_t, N_t, \Pi_t\}$.

Proof. We have already derived Eq. (Bond Market). For Eq. (Capital Market), we take Eq. (Capital), which is the capital Euler equation:

$$\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} (r_{t+1}^k + 1 - \delta) \right]$$

and substitute in the firm's first order condition for capital:

$$r_{t+1}^k = \frac{R_{t+1}^K}{P_{t+1}} = \frac{1}{P_{t+1}} \alpha P_{t+1} A_{t+1} K_t^{\alpha-1} N_{t+1}^{1-\alpha}$$

to get our desired result. We have already derived the money-market equation as well. For the labour-market condition, we take the household's labour first-order condition Eq. ([Hours Worked FOC](#)):

$$N_t^\psi = \lambda_t w_t$$

and substitute in the firm's labour first order condition Eq. ([Labour](#)):

$$w_t = \frac{W_t}{P_t} = \frac{1}{P_t} (1 - \alpha) P_t A_t K_{t-1}^\alpha N_t^{-\alpha}$$

Finally, for the goods market condition, we take the household's budget constraint:

$$P_t C_t + P_t I_t + B_t + M_t = (1 + i_{t-1}) B_{t-1} + M_{t-1} + R_t^k K_{t-1} + W_t N_t - T_t$$

and then substitute in the government budget constraint:

$$B_t = (1 + i_{t-1}) B_{t-1} - (T_t - P_t G_t) - (M_t - M_{t-1})$$

to end up with:

$$\begin{aligned} P_t C_t + P_t I_t + B_t + M_t &= (1 + i_{t-1}) B_{t-1} + M_{t-1} + R_t^k K_{t-1} + W_t N_t - T_t \\ \Leftrightarrow P_t C_t + P_t I_t + (1 + i_{t-1}) B_{t-1} - (T_t - P_t G_t) - (M_t - M_{t-1}) + M_t &= (1 + i_{t-1}) B_{t-1} + M_{t-1} + R_t^k K_{t-1} + W_t N_t - T_t \\ \Leftrightarrow P_t C_t + P_t I_t + P_t G_t &= R_t^k K_{t-1} + W_t N_t \end{aligned}$$

Substitute in the evolution of capital:

$$K_t = (1 - \delta) K_{t-1} + I_t$$

to end up with:

$$P_t C_t + P_t (K_t - (1 - \delta) K_{t-1}) + P_t G_t = R_t^k K_{t-1} + W_t N_t.$$

Now divide through by the price level P_t :

$$C_t + (K_t - (1 - \delta) K_{t-1}) + G_t = r_t^k K_{t-1} + w_t N_t.$$

We then substitute in the firm's labour first order condition Eq. ([Labour](#)) and firm's capital first order condition Eq. ([Capital](#)):

$$\begin{aligned} C_t + (K_t - (1 - \delta) K_{t-1}) + G_t &= r_t^k K_{t-1} + w_t N_t \\ \Leftrightarrow C_t + (K_t - (1 - \delta) K_{t-1}) + G_t &= \left(\alpha A_t K_{t-1}^{\alpha-1} N_t^{1-\alpha} \right) K_{t-1} + \left((1 - \alpha) A_t K_{t-1}^\alpha N_t^{-\alpha} \right) N_t \\ \Leftrightarrow C_t + (K_t - (1 - \delta) K_{t-1}) + G_t &= \alpha A_t K_{t-1}^\alpha N_t^{1-\alpha} + (1 - \alpha) A_t K_{t-1}^\alpha N_t^{1-\alpha} \\ \Leftrightarrow C_t + (K_t - (1 - \delta) K_{t-1}) + G_t &= A_t K_{t-1}^\alpha N_t^{1-\alpha} \end{aligned}$$

□

Remark 431. Note that there are 5 equations but 6 variables as we have not specified the equation for monetary policy.

We can specify fiscal policy G_t to be exogenous due to the fact that Ricardian equivalence holds and hence the combination of bonds and taxes the government chooses to finance spending is irrelevant.

20.3.1 Determination of Price Level and Inflation

Now, look at the money market equilibrium:

$$\frac{M_t}{P_t} = \chi \frac{1 + i_t}{i_t} C_t.$$

If monetary policy specifies the money stock M_t , we can solve for three real variables and two nominal variables, $\{C_t, K_t, N_t, P_t, i_t\}$. If policy instead specifies a rule for the nominal interest rate i_t , the model determines real money balances M_t/P_t , but not M_t and P_t separately. Whether inflation is determinate depends on the form of the interest-rate rule, as shown below.

Under a policy targeting the quantity of money, fixing the money supply M_t allows us to recover the price level P_t . Under an interest-rate policy, we cannot infer a level (the price level) from a rate (the interest rate). Hence, the price level is undetermined when monetary policy is specified in terms of a nominal interest rate. In practice, inflation and the nominal interest rate are usually more salient than the levels of prices or money.

Theorem 432 (Determination of Price Level Versus Inflation)

If monetary policy is specified in terms of the nominal quantity of money M_t :

1. The price level P_t is determined.
2. Inflation $\pi_t = (P_t - P_{t-1})/P_{t-1}$ is determined.

If monetary policy is specified by a rule for the nominal interest rate i_t :

1. Inflation may be determinate or indeterminate, depending on the rule.
2. The price level P_t is undetermined.

Remark 433. If we look at money market equilibrium:

$$\frac{M_t}{P_t} = \chi \frac{1 + i_t}{i_t} C_t$$

if we know M_t (by targeting the money supply), we can use M_t/P_t to solve for P_t . If we target i_t , we cannot separate M_t and P_t , so we cannot determine the price level.

20.3.2 Steady State

We now want to examine the model's long-run behaviour. To do so, we compute its steady state. We assume that real variables, the nominal interest rate, and gross inflation are constant at their steady-state levels.

Theorem 434 (Steady State Equilibrium of RBC Model with Money)

The steady state equilibrium of the RBC model with money is:

$$\begin{aligned}
 1 &= \beta(1 + r) && \text{(SS Bond Market)} \\
 1 &= \beta \left[\alpha A k^{\alpha-1} + 1 - \delta \right] && \text{(SS Capital Market)} \\
 m &= \chi \frac{1+i}{i} C && \text{(SS Money Market)} \\
 (1 - \alpha) A k^{\alpha} &= N^{\psi} C && \text{(SS Labour Market)} \\
 A k^{\alpha} &= \frac{C}{N} + \delta k + \frac{G}{N} && \text{(SS Goods Market)}
 \end{aligned}$$

whereby $(1 + r) \equiv \frac{1+i}{1+\pi}$ is the real interest rate, $m \equiv M/P$ is the real value of money, and $k \equiv K/N$ is the capital-labour ratio.

Proof. First, take Eq. (Bond Market) and plug in steady state variables. The bond market equilibrium can then be solved for by:

$$\begin{aligned}
 \frac{1}{C} &= \beta \mathbb{E}_t \left[\frac{1}{C} \cdot \frac{1+i}{\Pi} \right] \\
 \Leftrightarrow 1 &= \beta \frac{1+i}{\Pi} \\
 \Leftrightarrow 1 &= \beta(1 + r)
 \end{aligned}$$

whereby $(1 + r) \equiv (1 + i)/\Pi$.

Therefore, the bond market equilibrium is:

$$1 = \beta(1 + r) \quad (499)$$

Now, look at Eq. (Capital Market) and plug in the steady state variables:

$$\begin{aligned}
 \frac{1}{C} &= \beta \mathbb{E}_t \left[\frac{1}{C} (\alpha A K^{\alpha-1} N^{1-\alpha} + 1 - \delta) \right] \\
 \Leftrightarrow 1 &= \beta \mathbb{E}_t \left[(\alpha A \left(\frac{K}{N} \right)^{\alpha-1} + 1 - \delta) \right] \\
 \Leftrightarrow 1 &= \beta \left[(\alpha A \left(k \right)^{\alpha-1} + 1 - \delta) \right]
 \end{aligned}$$

whereby in the last line, we defined $k \equiv \frac{K}{N}$. Therefore, the capital markets equilibrium is given by:

$$1 = \beta \left[(\alpha A \left(k \right)^{\alpha-1} + 1 - \delta) \right] \quad (500)$$

Now, if we recall the demand for real money balances Eq. (492):

$$\begin{aligned}
 \frac{M}{P} &= \chi \frac{1+i}{i} C \\
 \Leftrightarrow m &= \chi \frac{1+i}{i} C
 \end{aligned}$$

whereby $m \equiv \frac{M}{P}$. Therefore, the money market equilibrium is:

$$m = \chi \frac{1+i}{i} C. \quad (501)$$

Now, if we look at the first order condition for hours worked, Eq. ([Labour Market](#)), we can plug in the consumption FOC Eq. ([Consumption FOC](#)) and Eq. ([Labour](#)) to get:

$$\begin{aligned} N^\psi &= \lambda w \\ \Leftrightarrow N^\psi &= \frac{w}{C} \\ \Leftrightarrow N^\psi &= \frac{(1-\alpha)AK^\alpha N^{-\alpha}}{C} \end{aligned}$$

and therefore the labour market equilibrium is:

$$(1-\alpha)Ak^\alpha = N^\psi C. \quad (502)$$

whereby $k \equiv \frac{K}{N}$.

Now, looking at Eq. ([Goods Market](#))

$$\begin{aligned} AK^\alpha N^{1-\alpha} &= C + K - (1-\delta)K + G \\ \Leftrightarrow AK^\alpha N^{-\alpha} &= \frac{C}{N} + \frac{K}{N} - (1-\delta)\frac{K}{N} + \frac{G}{N} \\ \Leftrightarrow Ak^\alpha &= \frac{C}{N} + k - (1-\delta)k + \frac{G}{N} \\ \Leftrightarrow Ak^\alpha &= \frac{C}{N} + \delta k + \frac{G}{N} \end{aligned}$$

Therefore, the goods market equilibrium is:

$$Ak^\alpha = \frac{C}{N} + \delta k + \frac{G}{N} \quad (503)$$

□

20.3.3 Superneutrality of Money

We now want to solve for variables in the system. First, we will need the equilibrium interest rate that will clear the savings and borrowing market.

Proposition 435 ([Equilibrium Interest Rate](#))

The equilibrium interest rate that clears the savings market is:

$$r = \frac{1}{\beta} - 1. \quad (504)$$

Proof. Re-arrange the steady state bond market equilibrium:

$$1 = \beta(1+r).$$

□

We now want to solve for the steady state variables as functions of only parameters.

Proposition 436 (Superneutrality Of Money)

The steady-state solutions for the real variables are:

$$k^* = \left[\frac{\alpha\beta A}{1 - \beta(1 - \delta)} \right]^{\frac{1}{1-\alpha}} \quad (505)$$

$$(1 - \alpha)A(k^*)^\alpha = (N^*)^\psi \left\{ [A(k^*)^\alpha - \delta k^*]N^* - G \right\} \quad (506)$$

$$C^* = [A(k^*)^\alpha - \delta k^*]N^* - G \quad (507)$$

where $k \equiv K/N$, A is the TFP process, C is consumption, ψ is the inverse Frisch-elasticity parameter, N is labour, G is government spending, α is the Cobb–Douglas production-function parameter, and β is the household's discount factor.

These real variables **do not** depend on nominal variables, including money. This is the superneutrality of money.

Proof. If we start off with the steady state for the bond market, we get that:

$$1 = \beta(1 + r^*)$$

$$r^* = \frac{1}{\beta} - 1$$

Therefore, the steady state for the real interest rate is:

$$r^* = \frac{1}{\beta} - 1. \quad (508)$$

We next look at the steady state for the capital market:

$$1 = \beta \left[\alpha A(k^*)^{\alpha-1} + 1 - \delta \right]$$

$$\Leftrightarrow 1 - \beta(1 - \delta) = \beta \alpha A(k^*)^{\alpha-1}$$

$$\Leftrightarrow \frac{1 - \beta(1 - \delta)}{\alpha\beta A} = (k^*)^{\alpha-1}$$

$$\Leftrightarrow k^* = \left[\frac{\alpha\beta A}{1 - \beta(1 - \delta)} \right]^{\frac{1}{1-\alpha}}$$

The steady state capital-labour ratio is therefore given by:

$$k^* = \left[\frac{\alpha\beta A}{1 - \beta(1 - \delta)} \right]^{\frac{1}{1-\alpha}} \quad (509)$$

Now, look at the steady state for the goods market equation:

$$A(k^*)^\alpha = \frac{C^*}{N^*} + \delta k^* + \frac{G}{N^*}$$

$$\Leftrightarrow [A(k^*)^\alpha - \delta k^*]N^* - G = C^*$$

We therefore have that consumption C^* is a function of labour N^* :

$$C^* = [A(k^*)^\alpha - \delta k^*]N^* - G \quad (510)$$

Finally, look at the labour market equilibrium:

$$(1 - \alpha)Ak^\alpha = N^\psi C$$

and plug in both the steady state solution for consumption and capital to get:

$$(1 - \alpha)A(k^*)^\alpha = (N^*)^\psi \left\{ [A(k^*)^\alpha - \delta k^*]N^* - G \right\}$$

which only has a closed form solution if $G = 0$ or $\psi = 0$. □

The important thing to notice is that all the real variables in steady state C^*, k^*, N^* do not depend on nominal variables. They only depend on exogenous variables and parameters. Therefore, this tells us that monetary policy is irrelevant for real variables in the steady state.

We now want to examine how monetary policy affects the nominal variables in the economy, which is money supply, nominal interest rates, and the price level.

Proposition 437 (Nominal Variables under Money Supply Targeting)

Suppose that monetary policy is specified in terms of quantity of money $M = \bar{M}$. Then, the nominal interest rate is:

$$i^* = \frac{1 - \beta}{\beta} \quad (511)$$

and the aggregate price level is:

$$P^* = \frac{(1 - \beta)\bar{M}}{\chi C^*}. \quad (512)$$

The price level is determined. As the price level is constant, then inflation is **zero**.

Proof. We guess that under equilibrium, inflation will be 0: $\pi = 0$. Now, look at the steady state bond market equilibrium:

$$r = \frac{1}{\beta} - 1$$

where we have defined that:

$$1 + r \equiv \frac{1 + i}{1 + \pi}$$

However, since $\pi = 0$ by assumption, we then have that:

$$r = \frac{1 + i - 1 - \pi}{1 + \pi}$$

which gives us:

$$r = i.$$

We then plug this into our steady state bond market:

$$i = \frac{1}{\beta} - 1$$

to get:

$$i^* = \frac{1 - \beta}{\beta}.$$

Now, look at the steady state money market equilibrium:

$$\frac{\bar{M}}{P^*} = \chi \frac{1 + i}{i} C^*$$

and plug in our result for the equilibrium level of interest:

$$\begin{aligned}
 \frac{1+i}{i} &= \frac{1 + \frac{1-\beta}{\beta}}{\frac{1-\beta}{\beta}} \\
 &= \left(1 + \frac{1-\beta}{\beta}\right) \left(\frac{\beta}{1-\beta}\right) \\
 &= \frac{\beta}{1-\beta} + 1 \\
 &= \frac{\beta + 1 - \beta}{1-\beta} \\
 &= \frac{1}{1-\beta}.
 \end{aligned}$$

Then, we go back to our steady state money market equilibrium:

$$\begin{aligned}
 \frac{\bar{M}}{P^*} &= \chi \frac{1+i}{i} C^* \\
 &= \chi \frac{1}{1-\beta} C^* \\
 \Rightarrow \frac{\bar{M}}{P^*} &= \chi \frac{1}{1-\beta} C^*
 \end{aligned}$$

and then solve for P^* to get our desired result. □

Remark 438. *As the price level is a constant, this implies that inflation is zero.*

Proposition 439 (Nominal Variables under Nominal Interest-Rate Targeting)

Suppose that monetary policy is specified in terms of a nominal interest rate and inflation is $\pi = 0$. Then, real money balances are:

$$m^* = \frac{\chi C^*}{1-\beta}. \quad (513)$$

The price level and nominal money balances are undetermined.

Proof. If inflation is $\pi = 0$, then the same argument as under money-supply targeting gives:

$$i^* = \frac{1-\beta}{\beta}.$$

Now, if we look at the money market equilibrium, we immediately have that:

$$m^* = \frac{\chi C^*}{1-\beta}$$

□

Remark 440. *When we target the nominal interest rate, we cannot pin down either nominal money balances or the price level. This is again because a rate does not determine a level.*

20.3.4 Friedman Rule

The Friedman rule asks if the government cares about the welfare of the representative household, should it choose zero inflation? Is zero inflation optimal?

The argument is as follows. When the marginal social cost of creating money is zero, efficiency requires the opportunity cost of holding money to be zero. This opportunity cost is the nominal interest rate. Setting the nominal interest rate to zero therefore requires deflation because the equilibrium real interest rate is positive.

Theorem 441 (Friedman Rule)

The optimal level of inflation π is mild deflation:

$$\pi = \beta - 1. \quad (514)$$

Proof. The marginal cost of producing money is zero, so efficiency requires the opportunity cost of holding it to be zero. Since this opportunity cost is the foregone return on bonds, set $i = 0$. The steady-state Fisher relation then gives:

$$\begin{aligned} r &= \frac{1}{\beta} - 1 \\ \Leftrightarrow \frac{1+i}{1+\pi} &= \frac{1}{\beta} \\ \Leftrightarrow \frac{1}{1+\pi} &= \frac{1}{\beta} \\ \Rightarrow \pi &= \beta - 1 < 0 \end{aligned}$$

as $\beta < 1$. □

In this money-in-the-utility model, money demand becomes unbounded as i approaches zero, so the exact Friedman rule is best understood as a limiting allocation. In addition, a government may trade off this objective against revenue from seigniorage.

20.4 Dynamics

We are now interested in the short-run dynamics of the RBC model with money. We ask how the model evolves outside the steady state and, in particular, how endogenous variables respond to exogenous variables and parameters. To do so, we use approximations.

20.4.1 Log-Linearisation

In this section, we first recall our five equilibrium conditions that we derived earlier.

Theorem 442 (Equilibrium of RBC with Money)

The equilibrium of the model is given by the equations:

$$\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \frac{1+i_t}{\Pi_{t+1}} \right] \quad (\text{Bond Market})$$

$$\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} (\alpha A_{t+1} K_t^{\alpha-1} N_{t+1}^{1-\alpha} + 1 - \delta) \right] \quad (\text{Capital Market})$$

$$\frac{M_t}{P_t} = \chi \frac{1+i_t}{i_t} C_t \quad (\text{Money Market})$$

$$(1-\alpha) A_t K_{t-1}^\alpha N_t^{-\alpha} = N_t^\psi C_t \quad (\text{Labour Market})$$

$$A_t K_{t-1}^\alpha N_t^{1-\alpha} = C_t + K_t - (1-\delta) K_{t-1} + G_t \quad (\text{Goods Market})$$

where A_t and G_t are exogenous processes. Given the predetermined capital stock, the endogenous sequences are $\{C_t, i_t, M_t/P_t, K_t, N_t, \Pi_t\}$.

We will now apply a log-linear approximation to each of these equations and collect all our log-linear approximations.

Theorem 443 (Univariate Taylor Expansion)

The Taylor expansion of the function $f(\cdot)$ around the point X is given by:

$$f(X_t) = f(X) + f'(X)(X_t - X) + \mathcal{O}(\|\epsilon_t\|^2). \quad (515)$$

We also have a multivariate version of the Taylor expansion.

Theorem 444 (Multivariate Taylor Expansion)

A first-order multivariate Taylor expansion of the function $f(\cdot, \cdot, \cdot)$ around the point (x^*, y^*, z^*) is given by:

$$f(x, y, z) = f(x^*, y^*, z^*) + f_x(x^*, y^*, z^*)(x - x^*) + f_y(x^*, y^*, z^*)(y - y^*) + f_z(x^*, y^*, z^*)(z - z^*) + \mathcal{O}(\|\epsilon\|^2) \quad (516)$$

where $\mathcal{O}(\|\epsilon\|^2)$ collects higher-order terms.

We now state a result that tells us why we will soon be interested in log deviations around steady state.

Theorem 445 (Log Deviation Around Steady State)

A log deviation around the steady state is approximately equal to the corresponding percentage deviation:

$$\frac{X_t - X}{X} \approx \ln \left(\frac{X_t}{X} \right) + \mathcal{O}(\|\epsilon_t\|^2) \quad (517)$$

Proof. First write:

$$X_t = e^{\ln(X_t)}$$

We then take a first order Taylor expansion:

$$e^{\ln(X_t)} \approx e^{\ln(X)} + e^{\ln(X)} (\ln(X_t) - \ln(X)) + \mathcal{O}(\|\epsilon_t\|^2) = X(1 + \hat{X}_t) + \mathcal{O}(\|\epsilon_t\|^2)$$

where $\hat{X}_t \equiv \ln(X_t) - \ln(X)$. □

We now state our first log-linear approximation.

Proposition 446 (Log-Linear Approximation of Bond Market Equilibrium)

The log-linear approximation of the bond market equilibrium is:

$$\hat{C}_t = -(\hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]) + \mathbb{E}_t[\hat{C}_{t+1}] \quad (\text{Bond Market})$$

where we denote:

$$\begin{aligned} \hat{C}_t &\equiv \frac{C_t - C}{C} \\ \hat{\pi}_{t+1} &\equiv \frac{(1 + \pi_{t+1}) - (1 + \pi)}{1 + \pi} \\ \hat{i}_t &\equiv \frac{(1 + i_t) - (1 + i)}{1 + i} \end{aligned}$$

with C , π , and i denoting the steady-state values.

Proof. First, recall the bond market equilibrium.

$$\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \frac{1 + i_t}{\Pi_{t+1}} \right]$$

We now shall take first order Taylor expansions of both sides. For the left hand side, we can apply the *univariate* Taylor expansion of C_t where $f(C_t) = \frac{1}{C_t}$ around the steady state C :

$$\frac{1}{C_t} = \frac{1}{C} - \frac{1}{C^2}(C_t - C) + \mathcal{O}(\|\epsilon\|^2)$$

For the right hand side, we apply the *multivariate* Taylor expansion with three variables Eq. (516) to the expression inside the expectation operator:

$$\left[\frac{1}{C_{t+1}} \frac{1 + i_t}{\Pi_{t+1}} \right] = \left[\frac{1}{C} \frac{1 + i}{\Pi} \right] - \frac{1}{C^2} \frac{1 + i}{\Pi} (C_{t+1} - C) + \frac{1}{C} \frac{1}{\Pi} (i_t - i) - \frac{1}{C} \frac{1 + i}{\Pi^2} (\Pi_{t+1} - \Pi) + \mathcal{O}(\|\epsilon\|^2)$$

whereby we used the fact that the partial derivatives are:

$$\begin{aligned} f_C(C, \Pi, i) &= \frac{\partial}{\partial C} \left[\frac{1}{C} \frac{1 + i}{\Pi} \right] = -\frac{1}{C^2} \frac{1 + i}{\Pi} \\ f_i(C, \Pi, i) &= \frac{\partial}{\partial i} \left[\frac{1}{C} \frac{1 + i}{\Pi} \right] = \frac{1}{C} \frac{1}{\Pi} \\ f_\Pi(C, \Pi, i) &= \frac{\partial}{\partial \Pi} \left[\frac{1}{C} \frac{1 + i}{\Pi} \right] = -\frac{1}{C} \frac{1 + i}{\Pi^2} \end{aligned}$$

We then equate our expressions for the left hand side and right hand side to get (dropping second order remainder terms):

$$\frac{1}{C} - \frac{1}{C^2}(C_t - C) = \beta \mathbb{E}_t \left[\left[\frac{1}{C} \frac{1 + i}{\Pi} \right] - \frac{1}{C^2} \frac{1 + i}{\Pi} (C_{t+1} - C) + \frac{1}{C} \frac{1}{\Pi} (i_t - i) - \frac{1}{C} \frac{1 + i}{\Pi^2} (\Pi_{t+1} - \Pi) \right]$$

$$\underbrace{\left\{ \frac{1}{C} - \beta \mathbb{E}_t \left[\left[\frac{1}{C} \frac{1 + i}{\Pi} \right] \right] \right\}}_{=0} - \frac{1}{C^2}(C_t - C) = \beta \mathbb{E}_t \left[-\frac{1}{C^2} \frac{1 + i}{\Pi} (C_{t+1} - C) + \frac{1}{C} \frac{1}{\Pi} (i_t - i) - \frac{1}{C} \frac{1 + i}{\Pi^2} (\Pi_{t+1} - \Pi) \right]$$

$$-\frac{1}{C^2}(C_t - C) = \beta \mathbb{E}_t \left[-\frac{1}{C^2} \frac{1+i}{\Pi} (C_{t+1} - C) + \frac{1}{C} \frac{1}{\Pi} (i_t - i) - \frac{1}{C} \frac{1+i}{\Pi^2} (\Pi_{t+1} - \Pi) \right]$$

We now plug back in:

$$\Pi \equiv 1 + \pi$$

to get:

$$\begin{aligned} -\frac{1}{C^2}(C_t - C) &= \beta \mathbb{E}_t \left[-\frac{1}{C^2} \frac{1+i}{(1+\pi)} (C_{t+1} - C) + \frac{1}{C} \frac{1}{(1+\pi)} (i_t - i) - \frac{1}{C} \frac{1+i}{(1+\pi)^2} (1 + \pi_{t+1} - 1 - \pi) \right] \\ \Leftrightarrow -\frac{1}{C^2}(C_t - C) &= \beta \mathbb{E}_t \left[-\frac{1}{C^2} \frac{1+i}{(1+\pi)} (C_{t+1} - C) + \frac{1}{C} \frac{1}{(1+\pi)} (i_t - i) - \frac{1}{C} \frac{1+i}{(1+\pi)^2} (\pi_{t+1} - \pi) \right] \end{aligned}$$

We now multiply both sides by steady-state consumption C :

$$-\left(\frac{C_t - C}{C}\right) = \beta \mathbb{E}_t \left[-\frac{1+i}{(1+\pi)} \left(\frac{C_{t+1} - C}{C}\right) + \frac{1}{(1+\pi)} (i_t - i) - \frac{1+i}{(1+\pi)^2} (\pi_{t+1} - \pi) \right]$$

Now, we note that we can write the deviation of the nominal rate as the deviation of the *gross rate* from steady state:

$$\begin{aligned} i_t - i &= (1 + i_t) - (1 + i) \\ \pi_t - \pi &= (1 + \pi_t) - (1 + \pi) \end{aligned}$$

So we plug these in to get:

$$-\left(\frac{C_t - C}{C}\right) = \beta \mathbb{E}_t \left[-\frac{1+i}{(1+\pi)} \left(\frac{C_{t+1} - C}{C}\right) + \frac{1+i}{1+\pi} \left[\frac{1 + i_t - (1 + i)}{1 + i} \right] - \frac{1+i}{1+\pi} \left[\frac{1 + \pi_{t+1} - (1 + \pi)}{1 + \pi} \right] \right]$$

Now, recall that in the steady state, we have shown that the steady state of the bond market equilibrium was:

$$\begin{aligned} 1 &= \beta(1 + r) \\ \Leftrightarrow 1 &= \beta \left(\frac{1+i}{1+\pi} \right) \end{aligned}$$

so then we have that:

$$\begin{aligned}
-\left(\frac{C_t - C}{C}\right) &= \beta \mathbb{E}_t \left[-\frac{1+i}{(1+\pi)} \left(\frac{C_{t+1} - C}{C} \right) + \frac{1+i}{1+\pi} \left[\frac{1+i_t - (1+i)}{1+i} \right] - \frac{1+i}{1+\pi} \left[\frac{1+\pi_{t+1} - (1+\pi)}{1+\pi} \right] \right] \\
\leftrightarrow \left(\frac{C_t - C}{C}\right) &= \beta \mathbb{E}_t \left[\frac{1+i}{(1+\pi)} \left(\frac{C_{t+1} - C}{C} \right) - \frac{1+i}{1+\pi} \left[\frac{1+i_t - (1+i)}{1+i} \right] + \frac{1+i}{1+\pi} \left[\frac{1+\pi_{t+1} - (1+\pi)}{1+\pi} \right] \right] \\
\leftrightarrow \left(\frac{C_t - C}{C}\right) &= \beta \mathbb{E}_t \left[\frac{1+i}{(1+\pi)} \left(\frac{C_{t+1} - C}{C} \right) \right] - \beta \mathbb{E}_t \left[\frac{1+i}{1+\pi} \left[\frac{1+i_t - (1+i)}{1+i} \right] \right] + \beta \mathbb{E}_t \left[\frac{1+i}{1+\pi} \left[\frac{1+\pi_{t+1} - (1+\pi)}{1+\pi} \right] \right] \\
\leftrightarrow \left(\frac{C_t - C}{C}\right) &= \beta \frac{1+i}{(1+\pi)} \mathbb{E}_t \left[\left(\frac{C_{t+1} - C}{C} \right) \right] - \beta \frac{1+i}{1+\pi} \mathbb{E}_t \left[\left[\frac{1+i_t - (1+i)}{1+i} \right] \right] + \beta \frac{1+i}{1+\pi} \mathbb{E}_t \left[\left[\frac{1+\pi_{t+1} - (1+\pi)}{1+\pi} \right] \right] \\
\leftrightarrow \left(\frac{C_t - C}{C}\right) &= \mathbb{E}_t \left[\left(\frac{C_{t+1} - C}{C} \right) \right] - \mathbb{E}_t \left[\left[\frac{1+i_t - (1+i)}{1+i} \right] \right] + \mathbb{E}_t \left[\left[\frac{1+\pi_{t+1} - (1+\pi)}{1+\pi} \right] \right] \\
\leftrightarrow \left(\frac{C_t - C}{C}\right) &= \mathbb{E}_t \left[\left(\frac{C_{t+1} - C}{C} \right) \right] - \mathbb{E}_t \left[\left[\frac{1+i_t - (1+i)}{1+i} \right] \right] + \mathbb{E}_t \left[\left[\frac{1+\pi_{t+1} - (1+\pi)}{1+\pi} \right] \right] \\
\leftrightarrow \left(\frac{C_t - C}{C}\right) &= \mathbb{E}_t \left[\left(\frac{C_{t+1} - C}{C} \right) \right] - \left[\frac{1+i_t - (1+i)}{1+i} \right] + \mathbb{E}_t \left[\left[\frac{1+\pi_{t+1} - (1+\pi)}{1+\pi} \right] \right]
\end{aligned}$$

To conclude, we use our result that a percentage deviation from steady state correspond to a log deviation from steady state:

$$\frac{X_t - X}{X} \approx \ln \left(\frac{X_t}{X} \right) \equiv \hat{X}_t$$

and therefore the log-linearisation is:

$$\hat{C}_t = \mathbb{E}_t[\hat{C}_{t+1}] - \hat{i}_t + \mathbb{E}_t[\hat{\pi}_{t+1}]$$

□

Proposition 447 (Log-Linear Approximation of Capital Market Equilibrium)

The log-linear approximation of the capital market equilibrium is:

$$\hat{C}_t = -[1 - \beta(1 - \delta)] \mathbb{E}_t[\hat{A}_{t+1} - (1 - \alpha)(\hat{K}_t - \hat{N}_{t+1})] + \mathbb{E}_t[\hat{C}_{t+1}] \quad (518)$$

Proof. We have the capital market equilibrium:

$$\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} (\alpha A_{t+1} K_t^{\alpha-1} N_{t+1}^{1-\alpha} + 1 - \delta) \right]$$

We now shall take first order Taylor expansions of both sides. For the left hand side, we can apply the *univariate* Taylor expansion of C_t where $f(C_t) = \frac{1}{C_t}$ around the steady state C :

$$\frac{1}{C_t} = \frac{1}{C} - \frac{1}{C^2} (C_t - C) + \mathcal{O}(\|\epsilon\|^2)$$

Now, for the right hand side, we recall the firm's factor product condition:

$$r_t^k = \alpha A_t K_{t-1}^{\alpha-1} N_t^{1-\alpha}$$

and we plug this in to get:

$$\beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} (\alpha A_{t+1} K_t^{\alpha-1} N_{t+1}^{1-\alpha} + 1 - \delta) \right] = \beta \mathbb{E}_t \left[\underbrace{\frac{1}{C_{t+1}} (r_{t+1}^k + 1 - \delta)}_{(*)} \right]$$

We now take multivariate Taylor expansion of (*) around steady states (C, r^k) to get:

$$\frac{1}{C_{t+1}}(r_{t+1}^k + 1 - \delta) = \frac{1}{C}(r^k + 1 - \delta) - \frac{1}{C^2}(r^k + 1 - \delta)(C_{t+1} - C) + \frac{1}{C}(r_{t+1}^k - r^k) + \mathcal{O}(\|\epsilon\|^2)$$

whereby we used the partial derivatives:

$$\begin{aligned}\frac{\partial}{\partial C} \left(\frac{1}{C}(r^k + 1 - \delta) \right) &= -\frac{1}{C^2}(r^k + 1 - \delta) \\ \frac{\partial}{\partial r^k} \left(\frac{1}{C}(r^k + 1 - \delta) \right) &= \frac{1}{C}\end{aligned}$$

We then equate the Taylor expansion sides together to get:

$$\begin{aligned}\frac{1}{C} - \frac{1}{C^2}(C_t - C) &= \beta \mathbb{E}_t \left[\frac{1}{C}(r^k + 1 - \delta) - \frac{1}{C^2}(r^k + 1 - \delta)(C_{t+1} - C) + \frac{1}{C}(r_{t+1}^k - r^k) \right] \\ \Leftrightarrow \underbrace{\frac{1}{C} - \beta \mathbb{E}_t \left[\frac{1}{C}(r^k + 1 - \delta) \right]}_{=0} - \frac{1}{C^2}(C_t - C) &= \beta \mathbb{E}_t \left[-\frac{1}{C^2}(r^k + 1 - \delta)(C_{t+1} - C) + \frac{1}{C}(r_{t+1}^k - r^k) \right] \\ \Leftrightarrow -\frac{1}{C^2}(C_t - C) &= \beta \mathbb{E}_t \left[-\frac{1}{C^2}(r^k + 1 - \delta)(C_{t+1} - C) + \frac{1}{C}(r_{t+1}^k - r^k) \right]\end{aligned}$$

Now, recall the steady state capital market equilibrium Eq. (SS Capital Market):

$$1 = \beta \left[\alpha A k^{\alpha-1} + 1 - \delta \right] \quad (\text{SS Capital Market})$$

whereby we can again plug in the firm's rental rate of capital:

$$\begin{aligned}1 &= \beta \left[\alpha A k^{\alpha-1} + 1 - \delta \right] \\ \Leftrightarrow 1 &= \beta \left[\alpha A \left(\frac{K}{N} \right)^{\alpha-1} + 1 - \delta \right] \\ \Leftrightarrow 1 &= \beta \left[\alpha A K^{\alpha-1} N^{1-\alpha} + 1 - \delta \right] \\ \Leftrightarrow 1 &= \beta \underbrace{\left[r^k + 1 - \delta \right]}_{(**)}\end{aligned}$$

So we can plug in (**) into:

$$\begin{aligned}-\frac{1}{C^2}(C_t - C) &= \beta \mathbb{E}_t \left[-\frac{1}{C^2}(r^k + 1 - \delta)(C_{t+1} - C) + \frac{1}{C}(r_{t+1}^k - r^k) \right] \\ \Leftrightarrow -\frac{1}{C^2}(C_t - C) &= \underbrace{\beta(r^k + 1 - \delta)}_{(**)} \mathbb{E}_t \left[-\frac{1}{C^2}(C_{t+1} - C) \right] + \beta \mathbb{E}_t \left[\frac{1}{C}(r_{t+1}^k - r^k) \right] \\ \Leftrightarrow -\frac{1}{C^2}(C_t - C) &= \mathbb{E}_t \left[-\frac{1}{C^2}(C_{t+1} - C) \right] + \beta \mathbb{E}_t \left[\frac{1}{C}(r_{t+1}^k - r^k) \right]\end{aligned}$$

Now multiply through by $-C$:

$$\begin{aligned}\Leftrightarrow \frac{C_t - C}{C} &= \mathbb{E}_t \left[\frac{C_{t+1} - C}{C} \right] - \beta \mathbb{E}_t \left[(r_{t+1}^k - r^k) \right] \\ \Leftrightarrow \frac{C_t - C}{C} &= \mathbb{E}_t \left[\frac{C_{t+1} - C}{C} \right] - \beta \mathbb{E}_t \left[\frac{r^k}{r^k} (r_{t+1}^k - r^k) \right] \\ \Leftrightarrow \frac{C_t - C}{C} &= \mathbb{E}_t \left[\frac{C_{t+1} - C}{C} \right] - \beta r^k \mathbb{E}_t \left[\frac{r_{t+1}^k - r^k}{r^k} \right]\end{aligned}$$

Now introduce our hat notation to get:

$$\hat{C}_t = -\beta r^k \mathbb{E}_t[\hat{r}_{t+1}^k] + \mathbb{E}_t[\hat{C}_{t+1}].$$

Finally, we notice that r_{t+1}^k is exactly log-linear so we have that:

$$\hat{r}_{t+1}^k = \hat{A}_{t+1} + (\alpha - 1)(\hat{K}_t - \hat{N}_{t+1})$$

which we can plug into our earlier expression:

$$\begin{aligned}\hat{C}_t &= -\beta r^k \mathbb{E}_t[\hat{r}_{t+1}^k] + \mathbb{E}_t[\hat{C}_{t+1}] \\ \Leftrightarrow \hat{C}_t &= -\beta r^k \mathbb{E}_t[\hat{A}_{t+1} + (\alpha - 1)(\hat{K}_t - \hat{N}_{t+1})] + \mathbb{E}_t[\hat{C}_{t+1}] \\ \Leftrightarrow \hat{C}_t &= -[1 - \beta(1 - \delta)] \mathbb{E}_t[\hat{A}_{t+1} + (\alpha - 1)(\hat{K}_t - \hat{N}_{t+1})] + \mathbb{E}_t[\hat{C}_{t+1}]\end{aligned}$$

where the last line uses the steady state capital market equilibrium Eq. (SS Capital Market):

$$\beta(r^k + 1 - \delta) = 1$$

□

Proposition 448 (Log-Linear Approximation of Money Market Equilibrium)

The log-linear approximation of the money market equilibrium is:

$$\frac{\widehat{M}_t}{\widehat{P}_t} = -\left(\frac{\beta}{\Pi - \beta}\right) \hat{i}_t + \hat{C}_t \quad (519)$$

Proof. The money market equilibrium is given by:

$$\frac{M_t}{P_t} = \chi \frac{1 + i_t}{i_t} C_t$$

The trick to realise is that we are treating $\frac{M_t}{P_t}$ (real money balance) as a single variable as we do not take a stand yet on M_t or P_t . Therefore, a univariate Taylor expansion of left hand side gives us:

$$\frac{M_t}{P_t} = \frac{M}{P} + 1\left(\frac{M_t}{P_t} - \frac{M}{P}\right) + \mathcal{O}(\|\epsilon_t\|^2).$$

Multivariate Taylor expansion of right hand side gives us:

$$\chi \frac{1 + i_t}{i_t} C_t = \chi \frac{1 + i}{i} C - \chi \frac{C}{i^2} (i_t - i) + \chi \frac{1 + i}{i} (C_t - C) + \mathcal{O}(\|\epsilon_t\|^2)$$

whereby we used the partial derivatives:

$$\begin{aligned}\frac{\partial}{\partial i} \left(\chi \frac{1+i}{i} C \right) &= \frac{\partial}{\partial i} \left(\chi \left(1 + \frac{1}{i} \right) C \right) = -\chi \frac{C}{i^2} \\ \frac{\partial}{\partial C} \left(\chi \frac{1+i}{i} C \right) &= \chi \frac{1+i}{i}\end{aligned}$$

So by equating the two sides, we get:

$$\begin{aligned}\frac{M}{P} + \left(\frac{M_t}{P_t} - \frac{M}{P} \right) &= \chi \frac{1+i}{i} C - \chi \frac{C}{i^2} (i_t - i) + \chi \frac{1+i}{i} (C_t - C) \\ \Leftrightarrow \underbrace{\frac{M}{P} - \chi \frac{1+i}{i} C}_{=0} + \left(\frac{M_t}{P_t} - \frac{M}{P} \right) &= -\chi \frac{C}{i^2} (i_t - i) + \chi \frac{1+i}{i} (C_t - C) \\ \Leftrightarrow \frac{M_t}{P_t} - \frac{M}{P} &= -\chi \frac{C}{i^2} (i_t - i) + \chi \frac{1+i}{i} (C_t - C) \\ \Leftrightarrow \frac{M_t}{P_t} - m &= -\chi \frac{C}{i^2} (i_t - i) + \chi \frac{1+i}{i} (C_t - C)\end{aligned}$$

where we defined $m \equiv \frac{M}{P}$. Now, we note that we can write the deviation of the nominal rate as the deviation of the *gross rate* from steady state:

$$i_t - i = (1 + i_t) - (1 + i)$$

so we have that:

$$\begin{aligned}\Leftrightarrow \frac{M_t}{P_t} - m &= -\chi \frac{C}{i^2} (i_t - i) + \chi \frac{1+i}{i} (C_t - C) \\ \Leftrightarrow \frac{M_t}{P_t} - m &= -\chi \frac{C}{i^2} [(1 + i_t) - (1 + i)] + \chi \frac{1+i}{i} (C_t - C) \\ \Leftrightarrow \frac{M_t}{P_t} - m &= \chi \frac{1+i}{i} C \left[-\frac{1}{i} \left(\frac{1 + i_t - (1 + i)}{1 + i} \right) + \left(\frac{C_t - C}{C} \right) \right] \\ \Leftrightarrow m \left(\frac{M_t/P_t - m}{m} \right) &= \chi \frac{1+i}{i} C \left[-\frac{1}{i} \left(\frac{1 + i_t - (1 + i)}{1 + i} \right) + \left(\frac{C_t - C}{C} \right) \right]\end{aligned}$$

Now note that under steady, the money market equilibrium is:

$$\frac{M_t}{P_t} = \chi \frac{1+i_t}{i_t} C_t \quad \Leftrightarrow \quad m = \chi \frac{1+i}{i} C$$

so we can cancel out the terms to get:

$$\begin{aligned}\Leftrightarrow m \left(\frac{M_t/P_t - m}{m} \right) &= \chi \frac{1+i}{i} C \left[-\frac{1}{i} \left(\frac{1 + i_t - (1 + i)}{1 + i} \right) + \left(\frac{C_t - C}{C} \right) \right] \\ \Leftrightarrow \left(\frac{M_t/P_t - m}{m} \right) &= \left[-\frac{1}{i} \left(\frac{1 + i_t - (1 + i)}{1 + i} \right) + \left(\frac{C_t - C}{C} \right) \right] \\ \Leftrightarrow \left(\frac{m_t - m}{m} \right) &= \left[-\frac{1}{i} \left(\frac{1 + i_t - (1 + i)}{1 + i} \right) + \left(\frac{C_t - C}{C} \right) \right] \\ \Leftrightarrow \left(\frac{m_t - m}{m} \right) &= -\left(\frac{\beta}{1 + \pi - \beta} \right) \left(\frac{1 + i_t - (1 + i)}{1 + i} \right) + \left(\frac{C_t - C}{C} \right)\end{aligned}$$

where in the last line, we used the steady state bond market equilibrium:

$$1 = \beta \left(\frac{1+i}{1+\pi} \right) \Rightarrow \frac{1}{i} = \left(\frac{\beta}{1+\pi-\beta} \right)$$

Use our notation for steady state deviation to conclude that:

$$\frac{\widehat{M_t}}{\widehat{P_t}} = - \left(\frac{\beta}{1+\pi-\beta} \right) \hat{i}_t + \hat{C}_t.$$

□

Proposition 449 (Log-Linear Approximation of Labour Market Equilibrium)

The log-linear approximation of the labour market equilibrium is:

$$\hat{A}_t + \alpha(\hat{K}_{t-1} - \hat{N}_t) = \psi \hat{N}_t + \hat{C}_t \quad (520)$$

Proof. The labour market equilibrium is given by:

$$(1 - \alpha)A_t K_{t-1}^\alpha N_t^{-\alpha} = N_t^\psi C_t$$

This uses a different (and easier) technique as it is a log-linear equation (if we take logs, we get a linear equation). This time, we take the logarithm of both sides and Taylor expand. For the left hand side, taking logarithms gets us:

$$\ln(1 - \alpha) + \ln(A_t) + \alpha \ln(K_{t-1}) - \alpha \ln(N_t)$$

We then apply three single Taylor expansions around (A, K, N) and use our hat notation to get:

$$\begin{aligned} \ln(1 - \alpha) + \left[\ln(A) + \frac{1}{A}(A_t - A) \right] + \alpha \left[\ln(K) + \frac{1}{K}(K_{t-1} - K) \right] - \alpha \left[\ln(N) + \frac{1}{N}(N_t - N) \right] \\ \Rightarrow \ln(1 - \alpha) + \ln(A) + \alpha \ln(K) - \alpha \ln(N) + \hat{A}_t + \alpha \hat{K}_{t-1} - \alpha \hat{N}_t \end{aligned}$$

Now for the right hand side, we take the logarithm to get:

$$\begin{aligned} \psi \ln(N_t) + \ln(C_t) \\ \Leftrightarrow \psi \left[\ln(N) + \frac{1}{N}(N_t - N) \right] + \left[\ln(C) + \frac{1}{C}(C_t - C) \right] \\ \Leftrightarrow \psi \ln(N) + \ln(C) + \psi \hat{N}_t + \hat{C}_t \end{aligned}$$

We then equate the two sides together to get:

$$\ln(1 - \alpha) + \ln(A) + \alpha \ln(K) - \alpha \ln(N) + \hat{A}_t + \alpha \hat{K}_{t-1} - \alpha \hat{N}_t = \psi \ln(N) + \ln(C) + \psi \hat{N}_t + \hat{C}_t$$

We know that in the steady state the labour-market equilibrium implies:

$$(1 - \alpha)A_t K_{t-1}^\alpha N_t^{-\alpha} = N_t^\psi C_t \Rightarrow \ln(1 - \alpha) + \ln(A) + \alpha \ln(K) - \alpha \ln(N) = \psi \ln(N) + \ln(C)$$

so therefore we have that:

$$\begin{aligned} \ln(1 - \alpha) + \ln(A) + \alpha \ln(K) - \alpha \ln(N) + \hat{A}_t + \alpha \hat{K}_{t-1} - \alpha \hat{N}_t &= \psi \ln(N) + \ln(C) + \psi \hat{N}_t + \hat{C}_t \\ \leftrightarrow \hat{A}_t + \alpha(\hat{K}_{t-1} - \hat{N}_t) &= \psi \hat{N}_t + \hat{C}_t \end{aligned}$$

□

Proposition 450 (Log-Linear Approximation of Goods Market Equilibrium)

The log-linear approximation of the goods market equilibrium is:

$$\hat{A}_t + \alpha \hat{K}_{t-1} + (1 - \alpha) \hat{N}_t = \frac{C}{Y} \hat{C}_t + \frac{K}{Y} [\hat{K}_t - (1 - \delta) \hat{K}_{t-1}] + \frac{G}{Y} \hat{G}_t \quad (521)$$

Proof. We look at goods market equilibrium:

$$A_t K_{t-1}^\alpha N_t^{1-\alpha} = C_t + K_t - (1 - \delta) K_{t-1} + G_t$$

The term on the left hand side is the Cobb-Douglas production function so we can rewrite this as:

$$Y_t = C_t + K_t - (1 - \delta) K_{t-1} + G_t$$

We now apply a regular Taylor expansion to both sides to each term. On the left hand side, we therefore have:

$$Y_t \approx Y + (Y_t - Y)$$

On the right hand side, we have that:

$$\begin{aligned} C_t + K_t - (1 - \delta) K_{t-1} + G_t \\ \approx [C + (C_t - C)] + [K + (K_t - K)] - (1 - \delta)[K + (K_{t-1} - K)] + [G + (G_t - G)] \end{aligned}$$

We then combine the left and right hand side Taylor expansions to get:

$$Y + (Y_t - Y) = [C + (C_t - C)] + [K + (K_t - K)] - (1 - \delta)[K + (K_{t-1} - K)] + [G + (G_t - G)]$$

Now, if we look at the goods market equilibrium, we have that in steady state:

$$Y_t = C_t + K_t - (1 - \delta) K_{t-1} + G_t \Rightarrow Y = C + K - (1 - \delta) K + G$$

so that our Taylor expansion becomes:

$$\begin{aligned} Y + (Y_t - Y) &= [C + (C_t - C)] + [K + (K_t - K)] - (1 - \delta)[K + (K_{t-1} - K)] + [G + (G_t - G)] \\ \leftrightarrow \underbrace{(Y - C - K + (1 - \delta)K - G)}_{=0} + (Y_t - Y) &= (C_t - C) + (K_t - K) - (1 - \delta)(K_{t-1} - K) + (G_t - G) \\ \leftrightarrow (Y_t - Y) &= (C_t - C) + (K_t - K) - (1 - \delta)(K_{t-1} - K) + (G_t - G) \end{aligned}$$

We then divide by steady state output Y and multiply and divide by the steady state values needed to get log-

deviation:

$$\begin{aligned}\frac{Y_t - Y}{Y} &= \frac{C}{Y} \frac{C_t - C}{C} + \frac{K}{Y} \frac{K_t - K}{K} - (1 - \delta) \frac{K}{Y} \frac{K_{t-1} - K}{K} + \frac{G}{Y} \frac{G_t - G}{G} \\ \Leftrightarrow \hat{Y}_t &= \frac{C}{Y} \hat{C}_t + \frac{K}{Y} \hat{K}_t - (1 - \delta) \frac{K}{Y} \hat{K}_{t-1} + \frac{G}{Y} \hat{G}_t\end{aligned}\quad (*)$$

To finish, we want to find an expression for $\hat{Y}_t$ in terms of the Cobb-Douglas production function:

$$Y_t = A_t K_{t-1}^\alpha N_t^{1-\alpha}$$

We take logs of both sides and Taylor expand. On the left hand side, we get:

$$\ln(Y) + \frac{1}{Y}(Y_t - Y).$$

On the right hand side, we get:

$$\ln(A) + \frac{1}{A}(A_t - A) + \alpha \left[\ln(K) + \frac{1}{K}(K_{t-1} - K) \right] + (1 - \alpha) \left[\ln(N) + \frac{1}{N}(N_t - N) \right]$$

We combine these two sides to get:

$$\begin{aligned}\ln(Y) + \frac{1}{Y}(Y_t - Y) &= \ln(A) + \frac{1}{A}(A_t - A) + \alpha \left[\ln(K) + \frac{1}{K}(K_{t-1} - K) \right] + (1 - \alpha) \left[\ln(N) + \frac{1}{N}(N_t - N) \right] \\ \Leftrightarrow \underbrace{\left[\ln(Y) - \ln(A) - \alpha \ln(K) - (1 - \alpha) \ln(N) \right]}_{=0} + \frac{Y_t - Y}{Y} &= \frac{A_t - A}{A} + \alpha \frac{K_{t-1} - K}{K} + (1 - \alpha) \frac{N_t - N}{N} \\ \Rightarrow \frac{Y_t - Y}{Y} &= \frac{A_t - A}{A} + \alpha \frac{K_{t-1} - K}{K} + (1 - \alpha) \frac{N_t - N}{N} \\ \Rightarrow \hat{Y}_t &= \hat{A}_t + \alpha \hat{K}_{t-1} + (1 - \alpha) \hat{N}_t\end{aligned}$$

We can finally plug this into the left hand side of equation (*) to conclude:

$$\hat{A}_t + \alpha \hat{K}_{t-1} + (1 - \alpha) \hat{N}_t = \frac{C}{Y} \hat{C}_t + \frac{K}{Y} \hat{K}_t - (1 - \delta) \frac{K}{Y} \hat{K}_{t-1} + \frac{G}{Y} \hat{G}_t$$

□

We now collect all the equations that constitute the log-linearised equilibrium.

Theorem 451 (Log-Linearised Equilibrium)

The log-linearised equilibrium of the model is given by the equations:

$$\hat{C}_t = -(\hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]) + \mathbb{E}_t[\hat{C}_{t+1}] \quad (\text{Bond Market})$$

$$\hat{C}_t = -[1 - \beta(1 - \delta)]\mathbb{E}_t[\hat{A}_{t+1} - (1 - \alpha)(\hat{K}_t - \hat{N}_{t+1})] + \mathbb{E}_t[\hat{C}_{t+1}] \quad (\text{Capital Market})$$

$$\frac{\widehat{M}_t}{P_t} = -\left(\frac{\beta}{1 - \beta}\right)\hat{i}_t + \hat{C}_t \quad (\text{Money Market})$$

$$\hat{A}_t + \alpha(\hat{K}_{t-1} - \hat{N}_t) = \psi \hat{N}_t + \hat{C}_t \quad (\text{Labour Market})$$

$$\hat{A}_t + \alpha \hat{K}_{t-1} + (1 - \alpha) \hat{N}_t = \frac{C}{Y} \hat{C}_t + \frac{K}{Y} [\hat{K}_t - (1 - \delta) \hat{K}_{t-1}] + \frac{G}{Y} \hat{G}_t \quad (\text{Goods Market})$$

In the log-linearised bond market equilibrium, there now appears a new term that is quite important and used repeatedly so it is worth defining it.

Definition 452 (Fisher Equation)

The log-linearised Fisher equation is:

$$\hat{r}_t \equiv \hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]. \quad (522)$$

20.4.2 Special Case: No Capital

Given our log-linearised equilibrium, we want to derive analytical results showing how monetary policy affects real variables. For ease of analysis, we consider the case in which capital is fixed, $\hat{K}_t = 0$ for all t . We also assume that technology is linear in labour input, $\alpha = 0$, and that government spending is zero, $G_t = 0$.

Proposition 453 (Real Variables are Independent of Monetary Policy)

The real variables $\hat{C}_t$, $\hat{Y}_t$, $\hat{N}_t$, and $\hat{r}_t$ are independent of monetary policy. In particular, the deviation of hours worked from its steady state is zero:

$$\hat{N}_t = 0. \quad (523)$$

Steady state deviation of consumption and output is pinned down by productivity $\hat{A}_t$:

$$\hat{C}_t = \hat{Y}_t = \hat{A}_t. \quad (524)$$

The real interest rate $\hat{r}_t \equiv \hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]$ is pinned down by productivity:

$$\hat{r}_t = \mathbb{E}_t[\hat{A}_{t+1}] - \hat{A}_t. \quad (525)$$

Money is also neutral in the short run.

Proof. First, look at the log-linearised labour market:

$$\hat{A}_t + \alpha(\hat{K}_{t-1} - \hat{N}_t) = \psi \hat{N}_t + \hat{C}_t$$

We have that $\alpha = 0$ so we therefore have:

$$\hat{A}_t = \psi \hat{N}_t + \hat{C}_t \quad (*)$$

Now, look at the log-linearised goods market:

$$\hat{A}_t + \alpha \hat{K}_{t-1} + (1 - \alpha) \hat{N}_t = \frac{C}{Y} \hat{C}_t + \frac{K}{Y} [\hat{K}_t - (1 - \delta) \hat{K}_{t-1}] + \frac{G}{Y} \hat{G}_t$$

We have that $\alpha = 0$ and government spending is equal to 0:

$$\hat{A}_t + \hat{N}_t = \frac{C}{Y} \hat{C}_t$$

Now, since there is no capital or government spending, consumption is equal to output as from the goods market clearing condition:

$$Y = C + I + G$$

but $G = 0$ by assumption and $I = 0$ since capital was assumed to be constant so we have that:

$$Y = C.$$

Therefore, our previous expression simplifies to:

$$\hat{A}_t + \hat{N}_t = \hat{C}_t \quad (**)$$

We now combine (*) and (**)

$$\begin{aligned} \hat{A}_t &= \psi \hat{N}_t + \hat{C}_t \\ \hat{A}_t + \hat{N}_t &= \hat{C}_t \end{aligned}$$

and solving out, we get:

$$\begin{aligned} \hat{N}_t &= 0 \\ \hat{C}_t &= \hat{A}_t \end{aligned}$$

Now, recall that consumption equals output so this implies that:

$$\hat{Y}_t = \hat{A}_t$$

as well.

Now, look at the log-linearised bond market equilibrium:

$$\hat{C}_t = -(\hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]) + \mathbb{E}_t[\hat{C}_{t+1}]$$

and write $\hat{r}_t \equiv \hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]$ as the real interest rate. Then substitute in that consumption is equal to productivity to get:

$$\hat{A}_t = -\hat{r}_t + \mathbb{E}_t[\hat{A}_{t+1}]$$

and therefore we get:

$$\hat{r}_t = \mathbb{E}_t[\hat{A}_{t+1}] - \hat{A}_t.$$

□

We see that labour is constant relative to its steady state. In this model, production therefore takes place with a constant number of hours worked, so output and consumption move one for one with productivity. Productivity also pins down the real interest rate. Hence, money is **neutral** in the short run as well. This neutrality result also holds in the general case in which we do not impose $\hat{K}_t = 0$.

20.5 Price Level and Inflation Determination

So far, we have shown that money affects real variables in neither the long run (the steady state) nor the short run (the log-linear approximation). We now examine how money affects nominal variables.

We maintain the assumptions from the end of the previous subsection.

- Assumption** (Assumptions for Price Level and Inflation Determination)
1. Capital is fixed, $\hat{K}_t = 0$ for all t ;
 2. Technology is linear in labour $\alpha = 0$;
 3. There is zero government spending $G_t = 0$.

Then, recall that under this setting, we had that the real variables were given by:

$$\begin{aligned}\hat{N}_t &= 0 \\ \hat{Y}_t &= \hat{C}_t = \hat{w}_t = \hat{A}_t \\ \hat{r}_t &= \mathbb{E}_t[\hat{A}_{t+1}] - \hat{A}_t\end{aligned}$$

We now want to find out what the solution is for **nominal variables** $\hat{P}_t, \hat{\pi}_t, \hat{l}_t$. To do so, we shall analyse what happens when the government specifies monetary policy in terms of the quantity of money or the nominal interest rate.

20.5.1 Monetary Policy in Terms of Money Supply

We assume that the central bank controls the amount of money in circulation, M_t . We previously saw that if the steady-state money stock M is constant, then the steady-state price level is determined. Hence, the log deviation $\hat{P}_t$ is well defined, and we can decompose $\widehat{M_t/P_t}$ because $\hat{M}_t$ is known.

Now, recall the log-linearised money demand function:

$$\widehat{\frac{M_t}{P_t}} = -\left(\frac{\beta}{1-\beta}\right)\hat{l}_t + \hat{C}_t$$

We shall define:

$$\eta \equiv \frac{\beta}{1-\beta}$$

to be the **elasticity of money demand to nominal interest rates**. This gives us:

$$\widehat{\frac{M_t}{P_t}} = -\eta\hat{l}_t + \hat{C}_t$$

Because $\hat{P}_t$ is now well defined and hats denote log deviations, we can use $\widehat{X_t/Y_t} = \hat{X}_t - \hat{Y}_t$ to decompose the left-hand side:

$$\hat{M}_t - \hat{P}_t = -\eta\hat{l}_t + \hat{C}_t$$

Now, recall the **Fisher equation**:

$$\hat{r}_t = \hat{l}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]$$

and we can break up the inflation term since the price level is determined:

$$\hat{l}_t = \hat{r}_t + \mathbb{E}_t[\hat{P}_{t+1}] - \hat{P}_t$$

We now combine the Fisher equation with the money demand function to arrive at:

$$\hat{M}_t - \hat{P}_t = -\eta(\hat{r}_t + \mathbb{E}_t[\hat{P}_{t+1}] - \hat{P}_t) + \hat{C}_t.$$

We then write this as a forward looking difference equation:

$$\hat{P}_t = \frac{1}{1+\eta} \left(\hat{M}_t + \eta\hat{r}_t - \hat{C}_t \right) + \frac{\eta}{1+\eta} \mathbb{E}_t[\hat{P}_{t+1}]$$

Before we proceed, we need another assumption.

Assumption (Price Level Does not Explode)

We assume that the future aggregate price level does not explode:

$$\lim_{T \rightarrow \infty} \left(\frac{\eta}{1 + \eta} \right)^T \mathbb{E}_t[\hat{P}_{t+T}] = 0.$$

Under this assumption, repeated substitution and taking limits give the price-level solution.

Theorem 454 (Price Level Determination)

When the government specifies the quantity of money, the price level is given by:

$$\hat{P}_t = \frac{1}{1 + \eta} \mathbb{E}_t \left[\sum_{j=0}^{\infty} \left(\frac{\eta}{1 + \eta} \right)^j (\hat{M}_{t+j} + \eta \hat{r}_{t+j} - \hat{C}_{t+j}) \right]. \quad (526)$$

$\hat{P}_t$ is a forward-looking equation composed of exogenous terms. Therefore, the price level is uniquely determined by the expected path of the money supply $\hat{M}_t$.

Remark 455. Recall that $\hat{r}_t$ and $\hat{C}_t$ were both pinned down by productivity $\hat{A}_t$. Furthermore, the money supply $\hat{M}_t$ is also exogenously pinned down by the government. Therefore, the price level is determined uniquely by the expected path of the money supply $\hat{M}_t$.

We therefore see that monetary policy affects nominal variables (the price level in this case).

20.5.2 Monetary Policy in Terms of Nominal Interest Rates

We now look at the scenario whereby the central bank instead specifies monetary policy via controlling the nominal interest rate i_t . Here, we have already seen that when the central bank chooses interest rates i_t , the steady state price level P is undetermined. This implies that the log-deviation of price level $\hat{P}_t$ is not well-defined either. As a result, we need to characterize the equilibrium in terms of inflation.

We look at two different cases of how the central bank may specify nominal interest rates.

Assumption (Nominal Interest Rate Policy 1)

Suppose that the central bank sets a constant interest rate, so that $\hat{i}_t = 0$ for all t .

Now, if we recall the Fisher equation:

$$\hat{r}_t = \hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]$$

becomes:

$$\mathbb{E}_t[\hat{\pi}_{t+1}] = -\hat{r}_t. \quad (527)$$

We see here that expected inflation is pinned down as we have already argued that the real interest rate $\hat{r}_t$ is pinned down by technology shocks $\hat{A}_t$. However, the issue is that **actual inflation is indeterminate**. That is, we in fact have an **infinite** number of solutions that satisfy the general equilibrium condition Eq. (527). To see this, suppose we had the equation:

$$\hat{\pi}_t + \epsilon_t + \hat{r}_{t-1} = 0$$

whereby $\mathbb{E}_t[\epsilon_{t+1}] = 0$. Then, if we iterate one step forward:

$$\hat{\pi}_{t+1} + \epsilon_{t+1} + \hat{r}_t = 0$$

and take expectations:

$$\mathbb{E}_t[\hat{\pi}_{t+1} + \epsilon_{t+1} + \hat{r}_t] = 0$$

$$\mathbb{E}_t[\hat{\pi}_{t+1}] + \hat{r}_t = 0$$

$$\Rightarrow \mathbb{E}_t[\hat{\pi}_{t+1}] = -\hat{r}_t$$

This satisfies Eq. (527). Hence, infinitely many processes ϵ_t satisfy $\mathbb{E}_t[\epsilon_{t+1}] = 0$, so actual inflation is indeterminate. Here, ϵ_t is called a **sunspot**.

Proposition 456 (Monetary Policy Under a Constant Interest Rate)

Suppose that the central bank sets a constant interest rate, so that $\hat{i}_t = 0$ for all t . Then expected inflation is pinned down:

$$\mathbb{E}_t[\hat{\pi}_{t+1}] = -\hat{r}_t \quad (528)$$

by the exogenous process $-\hat{r}_t$.

However, **actual** inflation is indeterminate.

Remark 457. Monetary policy that pegs interest rates generates indeterminacy. The reason for this is that sunspot shocks are self-fulfilling.

We now consider a second policy in which the central bank sets the interest rate in response to inflation.

Assumption (Nominal Interest Rate Policy 2)

Suppose that the central bank sets an interest rate that is responsive to inflation:

$$\hat{i}_t = \psi_\pi \hat{\pi}_t. \quad (529)$$

Now, there are two possible cases depending on how aggressively the central bank responds to inflation via ψ_π .

Theorem 458 (Inflation Level Determination)

Suppose that the central bank sets an interest rate that is responsive to inflation:

$$\hat{i}_t = \psi_\pi \hat{\pi}_t. \quad (530)$$

When the government adopts a feedback rule for the nominal interest rate $\hat{i}_t = \psi_\pi \hat{\pi}_t$, where $\psi_\pi > 1$, inflation is given by:

$$\hat{\pi}_t = \frac{1}{\psi_\pi} \mathbb{E}_t \left[\sum_{j=0}^{\infty} \left(\frac{1}{\psi_\pi} \right)^j \hat{r}_{t+j} \right]. \quad (531)$$

The inflation level is **determinate**.

Proof. Recall the Fisher equation:

$$\hat{r}_t = \hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]$$

and then plug in the inflation rule:

$$\hat{r}_t = \psi_\pi \hat{\pi}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]$$

Assuming $\psi_\pi > 1$, one can then solve forward for the inflation level $\hat{\pi}_t$. □

This restriction on the strength of the response has a known name in the literature.

Definition 459 (Taylor Principle)

The Taylor principle requires the central bank to respond forcefully to inflation:

$$\psi_\pi > 1. \quad (532)$$

Theorem 460 (Inflation Level Indetermination)

Suppose that the central bank sets an interest rate that is responsive to inflation:

$$\hat{i}_t = \psi_\pi \hat{\pi}_t. \quad (533)$$

When the government adopts a feedback rule for the nominal interest rate $\hat{i}_t = \psi_\pi \hat{\pi}_t$, where $\psi_\pi \leq 1$, inflation is **indeterminate** and can follow:

$$\hat{\pi}_t = \psi_\pi \hat{\pi}_{t-1} - \hat{r}_{t-1} + \epsilon_t. \quad (534)$$

Proof. Recall the Fisher equation:

$$\hat{r}_t = \hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]$$

and then plug in the inflation rule:

$$\hat{r}_t = \psi_\pi \hat{\pi}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]$$

As $\psi_\pi \leq 1$, we need to solve **backwards** to get the level of inflation. □

Remark 461. *Here, we have that there are possible sunspot shocks for the actual level of inflation when the response to inflation is not strong enough.*

21 Empirical Evidence on Real Effects of Monetary Policy

We have that when we introduced money in the baseline RBC model, we had a classical dichotomy.

1. Monetary policy does not affect real variables in either the short or long-run.
2. Monetary policy only determines the price level (or inflation), and affects other nominal variables.

This implies that any attempt to affect output or unemployment through monetary policy is futile, so central banks should focus on price stability.

21.1 Monetary Policy Effect in the Long Run

McCandless and Weber (1995) find a strong correlation between money growth and inflation (0.95), indicating that long-run money growth is strongly correlated with a nominal variable. This is what our baseline RBC model with money predicts. The paper also finds that the long-run correlation between money growth and output is essentially zero, suggesting that money growth has no long-run effect on real variables. This provides evidence for **long-run monetary neutrality**. One criticism is that correlation does not establish causation because money growth is not exogenous. Furthermore, a subsample of OECD countries exhibits a positive correlation between output growth and money growth.

Mishkin (1992) also provides evidence on this issue. Recall that in our baseline RBC model with money, the long-run Fisher equation was:

$$i = r + \pi.$$

We also saw that the real interest rate r was pinned down by real factors such as productivity. Therefore, we should see that the correlation between the nominal interest rate and inflation should be quite high.

As a result, we shall take the view that **monetary policy is neutral and does not affect real variables in the long-run**.

21.2 Monetary Policy Effect in the Short Run

We now examine the case of the short-run effects of monetary policy. We shall use vector autoregressive models to help us establish this.

21.2.1 Christiano, Eichenbaum, and Evans (1999)

This paper asks *what happens after a monetary-policy shock*. It is important to establish whether monetary policy has real effects. This method provides a strong case that monetary policy causally affects output.

We now define a monetary policy shock.

Definition 462 (Monetary Policy Shock)

A monetary policy shock is a policy innovation that is not explained by the central bank's systematic response to the state of the economy.

Observed monetary policy usually reflects a systematic response to the state of the economy. This creates an endogeneity problem, so an exogenous policy innovation is needed to identify causal effects.

The paper uses a quarterly VAR with four lags ($p = 4$). All variables are in logs except the federal funds rate. The variables are:

1. Real GDP Y_t
2. GDP price deflator P_t
3. Change in commodity price index $PCOM_t$
4. Federal Funds Rate FF_t
5. Total Reserves TR_t
6. Non-borrowed reserves plus credit NBR_t
7. $M1/M2$ denoted by M_t .

All these variables are required in order to account for all the possible effects of monetary policy. Here, **a monetary policy shock is a shock to the federal funds rate FF_t** .

To proceed, the paper must specify the timing of monetary policy: which variables the Federal Reserve observes when choosing the federal funds rate, and which variables respond contemporaneously to its decision.

Assumption (Timing assumption)

The central bank observes Y_t , P_t , and $PCOM_t$ when choosing the federal funds rate FF_t . The remaining variables (TR_t , NBR_t , M_t) are then determined contemporaneously in money markets.

Mapping this into time series, we see that FF_t affects Y_t , P_t , $PCOM_t$ with a lag.

Definition 463 (VAR(4) Model)

The structural VAR(4) model used in the paper is:

$$\mathbf{X}_t = \mathbf{A}_0 + \mathbf{A}_1\mathbf{X}_{t-1} + \mathbf{A}_2\mathbf{X}_{t-2} + \mathbf{A}_3\mathbf{X}_{t-3} + \mathbf{A}_4\mathbf{X}_{t-4} + \mathbf{B}\epsilon_t \quad (535)$$

where $\mathbf{A}_0$ is a vector of constants and the ordering of the variable vector is:

$$\mathbf{X}_t = \begin{bmatrix} Y_t \\ P_t \\ PCOM_t \\ FF_t \\ TR_t \\ NBR_t \\ M_t \end{bmatrix}. \quad (536)$$

We now need to impose assumptions on the $\mathbf{B}$ matrix for identification.

Assumption (Block-Recursive Impact Matrix)

Partition the variables into the slow-moving block $\mathbf{X}_{1t} = (Y_t, P_t, PCOM_t)'$, the policy instrument FF_t , and the money-market block $\mathbf{X}_{2t} = (TR_t, NBR_t, M_t)'$. The contemporaneous impact matrix has the form

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{0}_{3 \times 1} & \mathbf{0}_{3 \times 3} \\ b'_{21} & b_{22} & \mathbf{0}_{1 \times 3} \\ \mathbf{B}_{31} & \mathbf{b}_{32} & \mathbf{B}_{33} \end{bmatrix} \quad (537)$$

where the diagonal blocks are respectively 3×3 , scalar, and 3×3 . Thus, the slow-moving variables do not respond contemporaneously to either the policy shock or money-market shocks; the policy instrument may respond contemporaneously to the slow-moving block; and money-market variables may respond contemporaneously to all three blocks. Additional restrictions within $\mathbf{B}_{11}$ and $\mathbf{B}_{33}$ are unnecessary if the object of interest is only the monetary-policy shock.

We can then use an impulse-response function to study the effects of a federal-funds-rate shock. Suppose that the interest rate rises because of a contractionary monetary-policy shock.

1. Output falls and remains below baseline for a while.
2. The price level initially changes little and eventually falls.
3. Commodity prices fall.
4. After its initial increase, the federal funds rate gradually returns towards its baseline.
5. Non-borrowed reserves and M1 fall immediately.

Monetary-policy tightening leads to an output contraction. Monetary policy therefore has real effects. We also see that **prices fall with a lag**, which is evidence of **nominal rigidities**. We will carry these results forward and argue that **nominal rigidities drive the real effects of monetary policy**. Furthermore, **long-run impulse responses are consistent with money neutrality**: output returns to trend, while the price level changes permanently.

To conclude, monetary policy has short-run real effects. However, monetary-policy shocks do not explain much of output and price volatility, suggesting that central banks rarely make random decisions and that systematic monetary policy has an important role to play.

22 Monopolistic Competition and Nominal Rigidities

22.1 Introduction

We consider two types of firms:

1. **Retailers:** They supply a single final good to households.
2. **Wholesale Producers:** Each supplies one differentiated intermediate good used by retailers to produce the final good.

We also specify the market structure for both firms:

1. Retailers: They operate in perfect competition and take prices of intermediate goods as given.
2. Wholesale Producers: They operate in monopolistic competition and hold market power.

Market power arises because the differentiated intermediate goods are imperfect substitutes. We now examine the first-order conditions characterising retailers' decisions and then those characterising wholesale producers' decisions.

22.2 Imperfect Competition

22.2.1 Retailers

Definition 464 (Dixit–Stiglitz Aggregator)

We assume that there is a continuum of measure one of differentiated goods $Y_t(i)$ aggregated according to:

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}} \quad (538)$$

where $\theta > 1$ is the elasticity of substitution among differentiated goods.

Remark 465. Y_t here represents a basket of goods.

The retailers take prices of differentiated goods $P_t(i)$ as given. We can then define the retailer's minimisation problem.

Definition 466 (Retailer's Cost Minimisation Problem)

The cost minimisation problem facing the retailer is given by:

$$\begin{aligned} \min_{Y_t(i)} \quad & \int_0^1 P_t(i) Y_t(i) di \\ \text{s.t.} \quad & Y_t = \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}} \end{aligned} \quad (539)$$

From this problem, we can derive the retailer's first-order condition.

Theorem 467 (Retailer's First-Order Condition)

The retailer's first-order condition is:

$$P_t(i) = \lambda_t Y_t^{\frac{1}{\theta}} \cdot Y_t(i)^{-\frac{1}{\theta}}. \quad (540)$$

Proof. The **Lagrangian** is given by:

$$\mathcal{L}_t = \int_0^1 P_t(i) Y_t(i) di - \lambda_t \left\{ \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}} - Y_t \right\} \quad (541)$$

We now compute the first-order condition for a generic quantity $Y_t(i)$:

$$\frac{\partial \mathcal{L}_t}{\partial Y_t(i)} = \underbrace{\frac{\partial}{\partial Y_t(i)} \left[\int_0^1 P_t(i) Y_t(i) di \right]}_{(*)} - \underbrace{\frac{\partial}{\partial Y_t(i)} \left[\lambda_t \left\{ \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}} - Y_t \right\} \right]}_{(**)}$$

We now analyse the two terms respectively.

$$\begin{aligned} (*) &= \frac{\partial}{\partial Y_t(i)} \left[\int_0^1 P_t(i) Y_t(i) di \right] \\ &= P_t(i). \end{aligned}$$

The remark following this proof gives intuition for differentiating the integral.

For the other term:

$$\begin{aligned} (**) &= \frac{\partial}{\partial Y_t(i)} \left[\lambda_t \left\{ \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}} - Y_t \right\} \right] \\ &= \lambda_t \frac{\theta}{\theta-1} \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}-1} \cdot \frac{\partial}{\partial Y_t(i)} \int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \\ &=_{(2)} \lambda_t \left\{ \frac{\theta}{\theta-1} \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}-1} \cdot \frac{\theta-1}{\theta} Y_t(i)^{\frac{\theta-1}{\theta}-1} \right\} \end{aligned}$$

where we use the chain rule and differentiate with respect to the quantity of variety i .

$$\begin{aligned} \frac{\partial \mathcal{L}_t}{\partial Y_t(i)} &= \frac{\partial}{\partial Y_t(i)} \left[\int_0^1 P_t(i) Y_t(i) di \right] - \frac{\partial}{\partial Y_t(i)} \left[\lambda_t \left\{ \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}} - Y_t \right\} \right] \\ &= P_t(i) - \lambda_t \left\{ \frac{\theta}{\theta-1} \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}-1} \cdot \frac{\theta-1}{\theta} Y_t(i)^{\frac{\theta-1}{\theta}-1} \right\} \\ &= P_t(i) - \lambda_t \left\{ \frac{\theta}{\theta-1} \frac{\theta-1}{\theta} \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}-1} \cdot Y_t(i)^{\frac{\theta-1}{\theta}-1} \right\} \\ &= P_t(i) - \lambda_t \left\{ \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}-1} \cdot Y_t(i)^{\frac{\theta-1}{\theta}-1} \right\} \\ &=_{(1)} P_t(i) - \lambda_t \left\{ Y_t^{\frac{1}{\theta}} \cdot Y_t(i)^{\frac{\theta-1}{\theta}-1} \right\} \\ &= P_t(i) - \lambda_t Y_t^{\frac{1}{\theta}} \cdot Y_t(i)^{-\frac{1}{\theta}} = 0 \end{aligned}$$

whereby $=_{(1)}$ uses the fact that:

$$Y_t^{\frac{1}{\theta}} = \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}-1}$$

from Eq. (538). □

Remark 468. In the previous proof, the step where we differentiated:

$$\frac{d}{dY_t(i)} \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]$$

may seem strange. But for intuition, replace this with an infinite sum and differentiate it with respect to Y_t :

$$\frac{d}{dY_t} \left[\sum_{j=0}^{\infty} Y_{t+j}^{\frac{\theta-1}{\theta}} \right] = \frac{d}{dY_t} \left[Y_t^{\frac{\theta-1}{\theta}} + Y_{t+1}^{\frac{\theta-1}{\theta}} + Y_{t+2}^{\frac{\theta-1}{\theta}} + \dots \right] = \frac{\theta-1}{\theta} Y_t^{-\frac{1}{\theta}}$$

This derivative applies to every variety. Replacing the sum with an integral gives:

$$\frac{d}{dY_t(i)} \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right] = \frac{\theta-1}{\theta} Y_t(i)^{-\frac{1}{\theta}}$$

which holds for all $i \in [0, 1]$ and hence we have a continuum of first order conditions.

Returning to the retailer's cost-minimisation problem, we can derive the value of the Lagrange multiplier.

Theorem 469 (Retailer's Lagrange Multiplier is Aggregate Price Level)

The Lagrange multiplier for the retailer is the aggregate price level P_t :

$$\lambda_t = P_t. \quad (542)$$

Proof. Because retailers are competitive, they earn zero profit, so revenue from the final good equals total expenditure on intermediate goods:

$$\underbrace{P_t Y_t}_{\text{Total Revenue}} - \underbrace{\int_0^1 P_t(i) Y_t(i) di}_{\text{Total Cost}} = 0$$

Now, plug in the retailer's FOC into the total cost:

$$\begin{aligned} P_t Y_t &= \int_0^1 P_t(i) Y_t(i) di \\ \Leftrightarrow P_t Y_t &= \int_0^1 [\lambda_t Y_t^{\frac{1}{\theta}} \cdot Y_t(i)^{-\frac{1}{\theta}}] Y_t(i) di \\ \Leftrightarrow P_t Y_t &= \lambda_t Y_t^{\frac{1}{\theta}} \int_0^1 Y_t(i)^{-\frac{1}{\theta}} Y_t(i) di \\ \Leftrightarrow P_t Y_t &= \lambda_t Y_t^{\frac{1}{\theta}} \int_0^1 Y_t(i)^{1-\frac{1}{\theta}} di \\ \Leftrightarrow P_t Y_t &= \lambda_t Y_t^{\frac{1}{\theta}} \int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \\ \Leftrightarrow P_t Y_t &=_{(1)} \lambda_t Y_t^{\frac{1}{\theta}} Y_t^{\frac{\theta-1}{\theta}} \\ \Leftrightarrow P_t Y_t &= \lambda_t Y_t^{\frac{1+\theta-1}{\theta}} \\ \Leftrightarrow P_t Y_t &= \lambda_t Y_t^{\frac{\theta}{\theta}} \\ \Leftrightarrow P_t Y_t &= \lambda_t Y_t \\ \Rightarrow P_t &= \lambda_t \end{aligned}$$

whereby $=_{(1)}$ uses Eq. (538). □

Remark 470. If we increase the aggregate output Y_t the firm wants to produce in the constraint, the cost will increase by the aggregate price level, P_t , which makes sense as this is the aggregate cost over all intermediate goods $Y_t(i)$.

We are now able to derive the demand for intermediate goods by the retailer.

Theorem 471 (Demand for Intermediate Goods)

The demand for intermediate goods by the retailer is given by:

$$Y_t(i) = \left[\frac{P_t(i)}{P_t} \right]^{-\theta} Y_t. \quad (543)$$

Proof. Take the retailer's FOC Eq. (540):

$$P_t(i) = \lambda_t Y_t^{\frac{1}{\theta}} \cdot Y_t(i)^{-\frac{1}{\theta}}$$

and then plug in the Lagrange multiplier being the aggregate price index Eq. (542):

$$P_t(i) = P_t Y_t^{\frac{1}{\theta}} \cdot Y_t(i)^{-\frac{1}{\theta}}$$

Now rearrange to obtain demand for intermediate good i , $Y_t(i)$:

$$\begin{aligned} P_t(i) &= P_t Y_t^{\frac{1}{\theta}} \cdot Y_t(i)^{-\frac{1}{\theta}} \\ \Leftrightarrow \frac{P_t(i)}{P_t} &= Y_t^{\frac{1}{\theta}} \cdot Y_t(i)^{-\frac{1}{\theta}} \\ \Leftrightarrow \left[\frac{P_t(i)}{P_t} \right]^{\theta} &= Y_t \cdot Y_t(i)^{-1} \\ \Rightarrow \left[\frac{P_t(i)}{P_t} \right]^{-\theta} Y_t &= Y_t(i) \end{aligned}$$

□

This demand has a nice interpretation.

1. The demand for individual good i depends positively on the total demand for output Y_t ;
2. The demand for individual good i depends negatively on the price of good i $P_t(i)$ as $\theta > 1$.

We are now interested in deriving the aggregate price level P_t as a function of the individual prices $P_t(i)$.

Theorem 472 (Aggregate Price Index)

The aggregate price index is a nonlinear combination of individual prices:

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta} di \right]^{\frac{1}{1-\theta}} \quad (544)$$

Proof. First, recall the Dixit–Stiglitz aggregator Eq. (538):

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}$$

Now plug in our demand for intermediate goods Eq. (543) into this expression:

$$\begin{aligned}
Y_t &= \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}} \\
\leftrightarrow Y_t &= \left\{ \int_0^1 \left(\left[\frac{P_t(i)}{P_t} \right]^{-\theta} Y_t \right)^{\frac{\theta-1}{\theta}} di \right\}^{\frac{\theta}{\theta-1}} \\
\leftrightarrow Y_t &= \left[\int_0^1 \left[\frac{P_t(i)}{P_t} \right]^{1-\theta} Y_t^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}} \\
\leftrightarrow Y_t &=_{(1)} \left[\int_0^1 \left[\frac{P_t(i)}{P_t} \right]^{1-\theta} di \right]^{\frac{\theta}{\theta-1}} Y_t \\
\leftrightarrow 1 &= \left[\int_0^1 \left[\frac{P_t(i)}{P_t} \right]^{1-\theta} di \right]^{\frac{\theta}{\theta-1}} \\
\leftrightarrow 1 &= \left[P_t^{-(1-\theta)} \int_0^1 [P_t(i)]^{1-\theta} di \right]^{\frac{\theta}{\theta-1}} \\
\leftrightarrow 1 &= \left[P_t^{\theta-1} \int_0^1 [P_t(i)]^{1-\theta} di \right]^{\frac{\theta}{\theta-1}} \\
\leftrightarrow 1 &= P_t^{\theta} \left[\int_0^1 P_t(i)^{1-\theta} di \right]^{\frac{\theta}{\theta-1}} \\
\Rightarrow P_t &= \left[\int_0^1 P_t(i)^{1-\theta} di \right]^{\frac{1}{1-\theta}}.
\end{aligned}$$

where $=_{(1)}$ uses the fact that the Dixit–Stiglitz aggregator is homogeneous of degree one. □

22.2.2 Wholesale Producers

We now turn to wholesale producers, which are the firms that produce the differentiated goods.

Assumption (Assumption on Production Function)

There is no capital K_t used in production and every wholesale producer gets access to the same technology A_t .

We can now state the production function for wholesale producers.

Definition 473 (Production Function for Wholesale Producers)

The production function for wholesale producers is given by:

$$Y_t(i) = A_t N_t(i) \tag{545}$$

where A_t is a stationary productivity process.

Because wholesale producers are monopolistic competitors, each chooses both its **price** $P_t(i)$ and the amount of labour $N_t(i)$ to hire at wage W_t to produce output $Y_t(i)$. However, each firm is sufficiently small that it does not internalise the effect of its price on the aggregate price index. Its revenue less total cost is therefore:

$$P_t(i)Y_t(i) - W_t N_t(i)$$

However, we want to express this in real terms so we shall divide by the aggregate price level P_t .

Definition 474 (Wholesale Producers' Profit Maximisation Problem)

The profit maximisation problem facing the wholesale producers in real terms is given by:

$$\begin{aligned} \max_{P_t(i), N_t(i)} \quad & (1 - \tau_t) \frac{P_t(i)}{P_t} Y_t(i) - \frac{W_t}{P_t} N_t(i) \\ \text{s.t.} \quad & Y_t(i) = A_t N_t(i) \\ & Y_t(i) = \left[\frac{P_t(i)}{P_t} \right]^{-\theta} Y_t \end{aligned} \quad (546)$$

where $\tau_t < 1$ is an **exogenous sales tax**; a negative value represents a subsidy.

Remark 475. The second constraint uses the demand curve faced by a monopolistically competitive firm. The price $P_t(i)$ affects demand for its good, $Y_t(i)$. We also incorporate a sales tax or subsidy because, later, we will use it to offset the monopolistic distortion.

We now derive the optimal relative price for wholesale producers.

Theorem 476 (Optimal Relative Price for Wholesale Producers)

The optimal relative price for the wholesale producer is to set relative prices:

$$\frac{P_t(i)}{P_t} = \frac{1 + \mu}{1 - \tau_t} MC_t \quad (547)$$

whereby $\mu \equiv 1/(\theta - 1)$ with θ being the elasticity of substitution of products and $MC_t \equiv W_t/(A_t P_t)$ is the real marginal cost.

Proof. Let us recall the firm profit maximisation problem:

$$\begin{aligned} \max_{P_t(i), N_t(i)} \quad & (1 - \tau_t) \frac{P_t(i)}{P_t} Y_t(i) - \frac{W_t}{P_t} N_t(i) \\ \text{s.t.} \quad & Y_t(i) = A_t N_t(i) \\ & Y_t(i) = \left[\frac{P_t(i)}{P_t} \right]^{-\theta} Y_t \end{aligned}$$

Looking at the production function constraint, we can rewrite this:

$$Y_t(i) = A_t N_t(i) \quad \leftrightarrow \quad N_t(i) = \frac{Y_t(i)}{A_t}$$

We then plug this into the objective function of the firm profit maximisation problem:

$$\begin{aligned} \max_{P_t(i), N_t(i)} \left[(1 - \tau_t) \frac{P_t(i)}{P_t} Y_t(i) - \frac{W_t}{P_t} N_t(i) \right] & \leftrightarrow \max_{P_t(i)} \left[(1 - \tau_t) \frac{P_t(i)}{P_t} Y_t(i) - \frac{W_t Y_t(i)}{A_t P_t} \right] \\ \Rightarrow \max_{P_t(i)} \left\{ \left[(1 - \tau_t) \frac{P_t(i)}{P_t} - \frac{W_t}{A_t P_t} \right] Y_t(i) \right\} & \text{s.t. } Y_t(i) = \left[\frac{P_t(i)}{P_t} \right]^{-\theta} Y_t \end{aligned}$$

Plug the constraint into the objective function as this is a static problem:

$$\max_{P_t(i)} \left\{ \left[(1 - \tau_t) \frac{P_t(i)}{P_t} - \frac{W_t}{A_t P_t} \right] \left[\frac{P_t(i)}{P_t} \right]^{-\theta} Y_t \right\} = \max_{P_t(i)} \left\{ (1 - \tau_t) \left(\frac{P_t(i)}{P_t} \right)^{1-\theta} Y_t - \frac{W_t}{A_t P_t} \left[\frac{P_t(i)}{P_t} \right]^{-\theta} Y_t \right\}$$

We now differentiate this with respect to $P_t(i)$ and impose the first-order condition:

$$\underbrace{(1 - \theta)(1 - \tau_t) \frac{P_t(i)^{-\theta}}{P_t^{1-\theta}} Y_t}_{MR} + \underbrace{\theta MC_t \frac{P_t(i)^{-\theta-1}}{P_t^{-\theta}} Y_t}_{MC} = 0$$

The first term is the effect on marginal revenue from changing the price whilst the second term is the effect on marginal costs from changing the price. We can now simplify further:

$$\begin{aligned} (1 - \theta)(1 - \tau_t) \frac{P_t(i)^{-\theta}}{P_t^{1-\theta}} Y_t + \theta MC_t \frac{P_t(i)^{-\theta-1}}{P_t^{-\theta}} Y_t &= 0 \\ \Leftrightarrow -(1 - \theta)(1 - \tau_t) \frac{P_t(i)^{-\theta}}{P_t^{1-\theta}} &= \theta MC_t \frac{P_t(i)^{-\theta-1}}{P_t^{-\theta}} \\ \Leftrightarrow \frac{P_t(i)^{-\theta}}{P_t^{1-\theta}} &= -\frac{\theta}{1 - \theta} \frac{1}{1 - \tau_t} MC_t \frac{P_t(i)^{-\theta-1}}{P_t^{-\theta}} \\ \Leftrightarrow \frac{P_t(i)^0}{P_t^1} &= -\frac{\theta}{1 - \theta} \frac{1}{1 - \tau_t} MC_t \frac{P_t(i)^{-1}}{P_t^0} \\ \Leftrightarrow \frac{P_t(i)}{P_t} &= -\frac{\theta}{1 - \theta} \frac{1}{1 - \tau_t} MC_t \\ \Rightarrow \frac{P_t(i)}{P_t} &= \frac{1 + \mu}{1 - \tau_t} MC_t \end{aligned}$$

whereby $\mu \equiv 1/(\theta - 1)$. □

Remark 477. The expression $MC_t \equiv W_t/(A_t P_t)$ is the real wage in units of productivity. This marginal cost is not firm-specific: it depends on the aggregate wage W_t , aggregate price level P_t , and technology A_t , rather than on any individual firm's characteristics.

We see that the optimal relative price is therefore a markup over real marginal cost:

$$\frac{P_t(i)}{P_t} = \frac{1 + \mu}{1 - \tau_t} MC_t$$

where the gross markup $1 + \mu = 1 + 1/(\theta - 1)$ is decreasing in the elasticity of substitution θ , while the price is increasing in the tax rate τ_t . As $\theta \rightarrow \infty$, the net markup μ approaches zero and the gross markup approaches one. This makes intuitive sense: as the goods become **perfect substitutes**, retailers avoid wholesalers whose relative prices are too high. The limiting case coincides with **perfect competition**, in which price equals marginal cost. Finally, the exogenous sales tax or subsidy τ_t allows the effective markup to vary over time. Setting $\tau_t = -\mu$ eliminates the distortion:

$$\frac{P_t(i)}{P_t} = \frac{1 + \mu}{1 - \tau_t} MC_t = \frac{1 + \mu}{1 + \mu} MC_t = MC_t.$$

This makes production efficient because price equals marginal cost.

Remark 478. Under flexible prices, wholesale producers set their optimal relative prices based on **current marginal costs**. With nominal rigidities, firms instead consider **future discounted marginal costs**.

22.3 Nominal Rigidities

22.3.1 Optimal Relative Price under Sticky Prices

We now combine imperfect competition with staggered price setting, giving rise to nominal rigidities and real effects of nominal variables. Because prices do not adjust instantaneously, output and employment become demand-determined, allowing monetary policy to affect expenditure. We first impose an important assumption.

Assumption (Calvo Fairy)

In each period, a firm keeps price unchanged with probability $\alpha \in (0, 1)$, independent of previous history.

Firm i sets a price $\tilde{P}_t(i)$ at time t , taking into account the probability that it cannot readjust in the future. In contrast to the flexible-price solution, which depends only on current marginal cost, the firm must account for future periods in which its price may remain fixed. The resulting markup can vary with **aggregate-demand** conditions. We also assume that firms unable to change their prices can still satisfy demand and earn positive profits.

First, recall that under flexible prices, the wholesaler's optimisation problem of maximising their profit was given by:

$$\max_{P_t(i)} \left\{ \left[(1 - \tau_t) \frac{P_t(i)}{P_t} - \frac{W_t}{A_t P_t} \right] Y_t(i) \right\} \quad \text{s.t.} \quad Y_t(i) = \left[\frac{P_t(i)}{P_t} \right]^{-\theta} Y_t$$

We now modify this problem. At time t , profit in period $t+j$, conditional on the price remaining unchanged, is:

$$\mathcal{P}_{t+j|t} = \left[(1 - \tau_{t+j}) \frac{\tilde{P}_t(i)}{P_{t+j}} - MC_{t+j} \right] Y_{t+j|t}(i)$$

where $\tilde{P}_t(i)$ is the price set at time t and left unchanged. It is the only variable indexed at t , while all other variables can evolve through $t+j$. The term $Y_{t+j|t}(i)$ is demand for good i , conditional on its price remaining unchanged:

$$Y_{t+j|t}(i) = \left[\frac{\tilde{P}_t(i)}{P_{t+j}} \right]^{-\theta} Y_{t+j}$$

Aggregate price P_{t+j} and aggregate demand Y_{t+j} can change, but the firm's price $\tilde{P}_t(i)$ remains fixed.

With this in mind, we can now define the problem for the wholesaler under sticky prices.

Definition 479 (Wholesale Producers' Profit Maximisation Problem)

The profit maximisation problem facing the wholesaler under sticky prices is given by:

$$\begin{aligned} \max_{\tilde{P}_t(i)} \quad & \mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \mathcal{P}_{t+j|t} \right] \\ \text{s.t.} \quad & \mathcal{P}_{t+j|t} = \left[(1 - \tau_{t+j}) \frac{\tilde{P}_t(i)}{P_{t+j}} - MC_{t+j} \right] Y_{t+j|t}(i) \\ & Y_{t+j|t}(i) = \left[\frac{\tilde{P}_t(i)}{P_{t+j}} \right]^{-\theta} Y_{t+j} \end{aligned} \tag{548}$$

where $\mathcal{M}_{t,t+j}$ is the stochastic discount factor.

We can now state the solution to this expected profit maximisation problem to derive the price that firms now set.

Proposition 480 (Optimal Relative Price for Wholesaler Under Sticky Prices)

The optimal relative price set by firm i depends only on **aggregate variables** and is given by:

$$\frac{\tilde{P}_t(i)}{P_t} = \frac{(1 + \mu) \mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \Pi_{t+j,t}^{\theta} MC_{t+j} Y_{t+j} \right]}{\mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} (1 - \tau_{t+j}) \Pi_{t+j,t}^{\theta-1} Y_{t+j} \right]} \quad (549)$$

where the numerator is the present discounted value of marginal costs, the denominator is the present discounted value of future revenues, and $\Pi_{t+j,t}$ is **cumulative gross inflation**. For $j \geq 1$:

$$\Pi_{t+j,t} = \frac{P_{t+j}}{P_t} = \frac{P_{t+j}}{P_{t+j-1}} \times \frac{P_{t+j-1}}{P_{t+j-2}} \times \dots \times \frac{P_{t+2}}{P_{t+1}} \times \frac{P_{t+1}}{P_t},$$

with $\Pi_{t,t} \equiv 1$.

Proof. We take the maximised objective function and substitute in our constraints:

$$\begin{aligned} & \max_{\tilde{P}_t(i)} \mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \mathcal{P}_{t+j|t} \right] \\ & \Leftrightarrow \max_{\tilde{P}_t(i)} \mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \left\{ \left[(1 - \tau_{t+j}) \frac{\tilde{P}_t(i)}{P_{t+j}} - MC_{t+j} \right] Y_{t+j|t}(i) \right\} \right] \\ & \Leftrightarrow \max_{\tilde{P}_t(i)} \mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \left[(1 - \tau_{t+j}) \frac{\tilde{P}_t(i)}{P_{t+j}} - MC_{t+j} \right] \left[\frac{\tilde{P}_t(i)}{P_{t+j}} \right]^{-\theta} Y_{t+j} \right] \\ & \Leftrightarrow \max_{\tilde{P}_t(i)} \mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \left\{ (1 - \tau_{t+j}) \left(\frac{\tilde{P}_t(i)}{P_{t+j}} \right)^{1-\theta} Y_{t+j} - MC_{t+j} \left(\frac{\tilde{P}_t(i)}{P_{t+j}} \right)^{-\theta} Y_{t+j} \right\} \right] \end{aligned}$$

We can then differentiate this and get the first order condition:

$$\mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \left\{ \underbrace{(1 - \tau_{t+j})(1 - \theta) \left(\frac{\tilde{P}_t(i)}{P_{t+j}} \right)^{-\theta} \frac{Y_{t+j}}{P_{t+j}}}_{\text{Revenue}} + \underbrace{\theta MC_{t+j} \left(\frac{\tilde{P}_t(i)}{P_{t+j}} \right)^{-1-\theta} \frac{Y_{t+j}}{P_{t+j}}}_{\text{Cost}} \right\} \right] = 0$$

The first-order condition trades off the effect of a price change on revenue against its effect on costs.

Now multiply by $\tilde{P}_t(i)^{\theta}/(\theta - 1)$ to get:

$$\begin{aligned} & \mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \left\{ (1 - \tau_{t+j})(1 - \theta) \left(\frac{\tilde{P}_t(i)}{P_{t+j}} \right)^{-\theta} \frac{\tilde{P}_t(i)^{\theta}}{(\theta - 1)} \frac{Y_{t+j}}{P_{t+j}} + MC_{t+j} \theta \left(\frac{\tilde{P}_t(i)}{P_{t+j}} \right)^{-1-\theta} \frac{\tilde{P}_t(i)^{\theta}}{(\theta - 1)} \frac{Y_{t+j}}{P_{t+j}} \right\} \right] = 0 \\ & \Leftrightarrow \mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \left\{ (1 - \tau_{t+j}) \frac{(1 - \theta)}{(\theta - 1)} \left(\frac{P_{t+j}^{\theta}}{\tilde{P}_t(i)^{\theta}} \right) \tilde{P}_t(i)^{-\theta} \tilde{P}_t(i)^{\theta} Y_{t+j} + MC_{t+j} \frac{\theta}{(\theta - 1)} \left(\frac{P_{t+j}^{1+\theta}}{\tilde{P}_t(i)^{1+\theta}} \right) \tilde{P}_t(i)^{-\theta-1} \tilde{P}_t(i)^{\theta} Y_{t+j} \right\} \right] = 0 \\ & \Leftrightarrow \mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \left\{ (1 - \tau_{t+j}) \frac{(1 - \theta)}{(\theta - 1)} P_{t+j}^{\theta-1} (1) Y_{t+j} + MC_{t+j} \frac{\theta}{(\theta - 1)} P_{t+j}^{\theta} \tilde{P}_t(i)^{-1} Y_{t+j} \right\} \right] = 0 \\ & \Leftrightarrow \mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \left\{ (1 - \tau_{t+j}) P_{t+j}^{\theta-1} Y_{t+j} - \frac{\theta}{(\theta - 1)} P_{t+j}^{\theta} \frac{MC_{t+j} Y_{t+j}}{\tilde{P}_t(i)} \right\} \right] = 0 \end{aligned}$$

Now, multiply by $\tilde{P}_t(i)/P_t$:

$$\mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \left\{ (1 - \tau_{t+j}) \frac{\tilde{P}_t(i)}{P_t} P_{t+j}^{\theta-1} Y_{t+j} - \frac{\theta}{(\theta-1)} P_{t+j}^{\theta} \frac{MC_{t+j} Y_{t+j}}{P_t} \right\} \right] = 0$$

Finally, divide by $P_t^{\theta-1}$ to get:

$$\mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \left\{ (1 - \tau_{t+j}) \frac{\tilde{P}_t(i)}{P_t} \left(\frac{P_{t+j}}{P_t} \right)^{\theta-1} Y_{t+j} - \frac{\theta}{(\theta-1)} \left(\frac{P_{t+j}}{P_t} \right)^{\theta} MC_{t+j} Y_{t+j} \right\} \right] = 0$$

Now, recall that markups are: $\mu \equiv 1/(\theta-1)$ so we have:

$$\mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \left\{ (1 - \tau_{t+j}) \frac{\tilde{P}_t(i)}{P_t} \left(\frac{P_{t+j}}{P_t} \right)^{\theta-1} Y_{t+j} - (1 + \mu) \left(\frac{P_{t+j}}{P_t} \right)^{\theta} MC_{t+j} Y_{t+j} \right\} \right] = 0$$

Now, re-arrange the expression to get:

$$\mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} (1 - \tau_{t+j}) \frac{\tilde{P}_t(i)}{P_t} \left(\frac{P_{t+j}}{P_t} \right)^{\theta-1} Y_{t+j} \right] = (1 + \mu) \mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \left(\frac{P_{t+j}}{P_t} \right)^{\theta} MC_{t+j} Y_{t+j} \right]$$

The left hand side of this equation is the present discounted value of future revenues and the right hand side is a markup on the present discounted value of future costs.

Now, pull out the $\frac{\tilde{P}_t(i)}{P_t}$ and divide through:

$$\frac{\tilde{P}_t(i)}{P_t} \mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} (1 - \tau_{t+j}) \left(\frac{P_{t+j}}{P_t} \right)^{\theta-1} Y_{t+j} \right] = (1 + \mu) \mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \left(\frac{P_{t+j}}{P_t} \right)^{\theta} MC_{t+j} Y_{t+j} \right]$$

and therefore we get:

$$\frac{\tilde{P}_t(i)}{P_t} = \frac{(1 + \mu) \mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \left(\frac{P_{t+j}}{P_t} \right)^{\theta} MC_{t+j} Y_{t+j} \right]}{\mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} (1 - \tau_{t+j}) \left(\frac{P_{t+j}}{P_t} \right)^{\theta-1} Y_{t+j} \right]}$$

Finally, P_{t+j}/P_t is cumulative gross inflation. For $j \geq 1$, define:

$$\Pi_{t+j,t} \equiv \frac{P_{t+j}}{P_t} = \frac{P_{t+j}}{P_{t+j-1}} \times \frac{P_{t+j-1}}{P_{t+j-2}} \times \dots \times \frac{P_{t+2}}{P_{t+1}} \times \frac{P_{t+1}}{P_t}$$

and set $\Pi_{t,t} \equiv 1$ for the $j = 0$ term.

We can plug this in to get our final result:

$$\frac{\tilde{P}_t(i)}{P_t} = \frac{(1 + \mu) \mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \Pi_{t+j,t}^{\theta} MC_{t+j} Y_{t+j} \right]}{\mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} (1 - \tau_{t+j}) \Pi_{t+j,t}^{\theta-1} Y_{t+j} \right]}$$

□

Remark 481. If $\alpha = 0$, prices are fully flexible and the optimal relative price is:

$$\frac{\tilde{P}_t(i)}{P_t} = \frac{1 + \mu}{1 - \tau_t} MC_t.$$

The important interpretation is that the optimal relative price for firm i to set is given only by aggregate variables.

The numerator in Eq. (549) can be interpreted as the markup on the present discounted value of marginal cost after adjusting for inflation. The denominator is the total net revenue for firm i after adjusting for inflation. Also, notice that this optimal relative price only depends on aggregate variables. This gives us the following result.

Proposition 482 (All Readjusting Firms Choose the Same Price)

All firms that readjust choose the same relative price:

$$\frac{\tilde{P}_t(i)}{P_t} = \frac{\tilde{P}_t}{P_t} = \frac{(1 + \mu) \mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \Pi_{t+j,t}^{\theta} MC_{t+j} Y_{t+j} \right]}{\mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} (1 - \tau_{t+j}) \Pi_{t+j,t}^{\theta-1} Y_{t+j} \right]}. \quad (550)$$

Proof. Each firm's optimal relative price depends only on aggregate variables, not on firm-specific characteristics. Hence, all readjusting firms choose the same relative price. $\square$

Proposition 483 (Recursive Form of Future Revenues and Costs)

The optimal relative price in Eq. (550) can be written recursively. The numerator (the present discounted value of future costs) is:

$$X_{1t} \equiv (1 + \mu) MC_t Y_t + \alpha \mathbb{E}_t \left[\mathcal{M}_{t,t+1} \Pi_{t+1,t}^{\theta} X_{1t+1} \right] \quad (551)$$

and the denominator (PDV of future revenues) is:

$$X_{2t} \equiv (1 - \tau_t) Y_t + \alpha \mathbb{E}_t \left[\mathcal{M}_{t,t+1} \Pi_{t+1,t}^{\theta-1} X_{2t+1} \right] \quad (552)$$

Proof. We start off with the numerator. First, use the numerator in Eq. (550):

$$X_{1t} = (1 + \mu) \mathbb{E}_t \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \Pi_{t+j,t}^{\theta} MC_{t+j} Y_{t+j} \right]$$

Take out the first term:

$$X_{1t} = (1 + \mu) MC_t Y_t + (1 + \mu) \mathbb{E}_t \left[\sum_{j=1}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \Pi_{t+j,t}^{\theta} MC_{t+j} Y_{t+j} \right]$$

We now want to see how we can rewrite:

$$(1 + \mu) \mathbb{E}_t \left[\sum_{j=1}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \Pi_{t+j,t}^{\theta} MC_{t+j} Y_{t+j} \right]$$

First, note that:

$$\begin{aligned} X_{1t+1} &= (1 + \mu) \mathbb{E}_{t+1} \left[\sum_{j=0}^{\infty} \alpha^j \mathcal{M}_{t+1,t+j+1} \Pi_{t+j+1,t+1}^{\theta} MC_{t+j+1} Y_{t+j+1} \right] \\ &= (1 + \mu) \mathbb{E}_{t+1} \left[\sum_{j=1}^{\infty} \alpha^{j-1} \mathcal{M}_{t+1,t+j} \Pi_{t+j,t+1}^{\theta} MC_{t+j} Y_{t+j} \right] \end{aligned}$$

where we shifted the index. Multiply by $\alpha \mathcal{M}_{t,t+1} \Pi_{t+1,t}^{\theta}$, take the expectation at time t , and use the law of iterated expectations together with $\mathcal{M}_{t,t+1} \mathcal{M}_{t+1,t+j} = \mathcal{M}_{t,t+j}$ and $\Pi_{t+1,t} \Pi_{t+j,t+1} = \Pi_{t+j,t}$. This gives:

$$\alpha \mathbb{E}_t [\mathcal{M}_{t,t+1} \Pi_{t+1,t}^\theta X_{1t+1}] = (1 + \mu) \mathbb{E}_t \left[\sum_{j=1}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \Pi_{t+j,t}^\theta MC_{t+j} Y_{t+j} \right].$$

Finally, note that $\Pi_{t+1,t} = \Pi_{t+1}$.

We plug this back into:

$$X_{1t} = (1 + \mu) MC_t Y_t + (1 + \mu) \mathbb{E}_t \left[\sum_{j=1}^{\infty} \alpha^j \mathcal{M}_{t,t+j} \Pi_{t+j,t}^\theta MC_{t+j} Y_{t+j} \right]$$

so we get the recursive form:

$$X_{1t} = (1 + \mu) MC_t Y_t + \alpha \mathbb{E}_t [\mathcal{M}_{t,t+1} \Pi_{t+1}^\theta X_{1t+1}].$$

You can follow a similar procedure to derive the recursive form for X_{2t} . □

We now wish to interpret these recursive forms. We look at the numerator which was the present discounted value of costs.

$$X_{1t} \equiv (1 + \mu) MC_t Y_t + \alpha \mathbb{E}_t [\mathcal{M}_{t,t+1} \Pi_{t+1}^\theta X_{1t+1}]$$

The red term represents current total cost plus a markup. The blue term is the expected continuation value of future costs, adjusted for inflation and weighted by the probability α that the firm cannot reset its price.

The denominator was the present discounted value of revenues:

$$X_{2t} \equiv (1 - \tau_t) Y_t + \alpha \mathbb{E}_t [\mathcal{M}_{t,t+1} \Pi_{t+1}^{\theta-1} X_{2t+1}]$$

The red term is current net revenue. The blue term is expected future revenue, adjusted for inflation and conditional on the firm being unable to reset its price.

Theorem 484 (Recursive Form of Optimal Relative Price for Wholesaler)

The optimal relative price for a wholesaler that readjusts its price at time t is:

$$\frac{\tilde{P}_t}{P_t} = \frac{X_{1t}}{X_{2t}}. \quad (553)$$

Proof. First take Eq. (550) and plug in our recursive expressions. □

We have characterised the optimal price for a wholesaler that is allowed to readjust in the current period but may be unable to do so in future periods.

22.3.2 Optimal Relative Price and Aggregate Inflation

We now link the optimal relative price chosen by a readjusting firm to the aggregate price level in order to close the firm's problem.

Proposition 485 (Link between Optimal Relative Price and Aggregate Inflation)

The link between the optimal relative price and aggregate inflation is fully characterised by the three equations:

$$1 = \alpha \Pi_t^{\theta-1} + (1 - \alpha) \left(\frac{X_{1t}}{X_{2t}} \right)^{1-\theta} \quad (554)$$

$$X_{1t} \equiv (1 + \mu) MC_t Y_t + \alpha \mathbb{E}_t \left[\mathcal{M}_{t,t+1} \Pi_{t+1}^\theta X_{1t+1} \right] \quad (555)$$

$$X_{2t} \equiv (1 - \tau_t) Y_t + \alpha \mathbb{E}_t \left[\mathcal{M}_{t,t+1} \Pi_{t+1}^{\theta-1} X_{2t+1} \right] \quad (556)$$

Proof. First, recall the aggregate price index in Eq. (544), which is a nonlinear combination of prices:

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta} di \right]^{\frac{1}{1-\theta}}$$

We can re-arrange this to get:

$$P_t^{1-\theta} = \int_0^1 P_t(i)^{1-\theta} di \quad (557)$$

Now, we can invoke the law of large numbers whereby the probability of readjusting $(1 - \alpha)$ is equal to the fraction of readjusting firms:

$$P_t^{1-\theta} = \int_0^1 P_t(i)^{1-\theta} di = \int_0^\alpha P_{t-1}(i)^{1-\theta} di + \int_\alpha^1 \tilde{P}_t(i)^{1-\theta} di$$

where the first term uses the fact that a fraction α of firms do not adjust their prices, while the second term refers to the fraction $1 - \alpha$ that can adjust. We have also shown that all adjusting firms choose the same reset price, so we can drop the i subscript:

$$\int_0^\alpha P_{t-1}(i)^{1-\theta} di + \int_\alpha^1 \tilde{P}_t(i)^{1-\theta} di = \underbrace{\int_0^\alpha P_{t-1}(i)^{1-\theta} di}_{(*)} + \underbrace{\int_\alpha^1 \tilde{P}_t^{1-\theta} di}_{(**)}$$

We first analyse (*). Since the opportunity to reset is independent of a firm's price history, the law of large numbers gives:

$$(*) = \int_0^\alpha P_{t-1}(i)^{1-\theta} di = \alpha \int_0^1 P_{t-1}(i)^{1-\theta} di =_{(1)} \alpha P_{t-1}^{1-\theta}$$

where $=_{(1)}$ uses Eq. (557). The distribution of prices among firms not adjusting at time t is therefore a fraction α of the distribution of prices at time $t - 1$.

We now analyse (**) which is easier. As $\tilde{P}_t$ does not depend on i , we can pull out of the integral:

$$(**) = \int_\alpha^1 \tilde{P}_t^{1-\theta} di = \tilde{P}_t^{1-\theta} \int_\alpha^1 di = \tilde{P}_t^{1-\theta} (1 - \alpha)$$

We now plug (*) and (**) back into Eq. (557):

$$P_t^{1-\theta} = \alpha P_{t-1}^{1-\theta} + (1 - \alpha) \tilde{P}_t^{1-\theta}$$

This says that today's aggregate price index combines the prices of the fraction α of firms that do not readjust with the common reset price chosen by the fraction $1 - \alpha$ that do readjust.

We then divide through by $P_t^{1-\theta}$:

$$1 = \alpha \Pi_t^{\theta-1} + (1 - \alpha) \left(\frac{\tilde{P}_t}{P_t} \right)^{1-\theta}$$

Finally, we recall the recursive form of the optimal relative price:

$$\frac{\tilde{P}_t}{P_t} = \frac{X_{1t}}{X_{2t}}$$

and plug this in to get:

$$1 = \alpha \Pi_t^{\theta-1} + (1 - \alpha) \left(\frac{X_{1t}}{X_{2t}} \right)^{1-\theta}.$$

□

Hence, we have shown how the optimal relative price relates to aggregate inflation. We now express the rest of the model in terms of aggregate variables.

Proposition 486 (Aggregate Wholesalers' Production)

The aggregate production function for wholesalers is given by:

$$\int_0^1 Y_t(i) di = A_t \int_0^1 N_t(i) di \quad (558)$$

Proof. Recall for each wholesaler, we had the linear production function:

$$Y_t(i) = A_t N_t(i)$$

Now integrate both sides over the continuum of wholesalers:

$$\begin{aligned} \int_0^1 Y_t(i) di &= \int_0^1 A_t N_t(i) di \\ &= A_t \int_0^1 N_t(i) di \end{aligned}$$

□

Before we proceed, we give a name to something important.

Definition 487 (Price Dispersion)

The **price dispersion** is given by:

$$\Delta_t \equiv \int_0^1 \left[\frac{P_t(i)}{P_t} \right]^{-\theta} di. \quad (559)$$

We go back to Eq. (558):

$$\int_0^1 Y_t(i) di = A_t \int_0^1 N_t(i) di$$

and ask whether we can write this production function using only aggregate variables.

Theorem 488 (Aggregate Production Function with Aggregate Variables)

The aggregate production function with aggregate variables is given by:

$$Y_t \Delta_t = A_t N_t \quad (560)$$

Proof. We start from Eq. (558):

$$\int_0^1 Y_t(i) di = A_t \int_0^1 N_t(i) di$$

The remaining task is to express the following two integrals in terms of aggregate variables:

$$\int_0^1 Y_t(i) di \quad \text{and} \quad \int_0^1 N_t(i) di$$

The labour market is homogeneous, so total labour demand by wholesale producers equals the total labour supply of the representative household:

$$\int_0^1 N_t(i) di = N_t$$

Therefore, we can rewrite Eq. (558) as:

$$\int_0^1 Y_t(i) di = A_t N_t$$

The right-hand side is now expressed in aggregate terms. It remains to express the left-hand side:

$$\int_0^1 Y_t(i) di$$

This term does not equal aggregate final output when prices are dispersed. To see this, recall the retailer's demand for the wholesaler's product in Eq. (543):

$$Y_t(i) = \left[\frac{P_t(i)}{P_t} \right]^{-\theta} Y_t.$$

We can plug this in:

$$\int_0^1 Y_t(i) di = \int_0^1 \left[\frac{P_t(i)}{P_t} \right]^{-\theta} Y_t di = Y_t \int_0^1 \left[\frac{P_t(i)}{P_t} \right]^{-\theta} di = \Delta_t Y_t$$

where Δ_t is price dispersion.

We have derived expressions in terms of aggregate variables for both sides of:

$$\int_0^1 Y_t(i) di = A_t \int_0^1 N_t(i) di$$

and hence we have that:

$$Y_t \Delta_t = A_t N_t$$

□

Notice for our aggregate production function:

$$Y_t \Delta_t = A_t N_t$$

if our price dispersion was $\Delta_t = 1$, so that we have no price dispersion, then this aggregate production function

$$Y_t = A_t N_t$$

corresponds to the individual production function:

$$Y_t(i) = A_t N_t(i)$$

Hence, we can see that due to price stickiness, we have this additional wedge which is the price dispersion Δ_t . One way to interpret this is that price dispersion is like a negative productivity shock:

$$Y_t \Delta_t = A_t N_t \quad \Rightarrow \quad Y_t = \frac{A_t N_t}{\Delta_t}$$

so as price dispersion increases, it is as if productivity is falling as output falls. We see that more price dispersion leads to production going down. Hence, we see that this price dispersion term is important and therefore we want to track its evolution over time.

Proposition 489 (Evolution of Price Dispersion)

Price dispersion evolves over time based on the previous price dispersion and aggregate inflation:

$$\Delta_t = \alpha \Pi_t^\theta \Delta_{t-1} + (1 - \alpha) \left(\frac{1 - \alpha \Pi_t^{\theta-1}}{1 - \alpha} \right)^{\frac{\theta}{\theta-1}} \quad (561)$$

Proof. First, split the price-dispersion term between firms that do and do not change their prices:

$$\Delta_t = \int_0^1 \left[\frac{P_t(i)}{P_t} \right]^{-\theta} di = \underbrace{\int_0^\alpha \left[\frac{P_{t-1}(i)}{P_t} \right]^{-\theta} di}_{(*)} + \underbrace{\int_\alpha^1 \left[\frac{\tilde{P}_t(i)}{P_t} \right]^{-\theta} di}_{(**)}$$

(*) represents price dispersion among firms that do not adjust their price, while (**) represents the contribution from firms that do adjust.

We first look at (*) and multiply by $(P_{t-1}/P_{t-1})^{-\theta}$

$$\begin{aligned} (*) &= \int_0^\alpha \left[\frac{P_{t-1}(i)}{P_t} \right]^{-\theta} di \\ &= \int_0^\alpha \left[\frac{P_{t-1}(i)}{P_{t-1}} \right]^{-\theta} \left(\frac{P_{t-1}}{P_t} \right)^{-\theta} di \\ &= \int_0^\alpha \left[\frac{P_{t-1}(i)}{P_{t-1}} \right]^{-\theta} \Pi_t^\theta di \\ &= \Pi_t^\theta \int_0^\alpha \left[\frac{P_{t-1}(i)}{P_{t-1}} \right]^{-\theta} di \\ &=_{(1)} \Pi_t^\theta \alpha \int_0^1 \left[\frac{P_{t-1}(i)}{P_{t-1}} \right]^{-\theta} di \\ &=_{(2)} \Pi_t^\theta \alpha \Delta_{t-1} \end{aligned}$$

where $=_{(1)}$ uses the law of large numbers and $=_{(2)}$ uses the definition of price dispersion.

We now look at (**) where we recall that all firms will choose the same relative price:

$$\begin{aligned}
(**) &= \int_{\alpha}^1 \left[\frac{\tilde{P}_t(i)}{P_t} \right]^{-\theta} di \\
&= \int_{\alpha}^1 \left[\frac{\tilde{P}_t}{P_t} \right]^{-\theta} di \\
&= \left[\frac{\tilde{P}_t}{P_t} \right]^{-\theta} \int_{\alpha}^1 di \\
&= \left[\frac{\tilde{P}_t}{P_t} \right]^{-\theta} (1 - \alpha) \\
&\stackrel{=_{(1)}}{=} \left[\frac{X_{1t}}{X_{2t}} \right]^{-\theta} (1 - \alpha) \\
&\stackrel{=_{(2)}}{=} (1 - \alpha) \left[\frac{1 - \alpha \Pi_t^{\theta-1}}{1 - \alpha} \right]^{\theta/(\theta-1)}
\end{aligned}$$

whereby $=_{(1)}$ uses Eq. (553) and $=_{(2)}$ uses the link between relative prices and aggregate inflation Eq. (554).

Combining (*) and (**) gives the evolution of the price-dispersion index:

$$\Delta_t = \alpha \Pi_t^{\theta} \Delta_{t-1} + (1 - \alpha) \left(\frac{1 - \alpha \Pi_t^{\theta-1}}{1 - \alpha} \right)^{\frac{\theta}{\theta-1}}$$

□

Looking at the evolution of the price dispersion index:

$$\Delta_t = \alpha \Pi_t^{\theta} \Delta_{t-1} + (1 - \alpha) \left(\frac{1 - \alpha \Pi_t^{\theta-1}}{1 - \alpha} \right)^{\frac{\theta}{\theta-1}}$$

we see that the current price dispersion depends on previous period's price dispersion and the aggregate inflation rate.

22.4 General Equilibrium

We now combine the results for firms with the household's RBC problem to obtain a general equilibrium.

Assumption (Simplifying Assumptions)

We assume that there is no capital, government, or money.

Furthermore, we assume that households have CRRA preferences:

$$U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma}.$$

We assume that firms maximise profits on shareholders' behalf, so the stochastic discount factor is:

$$\mathcal{M}_{t,t+j} \equiv \beta^j \left(\frac{C_{t+j}}{C_t} \right)^{-\sigma}$$

Theorem 490 (Household's FOC)

The household's first order conditions are:

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[C_{t+1}^{-\sigma} \frac{1 + i_t}{\Pi_{t+1}} \right] \quad (\text{Bond Market})$$

$$A_t MC_t = N_t^\psi C_t^\sigma \quad (\text{Labour Market})$$

$$Y_t = C_t \quad (\text{Goods Market})$$

What is important to notice is that the labour market equilibrium is different now:

$$A_t MC_t = N_t^\psi C_t^\sigma$$

The monopolistic distortion introduces a wedge, so marginal cost now appears in the labour-market condition.

Theorem 491 (Equilibrium)

Given the exogenous processes A_t and τ_t and a policy instrument i_t , the six endogenous variables $\{Y_t, MC_t, \Pi_t, \Delta_t, X_{1t}, X_{2t}\}$ are determined by the following six equations:

$$Y_t^{-\sigma} = \beta \mathbb{E}_t \left[Y_{t+1}^{-\sigma} \frac{1 + i_t}{\Pi_{t+1}} \right] \quad (562)$$

$$A_t^{1+\psi} MC_t = Y_t^{\sigma+\psi} \Delta_t^\psi \quad (563)$$

$$1 = \alpha \Pi_t^{\theta-1} + (1 - \alpha) \left(\frac{X_{1t}}{X_{2t}} \right)^{1-\theta} \quad (564)$$

$$X_{1t} = (1 + \mu) MC_t Y_t + \alpha \beta \mathbb{E}_t \left[\left(\frac{Y_{t+1}}{Y_t} \right)^{-\sigma} \Pi_{t+1}^\theta X_{1t+1} \right] \quad (565)$$

$$X_{2t} = (1 - \tau_t) Y_t + \alpha \beta \mathbb{E}_t \left[\left(\frac{Y_{t+1}}{Y_t} \right)^{-\sigma} \Pi_{t+1}^{\theta-1} X_{2t+1} \right] \quad (566)$$

$$\Delta_t = \alpha \Pi_t^\theta \Delta_{t-1} + (1 - \alpha) \left(\frac{1 - \alpha \Pi_t^{\theta-1}}{1 - \alpha} \right)^{\frac{\theta}{\theta-1}} \quad (567)$$

22.4.1 Efficient Steady State

We now consider a zero-inflation steady state, $\Pi = 1$. Zero steady-state inflation eliminates price dispersion from staggered price setting. Combined with the subsidy derived below, which offsets the monopolistic markup, this delivers the efficient steady state. We now state a series of propositions that follow from these conditions.

Proposition 492 (No Price Dispersion Under Zero Inflation)

Under zero-inflation steady state $\Pi = 1$, there is no price dispersion:

$$\Delta = 1.$$

Proof. Recall the formula for evolution of price dispersion at steady state:

$$\Delta = \alpha \Pi^\theta \Delta + (1 - \alpha) \left(\frac{1 - \alpha \Pi^{\theta-1}}{1 - \alpha} \right)^{\frac{\theta}{\theta-1}}$$

For $\Pi = 1$ we have:

$$\Delta = \alpha\Delta + (1 - \alpha)$$

which gives us:

$$\Delta = 1.$$

□

Proposition 493 (Aggregate Production)

Aggregate production under zero inflation steady state is:

$$Y = AN. \quad (568)$$

Proof. Since price dispersion $\Delta = 1$,

$$Y\Delta = AN \Rightarrow Y = AN.$$

□

Proposition 494 (Optimal Relative Price is One)

The optimal relative price is:

$$\frac{\tilde{P}_t}{P_t} = \frac{X_{1t}}{X_{2t}} = 1. \quad (569)$$

Proof. Recall that the link between optimal relative price and price dispersion is:

$$\begin{aligned} 1 &= \alpha\Pi^{\theta-1} + (1 - \alpha)\left(\frac{X_1}{X_2}\right)^{1-\theta} \\ \Leftrightarrow 1 &= \alpha + (1 - \alpha)\left(\frac{X_1}{X_2}\right)^{1-\theta} \\ \Leftrightarrow (1 - \alpha) &= (1 - \alpha)\left(\frac{X_1}{X_2}\right)^{1-\theta} \\ \Leftrightarrow 1 &= \left(\frac{X_1}{X_2}\right)^{1-\theta} \\ \Leftrightarrow X_1 &= X_2. \end{aligned}$$

Now, we use the fact that the optimal relative price is just the ratio of these two recursive variables to combine the proof:

$$\frac{\tilde{P}}{P} = \frac{X_1}{X_2} = 1$$

□

Proposition 495 (Subsidy to Eliminate Distortions)

The subsidy to eliminate monopolistic distortions is:

$$\tau = -\mu. \quad (570)$$

Proof. The steady state value of discounted costs are:

$$\begin{aligned}
 X_1 &= (1 + \mu)MC \cdot Y + \alpha\beta \mathbb{E} \left[\left(\frac{Y}{Y} \right)^{-\sigma} \Pi^\theta X_1 \right] \\
 \Leftrightarrow X_1 &= (1 + \mu)MC \cdot Y + \alpha\beta \left(\frac{Y}{Y} \right)^{-\sigma} \Pi^\theta X_1 \\
 \Leftrightarrow X_1 &= (1 + \mu)MC \cdot Y + \alpha\beta X_1 \\
 \Rightarrow X_1 &= \frac{(1 + \mu)MC \cdot Y}{1 - \alpha\beta}
 \end{aligned}$$

The steady state value of discounted revenues are:

$$\begin{aligned}
 X_2 &= (1 - \tau)Y + \alpha\beta \mathbb{E} \left[\left(\frac{Y}{Y} \right)^{-\sigma} \Pi^{\theta-1} X_2 \right] \\
 \Leftrightarrow X_2 &= (1 - \tau)Y + \alpha\beta X_2 \\
 \Rightarrow X_2 &= \frac{(1 - \tau)Y}{1 - \alpha\beta}
 \end{aligned}$$

We also have shown that $X_1 = X_2$:

$$X_1 = \frac{(1 + \mu)MC \cdot Y}{1 - \alpha\beta} = \frac{(1 - \tau)Y}{1 - \alpha\beta} = X_2.$$

We can then solve for this to get:

$$(1 + \mu)MC = 1 - \tau.$$

Hence the markup over the marginal cost is one minus the tax rate.

In an efficient steady state, the marginal cost $MC = 1$ as seen in the RBC model which is efficient. Therefore,

$$(1 + \mu) = 1 - \tau$$

and hence we get that:

$$\tau = -\mu.$$

□

Proposition 496 (Output is Pinned Down)

Output is a function of productivity shocks:

$$Y = A^{\frac{1+\psi}{\sigma+\psi}} C^\sigma \quad (571)$$

Proof. First, recall the labour market equilibrium:

$$A \cdot MC = N^\psi C^\sigma$$

Since $MC = 1$ in the efficient steady state:

$$A = N^\psi C^\sigma \Rightarrow N = A^{1/\psi} C^{-\sigma/\psi}.$$

Using $C = Y$ and $N = Y/A$, we obtain:

$$Y = A^{\frac{1+\psi}{\sigma+\psi}}.$$

□

In the next section, we analyse fluctuations around this steady state.

23 New Keynesian Model

23.1 Three-Equation New Keynesian Model

We now want to develop the 3-equation New Keynesian model. First, we recall the equilibrium we had from the last section.

Theorem 497 (Equilibrium)

Given a shock A_t and a policy instrument i_t , the sequence of 7 variables $\{Y_t, MC_t, i_t, \Pi_t, \Delta_t, X_{1t}, X_{2t}\}$ and the 6 equations form the equilibrium:

$$Y_t^{-\sigma} = \beta \mathbb{E}_t \left[Y_{t+1}^{-\sigma} \frac{1 + i_t}{\Pi_{t+1}} \right] \quad (572)$$

$$A_t^{1+\psi} MC_t = Y_t^{\sigma+\psi} \Delta_t^\psi \quad (573)$$

$$1 = \alpha \Pi_t^{\theta-1} + (1 - \alpha) \left(\frac{X_{1t}}{X_{2t}} \right)^{1-\theta} \quad (574)$$

$$X_{1t} = (1 + \mu) MC_t Y_t + \alpha \beta \mathbb{E}_t \left[\left(\frac{Y_{t+1}}{Y_t} \right)^{-\sigma} \Pi_{t+1}^\theta X_{1t+1} \right] \quad (575)$$

$$X_{2t} = (1 - \tau_t) Y_t + \alpha \beta \mathbb{E}_t \left[\left(\frac{Y_{t+1}}{Y_t} \right)^{-\sigma} \Pi_{t+1}^{\theta-1} X_{2t+1} \right] \quad (576)$$

$$\Delta_t = \alpha \Pi_t^\theta \Delta_{t-1} + (1 - \alpha) \left(\frac{1 - \alpha \Pi_t^{\theta-1}}{1 - \alpha} \right)^{\frac{\theta}{\theta-1}} \quad (577)$$

Remark 498. Notice that we have 7 variables but 6 equations. This is because we did not specify the monetary policy in this model.

Definition 499 (Aggregate Supply)

The aggregate supply equation is given by:

$$\pi_t = \frac{(1 - \alpha)(1 - \alpha\beta)}{\alpha} (mc_t - \hat{\tau}_t) + \beta \mathbb{E}_t [\pi_{t+1}] \quad (578)$$

where π_t , mc_t , and $\hat{\tau}_t$ are all expressed as log-deviations from the efficient steady state, where inflation is 0.

Remark 500. This aggregate supply equation is also known as the New Keynesian Phillips curve.

Sketch. Take the relation between optimal relative price and inflation:

$$1 = \alpha \Pi_t^{\theta-1} + (1 - \alpha) \left(\frac{X_{1t}}{X_{2t}} \right)^{1-\theta}.$$

Apply first order Taylor expansion and use the fact that in the steady state, $\Pi = 1$ and $X_1 = X_2$ to get:

$$\pi_t = \frac{1 - \alpha}{\alpha} (x_{1t} - x_{2t})$$

where lower-case letters denote log-deviations from the steady state. Note that x_{1t} is the deviation of the PDV of costs and x_{2t} is the deviation of the PDV of revenues.

We want an expression for $(x_{1t} - x_{2t})$. To do so, take log-linear approximations of the following:

$$\begin{aligned} X_{1t} &= (1 + \mu)MC_t Y_t + \alpha\beta \mathbb{E}_t \left[\left(\frac{Y_{t+1}}{Y_t} \right)^{-\sigma} \Pi_{t+1}^\theta X_{1t+1} \right] \\ X_{2t} &= (1 - \tau_t)Y_t + \alpha\beta \mathbb{E}_t \left[\left(\frac{Y_{t+1}}{Y_t} \right)^{-\sigma} \Pi_{t+1}^{\theta-1} X_{2t+1} \right] \end{aligned}$$

and you will get:

$$\begin{aligned} x_{1t} &= (1 - \alpha\beta)(mc_t + y_t) + \alpha\beta \mathbb{E}_t [m_{t,t+1} + \theta\pi_{t+1} + x_{1t+1}] \\ x_{2t} &= (1 - \alpha\beta)(\hat{\tau}_t + y_t) + \alpha\beta \mathbb{E}_t [m_{t,t+1} + (\theta - 1)\pi_{t+1} + x_{2t+1}] \end{aligned}$$

We can take the difference between these two equations to get:

$$x_{1t} - x_{2t} = (1 - \alpha\beta)(mc_t - \hat{\tau}_t) + \alpha\beta \mathbb{E}_t [\pi_{t+1} + x_{1t+1} - x_{2t+1}].$$

Now re-write:

$$\pi_t = \frac{1 - \alpha}{\alpha}(x_{1t} - x_{2t}) \Rightarrow x_{1t+1} - x_{2t+1} = \frac{\alpha}{1 - \alpha}\pi_{t+1}$$

and plug this in:

$$\begin{aligned} x_{1t} - x_{2t} &= (1 - \alpha\beta)(mc_t - \hat{\tau}_t) + (\alpha\beta) \mathbb{E}_t [\pi_{t+1} + x_{1t+1} - x_{2t+1}] \\ \Leftrightarrow x_{1t} - x_{2t} &= (1 - \alpha\beta)(mc_t - \hat{\tau}_t) + (\alpha\beta) \mathbb{E}_t [\pi_{t+1} + \frac{\alpha}{1 - \alpha}\pi_{t+1}] \\ \Leftrightarrow x_{1t} - x_{2t} &= (1 - \alpha\beta)(mc_t - \hat{\tau}_t) + (\alpha\beta) \mathbb{E}_t \left[\frac{1}{1 - \alpha}\pi_{t+1} \right] \end{aligned}$$

We therefore have that:

$$x_{1t} - x_{2t} = (1 - \alpha\beta)(mc_t - \hat{\tau}_t) + \frac{\alpha\beta}{1 - \alpha} \mathbb{E}_t [\pi_{t+1}]$$

We combine this with:

$$\pi_t = \frac{1 - \alpha}{\alpha}(x_{1t} - x_{2t})$$

to get our desired result:

$$\pi_t = \frac{(1 - \alpha)(1 - \alpha\beta)}{\alpha}(mc_t - \hat{\tau}_t) + \beta \mathbb{E}_t [\pi_{t+1}]$$

□

Let us interpret this New Keynesian Phillips curve.

$$\pi_t = \frac{(1 - \alpha)(1 - \alpha\beta)}{\alpha}(mc_t - \hat{\tau}_t) + \beta \mathbb{E}_t [\pi_{t+1}]$$

This relates current inflation π_t to current marginal cost net of taxes and to expected future inflation. This equation summarises aggregate supply in our model. We can immediately see the following factors that affect current inflation π_t .

1. Marginal Costs: Higher marginal costs for firms mean that there will be higher inflation.
2. Expected Future Inflation: Firms also take into account the evolution of future costs, and hence expected future inflation also matters.

3. Exogenous Cost-Push Shock: We have assumed that $\hat{\tau}_t$ represents an exogenous sales-tax disturbance, which causes markups to change over time. More broadly, it can represent any factor that changes firms' desired markups independently of marginal cost.

Before we proceed, we impose another assumption.

Assumption (No Hyperinflation)

$$\lim_{T \rightarrow \infty} \beta^T \mathbb{E}_t [\pi_{t+T+1}] = 0. \quad (579)$$

Notice that we can iterate the New Keynesian Phillips curve.

Theorem 501 (Forward Looking Expression for Inflation)

The forward looking expression for inflation is:

$$\pi_t = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j (mc_{t+j} - \hat{\tau}_{t+j}) \right] \quad (580)$$

Proof. We have that $\beta < 1$ so we can derive a forward looking equation based on:

$$\pi_t = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} (mc_t - \hat{\tau}_t) + \beta \mathbb{E}_t [\pi_{t+1}]$$

where we continually substitute for future inflation. The no-hyperinflation assumption eliminates the limiting term. $\square$

Current inflation is the expectation of all future marginal costs and cost push shocks. This links inflation to marginal costs and is why it is called an aggregate supply equation. We can therefore say that current inflation is a **jump variable** as it is a function of future variables.

We now look at the price dispersion effects on cross sectional prices.

Proposition 502 (Price Dispersion Effects)

Price dispersion has no first order effects on equilibrium dynamics:

$$\hat{\Delta}_t = 0. \quad (581)$$

Proof. Take the price dispersion equation

$$\Delta_t = \alpha \Pi_t^\theta \Delta_{t-1} + (1-\alpha) \left(\frac{1 - \alpha \Pi_t^{\theta-1}}{1-\alpha} \right)^{\frac{\theta}{\theta-1}}$$

Log-linearise and use the fact that in steady state, $\Delta = \Pi = 1$ to get:

$$\hat{\Delta}_t = \alpha \hat{\Delta}_{t-1}$$

If we substitute backwards for t periods, we get:

$$\hat{\Delta}_t = \alpha^t \hat{\Delta}_0$$

Because $\alpha < 1$, any initial first-order dispersion vanishes asymptotically. In the usual perturbation around a zero-dispersion steady state, $\hat{\Delta}_0 = 0$, so $\hat{\Delta}_t = 0$ for every t . $\square$

Intuitively, price dispersion is related to the variance of prices, which is a second-order term. Because we consider only first-order effects, price dispersion has no first-order effect on the equilibrium dynamics.

We now look at the rest of the model.

Proposition 503 (Production Function Dynamics)

The production function dynamics are:

$$y_t = a_t + n_t. \quad (582)$$

Proof. First, recall that the production function was given by:

$$Y_t \Delta_t = A_t N_t. \quad (583)$$

We then take logarithms and use the fact that price dispersion has no first-order effects, $\log(\Delta_t) = 0$, to conclude:

$$y_t = a_t + n_t.$$

□

Proposition 504 (Labour Market Dynamics)

The labour market equilibrium dynamics are:

$$a_t + mc_t = \psi n_t + \sigma c_t. \quad (584)$$

Proof. Use the labour market equilibrium:

$$A_t MC_t = N_t^\psi C_t^\sigma$$

and take logs to get the desired result.

□

Proposition 505 (Consumption-Output)

Consumption is equal to output:

$$y_t = c_t. \quad (585)$$

Proof. Use:

$$Y_t = C_t + I_t + G_t$$

but we have no government spending and capital is constant in steady state (so investment is zero):

$$Y_t = C_t.$$

Take logs of this to get the desired result.

□

From these three dynamics, we can derive how the firm's marginal costs evolve.

Proposition 506 (Marginal Cost Dynamics)

Marginal costs are given by:

$$mc_t = (\sigma + \psi)y_t - (1 + \psi)a_t. \quad (586)$$

Proof. Use the labour market dynamics:

$$a_t + mc_t = \psi n_t + \sigma c_t$$

and substitute the consumption-output dynamics $y_t = c_t$ and the production-function dynamics $n_t = y_t - a_t$:

$$a_t + mc_t = \psi[y_t - a_t] + \sigma y_t.$$

□

Remark 507. We see here that marginal cost mc_t increases with output y_t and decreases with productivity a_t .

Proposition 508 (Level of Output)

The evolution of output is:

$$y_t = -\sigma^{-1}(i_t - \mathbb{E}_t[\pi_{t+1}]) + \mathbb{E}_t[y_{t+1}]. \quad (587)$$

Proof. Use the bond-market equilibrium:

$$1 = \beta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{1 + i_t}{\Pi_{t+1}} \right]$$

Substitute in $C_t = Y_t$ and log-linearise to get the result.

□

We now define an important concept in New Keynesian economics.

Proposition 509 (Efficient Level of Output)

The efficient level of output is:

$$y_t^* = \frac{1 + \psi}{\sigma + \psi} a_t. \quad (588)$$

Proof. First, recall that in equilibrium with flexible prices:

$$1 = \frac{1 + \mu}{1 - \tau_t} MC_t.$$

In an efficient equilibrium, we can make the equilibrium efficient by setting:

$$\tau_t = \tau = -\mu$$

which therefore implies that the marginal cost:

$$MC_t^* = \frac{1 + \mu}{1 + \mu} = 1$$

which means that the log-deviation for marginal cost is 0:

$$mc_t^* = 0.$$

Now, recall the marginal cost dynamics Eq. (586):

$$mc_t = (\sigma + \psi)y_t - (1 + \psi)a_t.$$

and setting $mc_t = 0$ and re-arranging gets us the desired result.

□

Remark 510. The efficient level of output is a scaled level of productivity a_t , which is quite similar to the RBC model! Hence, output is pinned down by productivity.

The efficient level of output y_t^* can be interpreted as the ideal level of output if resources are fully utilised.

Definition 511 (Output Gap)

The output gap is defined as:

$$x_t \equiv y_t - y_t^* \quad (589)$$

We now want to rewrite the marginal cost dynamics:

$$mc_t = (\sigma + \psi)y_t - (1 + \psi)a_t$$

in terms of the output gap.

Proposition 512 (Marginal Cost in Terms of Output Gap)

Marginal cost in terms of the output gap is:

$$mc_t = (\sigma + \psi)x_t \quad (590)$$

where x_t is the output gap.

Proof. First, use the formula for the efficient level of output:

$$y_t^* = \frac{1 + \psi}{\sigma + \psi} a_t$$

and re-arrange for a_t :

$$a_t = \frac{\sigma + \psi}{1 + \psi} y_t^*.$$

Plug this into the marginal cost dynamics:

$$mc_t = (\sigma + \psi)y_t - (1 + \psi)a_t \quad \leftrightarrow \quad mc_t = (\sigma + \psi)y_t - (1 + \psi)\frac{\sigma + \psi}{1 + \psi}y_t^*.$$

which then becomes:

$$mc_t = (\sigma + \psi)(y_t - y_t^*) = (\sigma + \psi)x_t.$$

□

We can now rewrite the New Keynesian Phillips Curve (aggregate supply) equation in terms of the output gap.

Definition 513 (Aggregate Supply)

The aggregate supply equation is given by:

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t[\pi_{t+1}] + u_t \quad (591)$$

whereby

$$\kappa \equiv (1 - \alpha)(1 - \alpha\beta)(\sigma + \psi)/\alpha$$

$$u_t \equiv -\kappa \hat{\tau}_t / (\sigma + \psi)$$

Proof. Recall the New Keynesian Phillips curve we previously derived was:

$$\pi_t = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} (mc_t - \hat{\tau}_t) + \beta \mathbb{E}_t[\pi_{t+1}].$$

Substitute in our expression for marginal cost in terms of the output gap:

$$mc_t = (\sigma + \psi)x_t$$

which gives us:

$$\pi_t = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} ((\sigma + \psi)x_t - \hat{\tau}_t) + \beta \mathbb{E}_t[\pi_{t+1}].$$

which becomes:

$$\pi_t = \frac{(1-\alpha)(1-\alpha\beta)(\sigma + \psi)}{\alpha} x_t - \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} \hat{\tau}_t + \beta \mathbb{E}_t[\pi_{t+1}].$$

We then get the expression:

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t[\pi_{t+1}] + u_t$$

where we defined:

$$\kappa \equiv (1-\alpha)(1-\alpha\beta)(\sigma + \psi)/\alpha$$

$$u_t \equiv -\kappa \hat{\tau}_t / (\sigma + \psi)$$

□

We now move onto deriving the aggregate demand equation.

Definition 514 (Aggregate Demand)

The aggregate demand equation is given by:

$$x_t = -\sigma^{-1}(i_t - \mathbb{E}_t[\pi_{t+1}] - r_t^*) + \mathbb{E}_t[x_{t+1}]. \quad (592)$$

where the efficient real interest rate is:

$$r_t^* \equiv \sigma(\mathbb{E}_t[y_{t+1}^* - y_t^*]).$$

Proof. Take the level of output we derived from the bond market equilibrium:

$$y_t = -\sigma^{-1}(i_t - \mathbb{E}_t[\pi_{t+1}]) + \mathbb{E}_t[y_{t+1}].$$

Now add and subtract terms to get it in terms of the output gap:

$$y_t - y_t^* = -\sigma^{-1}(i_t - \mathbb{E}_t[\pi_{t+1}]) + \mathbb{E}_t[y_{t+1}] - y_t^* + \mathbb{E}_t[y_{t+1}^*] - \mathbb{E}_t[y_{t+1}^*]$$

$$y_t - y_t^* = -\sigma^{-1}(i_t - \mathbb{E}_t[\pi_{t+1}]) - \sigma^{-1}\sigma(y_t^* - \mathbb{E}_t[y_{t+1}^*]) + \mathbb{E}_t[y_{t+1} - y_{t+1}^*]$$

and use the definition of the output gap $x_t \equiv y_t - y_t^*$ to get:

$$x_t = -\sigma^{-1}(i_t - \mathbb{E}_t[\pi_{t+1}] - r_t^*) + \mathbb{E}_t[x_{t+1}]$$

where

$$r_t^* \equiv \sigma(\mathbb{E}_t[y_{t+1}^* - y_t^*])$$

□

A few things to note on the aggregate demand equation. First of all, it says that the current output gap is equal to the expected output gap less the difference between the real interest rate and the efficient real interest rate. Furthermore, looking at the efficient real interest rate:

$$r_t^* \equiv \sigma(\mathbb{E}_t[y_{t+1}^* - y_t^*])$$

we have shown that the efficient level of output y_t^* was a function of technology:

$$y_t^* = \frac{1 + \psi}{\sigma + \psi} a_t$$

which therefore implies that the efficient real interest rate is also a function of productivity.

To close this model, we also specify the monetary policy which states how the central bank sets interest rates.

Definition 515 (Monetary Policy)

Monetary policy is given by:

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \epsilon_t \quad (593)$$

Here, the central bank responds to the level of inflation and to the output gap. The shock ϵ_t can be interpreted as deviations from the systematic level of interest rates that the central bank wishes to set.

We can therefore summarise the three-equation New Keynesian model.

Theorem 516 (Three-Equation New Keynesian Model)

The three-equation New Keynesian model is given by:

1. Aggregate Supply:

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t[\pi_{t+1}] + u_t \quad (594)$$

2. Aggregate Demand:

$$x_t = -\sigma^{-1}(i_t - \mathbb{E}_t[\pi_{t+1}] - r_t^*) + \mathbb{E}_t[x_{t+1}]. \quad (595)$$

3. Monetary Policy:

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \epsilon_t \quad (596)$$

with

$$\kappa \equiv (1 - \alpha)(1 - \alpha\beta)(\sigma + \psi)/\alpha$$

$$u_t \equiv -\kappa \hat{\tau}_t / (\sigma + \psi)$$

$$r_t^* \equiv \sigma(\mathbb{E}_t[y_{t+1}^* - y_t^*])$$

$$y_t^* = \frac{1 + \psi}{\sigma + \psi} a_t.$$

We now summarise the model.

The aggregate supply comes from the pricing problem of the firm. The aggregate supply equation relates current inflation π_t to the current output gap x_t , weighted by a sensitivity parameter κ plus expected future inflation $\mathbb{E}_t[\pi_{t+1}]$, and a cost-push shock u_t , which generates inflation independently of what happens to the output gap x_t .

The aggregate demand equation comes from the household side of the RBC model. It relates the current output gap x_t negatively to the real interest rate $(i_t - \mathbb{E}_t[\pi_{t+1}])$ relative to the efficient real interest rate r_t^* , and positively to the expected future output gap.

Finally, we have a monetary policy rule that the central bank sets. The central bank sets the nominal interest rate in response to inflation π_t and the output gap x_t with a monetary policy shock ϵ_t .

The efficient real interest rate r_t^* is σ times the expected growth rate of efficient output, which in turn depends on productivity a_t .

23.2 Response to Exogenous Shocks

We now study three types of shocks in the three-equation New Keynesian model.

23.2.1 Monetary Policy Shock

We first look at a monetary policy shock.

Definition 517 (Monetary Policy Shock)

A monetary policy shock is given by:

$$\epsilon_t = \rho_m \epsilon_{t-1} + \epsilon_{mt}. \quad (597)$$

Specifically, a one-unit innovation ϵ_{mt} raises ϵ_t by one unit. The AR(1) structure then propagates the disturbance to future values of ϵ_t . The shock enters the monetary-policy equation directly:

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \epsilon_t.$$

This raises the nominal interest rate relative to the systematic component of policy. Recall the Fisher equation:

$$r_t = i_t - \mathbb{E}_t[\pi_{t+1}].$$

Provided expected inflation does not rise one-for-one, the real interest rate r_t rises. The aggregate-demand equation then implies

$$x_t = -\sigma^{-1}(i_t - \mathbb{E}_t[\pi_{t+1}] - r_t^*) + \mathbb{E}_t[x_{t+1}]$$

a higher real interest rate r_t depresses the output gap and therefore aggregate demand x_t falls. Then, looking at the aggregate supply curve:

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t[\pi_{t+1}] + u_t$$

a fall in x_t will cause a fall in inflation π_t . Now, recall that we have shown that the efficient level of output is given by:

$$y_t^* = \frac{1 + \psi}{\sigma + \psi} a_t$$

Because efficient output depends only on technology in this model, it is unchanged by the monetary-policy innovation. A lower output gap therefore translates directly into lower actual output. The policy rule also contains an endogenous response:

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \epsilon_t$$

The declines in the output gap and inflation lower the systematic component of the policy rate, partially offsetting the initial innovation. Forward-looking inflation expectations propagate the effects over time.

23.2.2 Productivity Shock

We now look at a shock to technology.

Definition 518 (Productivity Shock)

A productivity shock is given by:

$$a_t = \rho_a a_{t-1} + \epsilon_{at}. \quad (598)$$

Notice again the shock comes through ϵ_{at} and affects a_t . This then shows up directly in the equation for the efficient level of output:

$$y_t^* = \frac{1 + \psi}{\sigma + \psi} a_t.$$

This represents an increase in productivity, as in the RBC model. Efficient output rises immediately. Under the standard policy calibration illustrated here, actual output rises by less than efficient output, so the output gap $x_t = y_t - y_t^*$ becomes negative. This response is an equilibrium consequence of sticky prices and the monetary-policy rule, rather than a mechanical restriction that output cannot move. The aggregate-supply curve then implies

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t[\pi_{t+1}] + u_t$$

this pushes down inflation π_t . Monetary policy then responds

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \epsilon_t$$

by cutting the nominal rate because the output gap and inflation have both fallen. The corresponding real rate is

$$r_t = i_t - \mathbb{E}_t[\pi_{t+1}].$$

Its response also depends on the simultaneous response of expected inflation. Under standard calibrations it moves towards the lower natural rate, but this direction is an equilibrium result rather than a consequence of the Fisher equation alone.

23.2.3 Cost-Push Shock

We now consider a cost-push shock.

Definition 519 (Cost-Push Shock)

A cost push shock is given by:

$$u_t = \rho_u u_{t-1} + \epsilon_{ut} \quad (599)$$

whereby $0 \leq \rho_j < 1$ and $\epsilon_{jt} \sim \mathcal{N}(0, 1)$ for $j \in \{m, a, u\}$.

The cost-push shock immediately affects the aggregate supply curve:

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t[\pi_{t+1}] + u_t$$

and puts upward pressure on inflation. The central bank responds through

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \epsilon_t$$

but whether the real rate rises depends on how expected inflation responds:

$$r_t = i_t - \mathbb{E}_t[\pi_{t+1}].$$

Under a sufficiently strong policy response, the real rate rises relative to the passive-policy counterfactual, depressing the output gap $x_t = y_t - y_t^*$ through

$$x_t = -\sigma^{-1}(i_t - \mathbb{E}_t[\pi_{t+1}] - r_t^*) + \mathbb{E}_t[x_{t+1}].$$

Efficient output is unaffected because it depends only on technology, $y_t^* = \frac{1+\psi}{\sigma+\psi}a_t$. Actual output therefore falls with the output gap. This contraction puts downward pressure on inflation through

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t[\pi_{t+1}] + u_t$$

and partially offsets the shock's direct inflationary effect. Hence, the policy response dampens the transmission of the cost-push shock to inflation, at the cost of lower output.

To summarise the role of expected inflation:

1. Expected inflation can reinforce the propagation of monetary-policy and productivity shocks.
2. Following a cost-push shock, expected inflation and the real rate are jointly determined. Active policy dampens inflation relative to a passive-policy counterfactual, at the cost of a lower output gap.

23.3 Stability of Systems

Before proceeding, we need to introduce some definitions and ideas. We frequently refer to jump/forward-looking variables and predetermined/state variables. This section is therefore important for understanding equilibrium determinacy. It may be easiest to begin with examples.

Suppose we had a scalar equation:

$$x_{t+1} = \rho x_t. \tag{600}$$

Now, recall from time series, if we had an initial condition x_0 , we can back substitute in x_t into this equation to get:

$$x_{t+1} = \rho[\rho x_{t-1}] = \rho^2 x_{t-1}.$$

Clearly, we can repeat this until we reach the initial condition:

$$x_{t+1} = \rho^{t+1} x_0.$$

Now, if $|\rho| < 1$, we arrive at a **stable** solution since as we let $t \rightarrow \infty$, we get:

$$x_{t+1} = \rho^{t+1} x_0 \rightarrow 0.$$

In this scenario, x_t is called a state or predetermined variable because it starts from an initial value x_0 and evolves through the system. That is, x_t depends on lagged values of itself.

Now note that the only reason we could back substitute is because we were given an initial condition x_0 . If we were not given one, we couldn't do this process since there is no end to our back substitution since we have not defined what the initial value is of the system! This backward substitution is also known as the **backwards dynamics**.

Definition 520 (Predetermined/State Variable)

A variable x_t is a predetermined variable if it is expressed in a backward manner:

$$x_t = \rho x_{t-1} \quad (601)$$

where we have an initial condition x_0 .

So what do we do if we are not given an initial condition? We go forward! This may seem strange but bear with this for a bit. What we can instead do is first rewrite:

$$x_{t+1} = \rho x_t$$

as:

$$x_t = a x_{t+1}$$

where $a = \frac{1}{\rho}$. Now, we know that the equation for x_{t+2} is:

$$x_{t+2} = \rho x_{t+1}$$

which we can rewrite as: $x_{t+1} = a x_{t+2}$ where $a = 1/\rho$. We then plug this into our previous expression to get:

$$\begin{aligned} x_t &= a x_{t+1} \\ &= a \left[a x_{t+2} \right]. \end{aligned}$$

We continually repeat this process for T periods to get:

$$x_t = a^T x_{t+T}.$$

Now notice that x_t depends on future values of the variable. In this case, x_t is called a jump or forward-looking variable. We will explain the term “jump” below. If $|a| < 1$, which implies that $|\frac{1}{\rho}| < 1$ and hence $|\rho| > 1$, then, provided x_{t+T} remains bounded, as $T \rightarrow \infty$ we get:

$$x_t \rightarrow 0$$

and hence we have a stable system. This forward substitution is known as **forward dynamics**. Why can we repeatedly substitute forward here but not backwards? In an infinite-horizon model, as is common in macroeconomics, a boundedness or no-bubble condition can pin down the limiting future term. This is analogous to asset pricing, where a stock price is the discounted sum of all future dividends. Notice also that stability for this forward-looking equation requires $|\rho| > 1$, whereas stability for a backward-looking equation requires $|\rho| < 1$.

Definition 521 (Jump/Forward Looking Variable)

A variable x_t is a jump variable if it is expressed in a forward manner:

$$x_t = a x_{t+1} \quad (602)$$

where $a = \frac{1}{\rho}$.

Remark 522. These are also referred to as non-predetermined variables or controls.

We can now explain why these are called jump variables. Suppose we instead had a stochastic system such that, after forward substitution, we obtained:

$$x_t = \mathbb{E}_t[a^T x_{t+T}].$$

You can think of the $\mathbb{E}_t[\cdot]$ term as agents' expectations about the future. If those expectations change because beliefs or information change, then x_t can change immediately: it **jumps**. We do not always call such variables controls because, in the New Keynesian model, inflation π_t is a function of expected future inflation $\mathbb{E}_t[\pi_{t+1}]$ and is therefore a jump variable, even though calling inflation a control variable would be awkward. The term “jump variable” is thus broader.

To summarise: a predetermined variable is stable if $|\rho| < 1$. A jump variable in this formulation is stable if $|a| < 1$, which implies $|\rho| > 1$.

We also define another type of variable that can appear in our system.

Definition 523 (Sunspot Shock)

A sunspot shock ϵ_t is an extrinsic stochastic variable that is unrelated to the economy's fundamentals but can affect equilibrium outcomes. It is often represented by a mean-zero innovation.

Remark 524. *An exogenous fundamental shock is also determined outside the model, but unlike a sunspot it directly changes preferences, technology, policy, or another economic fundamental.*

For example, suppose we had a solution where capital k_t at time t depends on previous levels of capital and consumption:

$$k_t = f(k_{t-1}, k_{t-2}, \dots, c_{t-1}, c_{t-2}, \dots).$$

This is a **non-sunspot solution**. Suppose instead, we had that at every time t , there is a random ϵ_t shock and hence we instead have:

$$k_t = f(k_{t-1}, k_{t-2}, \dots, c_{t-1}, c_{t-2}, \dots, \epsilon_t)$$

where ϵ_t is an extrinsic random variable. A sunspot can affect allocations only when the underlying model is indeterminate: different beliefs about ϵ_t select among otherwise valid equilibria. Randomness by itself does not create multiple equilibria; the multiplicity must already be present in the model.

We now give more intuition for the Blanchard–Kahn conditions. Suppose we have a system with predetermined and jump variables. We want to know whether an equilibrium exists and is unique (determinate). The Blanchard–Kahn conditions compare the number of unstable roots with the number of jump variables; the precise inside/outside-unit-circle statement depends on how the system is written.

Suppose we have one predetermined (state) variable k_t , one jump (control) variable π_t , and an exogenous variable ϵ_t . We can stack the predetermined and jump variables into a vector $y_t = [k_t, \pi_t]^T$. There are **two** ways to write the first-order difference system (the first is used for the Michaelmas RBC model and the second for the Hilary New Keynesian model). The first is:

$$\mathbb{E}_t[\mathbf{C}y_{t+1}] = y_t + \mathbf{B}\epsilon_t$$

like in the RBC model in Michaelmas where $\mathbf{C}$ is a 2×2 coefficient matrix and $\mathbf{B}$ is a 2×1 coefficient matrix. Now, think back to our jump variable expression we had earlier:

$$x_t = ax_{t+1}.$$

Recall that we said this jump variable is stable if $|a| = |\frac{1}{\rho}| < 1$. Going back to our example:

$$\mathbb{E}_t[\mathbf{C}\mathbf{y}_{t+1}] = \mathbf{y}_t + \mathbf{B}\epsilon_t$$

the Blanchard Kahn condition said that for the matrix $\mathbf{C}$, the number of eigenvalues λ with modulus smaller than one has to be equal to the number of control/jump variables. Hence, we can see the similarity that the number of eigenvalues of $\mathbf{C}$ being less than 1 is equal to the number of jump variables is similar to what we said earlier about the stability of our jump variable where we needed $|a| < 1$!

Now, we look at the second way of writing this system. If the matrix $\mathbf{C}$ was invertible, we can invert it to get:

$$\begin{aligned}\mathbb{E}_t[\mathbf{C}\mathbf{y}_{t+1}] &= \mathbf{y}_t + \mathbf{B}\epsilon_t \\ \Leftrightarrow \mathbb{E}_t[\mathbf{y}_{t+1}] &= \mathbf{C}^{-1}\mathbf{y}_t + \underbrace{\mathbf{C}^{-1}\mathbf{B}}_{\equiv \mathbf{J}}\epsilon_t \\ \Leftrightarrow \mathbb{E}_t[\mathbf{y}_{t+1}] &= \mathbf{D}\mathbf{y}_t + \mathbf{J}\epsilon_t\end{aligned}$$

where $\mathbf{D} \equiv \mathbf{C}^{-1}$ (don't worry about matrix $\mathbf{J}$). Now, we have written our system to get:

$$\mathbb{E}_t[\mathbf{y}_{t+1}] = \mathbf{D}\mathbf{y}_t + \mathbf{J}\epsilon_t.$$

Think back to our jump-variable expression and recall that:

$$\begin{aligned}x_t &= ax_{t+1} \\ \Leftrightarrow \underbrace{\frac{1}{a}}_{\equiv \rho} x_t &= x_{t+1} \\ \Leftrightarrow \rho x_t &= x_{t+1}\end{aligned}$$

which was our original equation before solving it with forward dynamics! Now, recall that this original equation:

$$x_{t+1} = \rho x_t$$

is stable if $|a| < 1 \Leftrightarrow |\frac{1}{\rho}| < 1 \Leftrightarrow |\rho| > 1$. So the jump variable is stable if $|\rho| > 1$. Also notice that this jump variable looks like our system:

$$\mathbb{E}_t[\mathbf{y}_{t+1}] = \mathbf{D}\mathbf{y}_t + \mathbf{J}\epsilon_t.$$

Recall that the Blanchard–Kahn condition used in the Hilary determinacy handout says that the number of eigenvalues of $\mathbf{D}$ with modulus **greater** than 1 must equal the number of jump variables. The inequality is reversed relative to the first formulation because $\mathbf{D} = \mathbf{C}^{-1}$. This requirement is analogous to $|\rho| > 1$ for our jump variable. Notice, however, that the monetary-policy equation is static: all its variables are dated at the same time t , because it links the current interest rate to current inflation and the current output gap.

So what is all this saying? In our current setup, we have predetermined variables k_t and jump variables π_t , and exogenous variables ϵ_t . By definition, we said that predetermined variables must have an initial condition associated to them so that we could do backward substitutions. That is, all state/predetermined variables have initial conditions associated to them. From this, we can follow what we know in time series and get a stable solution to the difference equation for the state variables (assuming the coefficients/eigenvalues are less than 1). However, for control/jump variables, by definition, they do **not** have initial conditions associated with them. Therefore, it is theoretically possible

for them to have an infinite number of solutions (since the initial value could be anything) and therefore we can't necessarily arrive at a solution. Alternatively, we could have no solution either.

This is where the Blanchard–Kahn conditions come in. The broad idea is that the stability of the control-variable process depends on the number of stable and unstable eigenvalues. If the number of stable eigenvalues (those with modulus less than 1 in our first formulation) equals the number of jump variables, we can solve the forward difference equations for the jump variables by forward substitution. More technically, the forward solution and the transversality condition pin down a unique initial value for the control-variable process. We can then express the control variables as functions of the state and exogenous variables, just as a policy function in dynamic programming maps states into controls. Likewise, we can express the state variables as functions of their lagged values and the exogenous variables.

For the New Keynesian model below, we use the first formulation of the Blanchard–Kahn condition, as in the lecture slides and problem set; the determinacy handout uses the second formulation.

23.4 Equilibrium Properties

To summarise the three-equation New Keynesian model, we have three key elements:

1. Households: We constructed an RBC-style model without capital but introduced nominal bonds. Their one-period price is $1/(1 + i_t)$, where the central bank sets i_t .
2. Firm Side: We introduced monopolistic competition and staggered price setting. These two components allow monetary policy to generate real effects.
3. Monetary Policy: This sets the nominal interest rate and is a rule that is in response to deviations of inflation and output from target.

The nominal rigidities we introduced generate real effects of monetary policy. The intuition is as follows:

1. Since not all firms can change their prices (Calvo pricing), this leads to output being demand-determined. This is in contrast to the RBC model where output just fluctuated with productivity.
2. By changing the nominal interest rate, monetary policy affects the intertemporal consumption-savings decisions of the households. If the interest rate is higher, it is more tempting for the household to save rather than consume.
3. As a result of this, the central bank is now able to influence **both** output and inflation. Previously, in a standard RBC setting, the central bank was only able to influence inflation (this was the neutrality of money). However, the New Keynesian Phillips curve is what trades off between inflation and output here.

Now, recall the three parts of the model. The three-equation New Keynesian model is given by:

1. Aggregate Demand:

$$x_t = -\sigma^{-1}(i_t - \mathbb{E}_t[\pi_{t+1}] - r_t^*) + \mathbb{E}_t[x_{t+1}].$$

2. Aggregate Supply:

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t[\pi_{t+1}] + u_t$$

3. Monetary Policy:

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \epsilon_t$$

The endogenous variables in these three equations are **purely forward looking**: there is no endogenous state variable. (Persistent exogenous shocks can still act as states.) The output gap and inflation relate current values to

expected future values and are therefore **jump variables**.

Hence, the monetary-policy equation is a **static equation**. In contrast, the aggregate-demand and aggregate-supply equations are forward-looking equations.

Now, as the aggregate demand is a forward looking equation, with no initial condition, we can solve it forward using forward iteration to arrive at:

$$x_t = -\sigma^{-1} \mathbb{E}_t \left[\sum_{j=0}^{\infty} (i_{t+j} - \pi_{t+j+1} - r_{t+j}^*) \right]. \quad (603)$$

We can do something similar to solve forward for aggregate supply:

$$\pi_t = \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j (\kappa x_{t+j} + u_{t+j}) \right]. \quad (604)$$

The aggregate-demand equation shows that the current output gap depends negatively on expected **future** real-interest-rate gaps $(i_{t+j} - \pi_{t+j+1} - r_{t+j}^*)$. Households look into the future and adjust their spending according to their expectations of these gaps. For example, if households expect the real interest rate to equal the efficient interest rate in every future period, then today's output gap is zero.

Similarly, the aggregate-supply equation shows that current inflation depends on expected future output gaps x_{t+j} and future cost-push shocks u_{t+j} . If all future cost-push shocks are zero and all future real interest rates equal the efficient rate, so that every output gap is zero, current inflation is also zero.

Now, go back to the forward iterations for aggregate demand and supply:

$$x_t = -\sigma^{-1} \mathbb{E}_t \left[\sum_{j=0}^{\infty} (i_{t+j} - \pi_{t+j+1} - r_{t+j}^*) \right]$$

$$\pi_t = \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j (\kappa x_{t+j} + u_{t+j}) \right].$$

These forward solutions require transversality conditions on both the future output gap and inflation.

Assumption (Transversality Condition on Output Gap and Inflation)

We assume a transversality condition on the output gap:

$$\lim_{T \rightarrow \infty} \mathbb{E}_t [x_{t+T}] = 0 \quad (605)$$

and a transversality condition on inflation:

$$\lim_{T \rightarrow \infty} \beta^T \mathbb{E}_t [\pi_{t+T}] = 0. \quad (606)$$

Remark 525. Notice that the transversality condition on inflation is a milder condition as we have a discount factor β^T term.

Intuitively, these conditions mean that output gaps eventually close and inflation does not explode. This local, log-linear New Keynesian model is therefore most suitable for analysing economies with *relatively mild* inflation.

So far, we have shown that we can write forward-looking solutions for aggregate demand and aggregate supply under a **given monetary-policy** regime. This raises the question of whether different monetary-policy specifications alter the dynamics of the system. If the central bank does not react strongly enough, sunspot equilibria may arise. A similar result applies in the three-equation New Keynesian model.

We are now interested in how monetary policy affects the dynamics of inflation and the output gap, and in particular the stability of equilibrium. First recall the monetary-policy rule:

$$i_t = \phi_\pi \pi_t + \phi_x x_t + \epsilon_t$$

and the aggregate demand equation:

$$x_t = -\sigma^{-1}(i_t - \mathbb{E}_t[\pi_{t+1}] - r_t^*) + \mathbb{E}_t[x_{t+1}].$$

If we substitute in the monetary policy rule into the aggregate demand equation, we get:

$$\sigma x_t = -\underbrace{(\phi_\pi \pi_t + \phi_x x_t + \epsilon_t)}_{=i_t} - \mathbb{E}_t[\pi_{t+1}] - r_t^* + \sigma \mathbb{E}_t[x_{t+1}].$$

We don't do anything to the aggregate supply equation as it does not depend directly on nominal interest rates i_t . We therefore have the system:

$$\begin{aligned}\sigma x_t &= -(\phi_\pi \pi_t + \phi_x x_t + \epsilon_t - \mathbb{E}_t[\pi_{t+1}] - r_t^*) + \sigma \mathbb{E}_t[x_{t+1}] \\ \pi_t &= \kappa x_t + \beta \mathbb{E}_t[\pi_{t+1}] + u_t\end{aligned}$$

which consists of two forward-looking equations with two unknown jump variables (x_t, π_t) and exogenous variables (ϵ_t, r_t^*, u_t) . We now create the vectors:

$$\mathbf{z}_t \equiv \begin{bmatrix} x_t & \pi_t \end{bmatrix}^T$$

and

$$\mathbf{v}_t \equiv \begin{bmatrix} \epsilon_t & r_t^* & u_t \end{bmatrix}^T.$$

We then rearrange the system so that the variables dated t are on the left-hand side:

$$\begin{aligned}(\sigma + \phi_x)x_t + \phi_\pi \pi_t &= \sigma \mathbb{E}_t[x_{t+1}] + \mathbb{E}_t[\pi_{t+1}] - \epsilon_t + r_t^* \\ -\kappa x_t + \pi_t &= \beta \mathbb{E}_t[\pi_{t+1}] + u_t\end{aligned}$$

Our goal now is to get our system into the form:

$$\mathbf{z}_t = \mathbf{A} \mathbb{E}_t[\mathbf{z}_{t+1}] + \mathbf{B} \mathbf{v}_t$$

which is called the **linear rational expectation model** as we can then analyse the equilibrium of the system. Recall from our previous discussion that the stability of the system depends on the eigenvalues of the matrix $\mathbf{A}$ in this system. We now outline how to get there.

First, write the system into matrix form as seen in Fig. 44.

$$\underbrace{\begin{bmatrix} \sigma + \phi_x & \phi_\pi \\ -\kappa & 1 \end{bmatrix}}_{\equiv \mathbf{A}_0} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma & 1 \\ 0 & \beta \end{bmatrix}}_{\equiv \mathbf{A}_1} \begin{bmatrix} \mathbb{E}_t x_{t+1} \\ \mathbb{E}_t \pi_{t+1} \end{bmatrix} + \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\equiv \tilde{\mathbf{B}}} \begin{bmatrix} \epsilon_t \\ r_t^* \\ u_t \end{bmatrix}$$

Figure 44: State space form.

The matrix form of our system is therefore:

$$\mathbf{A}_0 \mathbf{z}_t = \mathbf{A}_1 \mathbb{E}_t[\mathbf{z}_{t+1}] + \tilde{\mathbf{B}} \mathbf{v}_t.$$

We now want to get rid of the $\mathbf{A}_0$ on the left-hand side. To do so, we need to figure out what the inverse of the matrix $\mathbf{A}_0$ is, which we show in Fig. 45.

Need to invert $\mathbf{A}_0$ to get in form $\mathbf{z}_t = \mathbf{A} \mathbb{E}_t \mathbf{z}_{t+1} + \mathbf{B} \mathbf{v}_t$

$$\mathbf{A}_0^{-1} = d \begin{bmatrix} 1 & -\phi_\pi \\ \kappa & \sigma + \phi_x \end{bmatrix} \quad \text{where} \quad d \equiv \frac{1}{\sigma + \phi_x + \kappa \phi_\pi}$$

Figure 45: Invert matrix $\mathbf{A}_0$ to get $\mathbf{A}_0^{-1}$ where d is the determinant of $\mathbf{A}_0$. Notice that the determinant depends on the coefficient of feedback of the output gap on the Taylor rule ϕ_x plus the slope of the Phillips curve κ times the feedback of inflation on the Taylor rule ϕ_π .

So now, we can take the inverse of $\mathbf{A}_0$ in:

$$\mathbf{A}_0 \mathbf{z}_t = \mathbf{A}_1 \mathbb{E}_t[\mathbf{z}_{t+1}] + \tilde{\mathbf{B}} \mathbf{v}_t.$$

to get:

$$\mathbf{z}_t = \mathbf{A}_0^{-1} \mathbf{A}_1 \mathbb{E}_t[\mathbf{z}_{t+1}] + \mathbf{A}_0^{-1} \tilde{\mathbf{B}} \mathbf{v}_t.$$

We now compute the matrix $\mathbf{A} = \mathbf{A}_0^{-1} \mathbf{A}_1$ as seen in Fig. 46.

Calculate A

$$\mathbf{A} = \mathbf{A}_0^{-1} \mathbf{A}_1 = d \begin{bmatrix} 1 & -\phi_\pi \\ \kappa & \sigma + \phi_x \end{bmatrix} \begin{bmatrix} \sigma & 1 \\ 0 & \beta \end{bmatrix} = d \begin{bmatrix} \sigma & 1 - \beta \phi_\pi \\ \sigma \kappa & \kappa + \beta(\sigma + \phi_x) \end{bmatrix}$$

Figure 46: Calculate the matrix $\mathbf{A} = \mathbf{A}_0^{-1} \mathbf{A}_1$.

Writing $\mathbf{B} \equiv \mathbf{A}_0^{-1} \tilde{\mathbf{B}}$, we therefore have the system:

$$\mathbf{z}_t = \mathbf{A} \mathbb{E}_t[\mathbf{z}_{t+1}] + \mathbf{B} \mathbf{v}_t$$

which is what we were after!

Definition 526 (Equilibrium System)

Equilibrium determinacy depends on the eigenvalues of the matrix $\mathbf{A}$ in the system:

$$\mathbf{z}_t = \mathbf{A} \mathbb{E}_t[\mathbf{z}_{t+1}] + \mathbf{B} \mathbf{v}_t \quad (607)$$

where $\mathbf{z}_t \equiv [x_t, \pi_t]^T$ and $\mathbf{v}_t \equiv [\epsilon_t, r_t^*, u_t]^T$.

We now recall the condition for equilibrium stability in this system.

Proposition 527 (Blanchard–Kahn Condition)

Uniqueness and determinacy of equilibrium require that the number of eigenvalues of $\mathbf{A}$ with modulus less than one equals the number of non-predetermined (jump) variables.

Remark 528. Refer back to previous subsection **stability of systems** if this doesn't make sense.

For our vector $\mathbf{z} \equiv [x_t, \pi_t]$, both variables are non-predetermined. Therefore, exactly two eigenvalues must have modulus less than one. We could compute the eigenvalues of $\mathbf{A}$ directly, but this can be tedious. Instead, we use the following equivalent conditions.

Theorem 529 (Bullard and Mitra (2002) Conditions)

The necessary and sufficient conditions for a 2×2 system with two jump variables to have a determinate equilibrium are:

$$|\det(\mathbf{A})| < 1 \quad (608)$$

$$|\text{tr}(\mathbf{A})| < 1 + \det(\mathbf{A}) \quad (609)$$

The next few steps are a matter of algebra. Fig. 47 shows how to compute the determinant and trace of $\mathbf{A}$.

Recall

$$\mathbf{A} = d \begin{bmatrix} \sigma & 1 - \beta\phi_\pi \\ \sigma\kappa & \kappa + \beta(\sigma + \phi_x) \end{bmatrix} \quad \text{where} \quad d = \frac{1}{\sigma + \phi_x + \kappa\phi_\pi}$$

Determinant

$$\det(\mathbf{A}) = d^2 \sigma [\kappa + \beta(\sigma + \phi_x)] - d^2 \sigma \kappa (1 - \beta\phi_\pi) = \beta\sigma d \in (0, 1)$$

Trace

$$\text{tr}(\mathbf{A}) = d\sigma + d[\kappa + \beta(\sigma + \phi_x)] = d[\sigma + \kappa + \beta(\sigma + \phi_x)] > 0$$

Figure 47: Determinant and trace of $\mathbf{A}$. Notice d^2 since the matrix $\mathbf{A}$ already has a determinant in it so we take the determinant again and hence d^2 .

We now check the conditions.

First, we check that $|\det(\mathbf{A})| < 1$. We see that:

$$|\det(\mathbf{A})| = \beta\sigma d = \frac{\beta\sigma}{\sigma + \phi_x + \kappa\phi_\pi} < 1$$

because $\beta < 1$ and $\phi_x, \phi_\pi \geq 0$, so the denominator exceeds $\beta\sigma$.

We now check $|\text{tr}(\mathbf{A})| < 1 + \det(\mathbf{A})$. To check this, we need to break down the absolute sign on the left into positive and negative part. First, you check that:

$$-\text{tr}(\mathbf{A}) < 1 + \det(\mathbf{A})$$

which you can do so using algebra for the results in Fig. 47.

The interesting case is to check the other sign:

$$\text{tr}(\mathbf{A}) < 1 + \det(\mathbf{A}).$$

Plug in the trace and determinant to get Fig. 48.

$$\text{tr}(\mathbf{A}) < 1 + \det(\mathbf{A}) \Rightarrow 1 + \beta\sigma d > d[\sigma + \kappa + \beta(\sigma + \phi_x)]$$

Figure 48: Second condition for Bullard and Mitra.

You can then further re-arrange this expression to arrive at an important condition called the **generalised Taylor principle**.

Theorem 530 (Generalised Taylor Principle)

The generalised Taylor principle is:

$$\phi_\pi > 1 - \frac{(1-\beta)\phi_x}{\kappa}. \quad (610)$$

Remark 531. Recall that β is close to one, so the term subtracted on the right-hand side is close to zero.

Let us interpret this generalised Taylor principle:

$$\phi_\pi > 1 - \frac{(1-\beta)\phi_x}{\kappa}.$$

If we set $\phi_x = 0$, we then get:

$$\phi_\pi > 1$$

which is the **strict** version of the Taylor principle which says that in the monetary policy equation for the central bank, the feedback rule coefficient for inflation, ϕ_π , must be greater than one. That is, if inflation rises by a percentage point, then the central bank must raise interest rates by more than a percentage point.

If we relax it by letting $\phi_x > 0$, then this says that the central bank also responds to the output gap and therefore, if inflation rises by a percentage point, the central bank does not necessarily raise interest rate by more than a percentage point now and instead responds to output too.

We now go through the intuition for this. Recall the Phillips curve:

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t[\pi_{t+1}] + u_t.$$

Suppose agents try to coordinate on a sunspot belief that raises expected inflation $\mathbb{E}_t[\pi_{t+1}]$. The Phillips curve puts upward pressure on current inflation. If the Taylor principle holds, the monetary-policy equation

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \epsilon_t$$

raises the nominal interest rate by enough to increase the real rate. Aggregate demand then contracts through

$$x_t = -\sigma^{-1}(i_t - \mathbb{E}_t[\pi_{t+1}] - r_t^*) + \mathbb{E}_t[x_{t+1}].$$

The lower output gap creates countervailing downward pressure on inflation through

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t[\pi_{t+1}] + u_t.$$

Under an **aggressive** policy response, this feedback rules out a bounded self-fulfilling inflation path, so the sunspot belief cannot select an additional equilibrium. With a sufficiently weak response, by contrast, such beliefs may be self-fulfilling.

Therefore, equilibrium in this three-equation NK model is unique and determinate if and only if policy satisfies the generalised Taylor principle above. Determinacy can come from the responses to inflation and the output gap together. The exact condition is model-specific because it was derived from this monetary-policy rule and aggregate-demand equation.

For a final discussion on this section, the handout on American inflation applies this section to a famous historical example.

24 Lower Bound on Nominal Interest Rate

In many economies, nominal interest rates have been near zero. We now explain why this is an issue. Recall that aggregate demand and the nominal interest rate chosen by the central bank are described by:

1. Aggregate Demand:

$$x_t = -\sigma^{-1}(i_t - \mathbb{E}_t[\pi_{t+1}] - r_t^*) + \mathbb{E}_t[x_{t+1}].$$

2. Monetary Policy:

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \epsilon_t$$

with the natural interest rate:

$$r_t^* \equiv \sigma(\mathbb{E}_t[y_{t+1}^*] - y_t^*)$$

and efficient level of output:

$$y_t^* = \frac{1 + \psi}{\sigma + \psi} a_t.$$

Suppose that there is a demand shock so that aggregate demand x_t falls. According to the monetary-policy equation, the central bank should lower the nominal interest rate i_t to stimulate the economy. However, if the nominal rate is already at 0, the central bank cannot lower it much further because households can substitute into cash. This constraint is known as the **zero lower bound** (ZLB). In practice, rates can fall slightly below zero, but for this course we assume they cannot. Conventional interest-rate policy therefore cannot provide further stabilisation. We now explore the implications within our model.

24.1 Demand-driven Recessions

We first look at how the ZLB is closely associated with demand-driven recessions. We explore why **financial crises** can be modelled as demand-driven recessions in which the ZLB binds.

Historically, short-term interest rates in the United States were near zero during the Great Depression and after the Great Recession. Japan also experienced near-zero rates following the collapse of its asset-price bubble in the early 1990s. With this motivation, we give a characterisation of financial crises.

Definition 532 (Financial Crises)

Financial crises are typically associated with falling output and prices.

With this characterisation, this is why we can think of financial crises as an example of a **demand shock**. This is why demand-driven recessions can be thought of as financial crises. We can also link this to what we have seen before when analysing the shocks to the 3-equation NK model. First, recall the aggregate demand equation:

$$x_t = -\sigma^{-1}(i_t - \mathbb{E}_t[\pi_{t+1}] - r_t^*) + \mathbb{E}_t[x_{t+1}].$$

A demand shock in this setting is a decline in the natural interest rate $r_t^* \equiv \sigma(\mathbb{E}_t[y_{t+1}^*] - y_t^*)$. As a result, this causes the output gap x_t to fall. Then, recalling the Phillips curve:

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t[\pi_{t+1}] + u_t$$

this causes inflation π_t to fall. Now, in normal circumstances, the central bank would lower nominal interest rates due

to the monetary policy rule:

$$i_t = \psi_\pi \pi_t + \psi_x x_t + \epsilon_t.$$

However, if the central bank is at the ZLB, they cannot lower nominal interest rates. Therefore, they cannot stabilise the demand shock! Hence, we can think of financial crises as large shocks to the natural interest rate.

Definition 533 (Financial Crises)

Financial crises are large negative demand shocks via the natural interest rates:

$$r_t^* \ll 0 \quad (611)$$

such that the nominal interest rate i_t hits the ZLB.

This interpretation poses a problem for the baseline New Keynesian model. To see what a large negative demand shock means there, recall that productivity follows

$$a_t = \rho_a a_{t-1} + \epsilon_{at}.$$

This feeds into the efficient level of output y_t^* :

$$y_t^* = \frac{1 + \psi}{\sigma + \psi} a_t.$$

This then feeds into the natural interest rate r_t^* :

$$r_t^* \equiv \sigma(\mathbb{E}_t[y_{t+1}^*] - y_t^*).$$

Hence, we can see that the natural interest rate depends on the productivity shocks a_t , and through substituting the AR(1) productivity shock into the efficient level of output and then into the natural interest rate, we get the following expression for the natural interest rate.

Definition 534 (Natural Interest Rate in Terms of Productivity Shock)

The natural interest rate r_t^* in terms of an AR(1) productivity process is given by:

$$r_t^* = -\frac{\sigma(1 + \psi)(1 - \rho_a)}{\sigma + \psi} a_t. \quad (612)$$

Now, recall that this natural interest rate feeds in **negatively** into the aggregate demand equation:

$$x_t = -\sigma^{-1}(i_t - \mathbb{E}_t[\pi_{t+1}] - r_t^*) + \mathbb{E}_t[x_{t+1}].$$

Thus generating a crisis-like fall in aggregate demand through productivity alone requires a large **positive** productivity shock. This is an awkward economic interpretation.

As a result, we need to do something else and instead introduce explicit demand shocks. The first one is a reduced form approach called discount factor shocks, which we shall cover next. The second way with microfoundations is to introduce a tightening of borrowing constraints, which we shall see later.

24.1.1 Discount Factor Shocks

The intuition behind discount-factor shocks is that they can make agents more patient. Households then want to save more by buying bonds and consume less, generating a negative demand shock.

To model this discount-factor shock, we introduce a new term Ξ_t that multiplies the discount factor β^t in the household's optimisation problem.

Definition 535 (Discount Factor Shock)

A discount-factor shock $\Xi_t > 0$ is a multiplicative preference shifter:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \Xi_t \left(\ln(C_t) - \frac{N_t^{1+\psi}}{1+\psi} \right) \right]. \quad (613)$$

This discount factor shock now affects the entire per-period utility function.

For our budget constraint, we shall use the same budget constraint that we saw in Section 20.1 when we introduced money into the utility function:

$$C_t + I_t + \frac{B_t}{P_t} + \frac{M_t}{P_t} = (1 + i_{t-1}) \frac{B_{t-1}}{P_t} + \frac{M_{t-1}}{P_t} + \frac{R_t^k}{P_t} K_{t-1} + \frac{W_t}{P_t} N_t - \frac{T_t}{P_t}.$$

We can see that when we derived the first order conditions with this, this new discount factor shock now shows up.

Proposition 536 (First-Order Conditions)

The first-order conditions with a discount-factor shock are:

$$\frac{\Xi_t}{C_t} = \lambda_t \quad (614)$$

$$\lambda_t \frac{W_t}{P_t} = \Xi_t N_t^\psi. \quad (615)$$

Proof. The current-value Lagrangian is:

$$\mathcal{L} = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left\{ \Xi_t \left(\ln(C_t) - \frac{N_t^{1+\psi}}{1+\psi} \right) + \lambda_t \left((1 + i_{t-1}) \frac{B_{t-1}}{P_t} + \frac{M_{t-1}}{P_t} + \frac{R_t^k}{P_t} K_{t-1} + \frac{W_t}{P_t} N_t - \frac{T_t}{P_t} - C_t - I_t - \frac{B_t}{P_t} - \frac{M_t}{P_t} \right) \right\} \right]$$

Now take first order conditions with respect to C_t :

$$\mathbb{E}_t \left[\beta^t \Xi_t \frac{1}{C_t} - \beta^t \lambda_t \right] = 0.$$

Re-arrange to get:

$$\frac{\Xi_t}{C_t} = \lambda_t.$$

Now for N_t , we get:

$$\mathbb{E}_t \left[-\beta^t \Xi_t N_t^\psi + \beta^t \lambda_t \frac{W_t}{P_t} \right] = 0$$

Re-arrange to get:

$$\lambda_t \frac{W_t}{P_t} = \Xi_t N_t^\psi.$$

□

The intratemporal labour condition is unchanged after adding the discount-factor shock.

Proposition 537 (Labour Condition Is Unchanged)

The first-order condition for labour is unchanged:

$$\frac{W_t}{P_t C_t} = N_t^\psi. \quad (616)$$

Proof. Combine the two first order conditions for consumption and labour together:

$$\begin{aligned} \frac{\Xi_t}{C_t} &= \lambda_t \\ \lambda_t \frac{W_t}{P_t} &= \Xi_t N_t^\psi. \end{aligned}$$

to get:

$$\frac{\Xi_t}{C_t} \frac{W_t}{P_t} = \Xi_t N_t^\psi.$$

□

The discount-factor shock does, however, appear in the Euler equation for bonds.

Proposition 538 (Euler Equation for Bonds)

The Euler equation for bonds is:

$$1 = \beta \mathbb{E}_t \left[\frac{\Xi_{t+1}}{\Xi_t} \frac{C_t}{C_{t+1}} \frac{1 + i_t}{\Pi_{t+1}} \right], \quad (617)$$

where $\Pi_{t+1} \equiv P_{t+1}/P_t$ is gross inflation.

Proof. The first-order condition for B_t equates its current real cost to its discounted real payoff:

$$\frac{\lambda_t}{P_t} = \beta \mathbb{E}_t \left[\lambda_{t+1} \frac{1 + i_t}{P_{t+1}} \right].$$

Substituting $\lambda_t = \Xi_t/C_t$, multiplying by $P_t C_t/\Xi_t$, and using $P_t/P_{t+1} = 1/\Pi_{t+1}$ gives the result. Equivalently, the consumption first-order condition is

$$\frac{\Xi_t}{C_t} = \lambda_t.$$

□

These discount factor shocks appear in the Euler equation for bonds.

$$1 = \beta \mathbb{E}_t \left[\frac{\Xi_{t+1}}{\Xi_t} \frac{C_t}{C_{t+1}} \frac{1 + i_t}{\Pi_{t+1}} \right].$$

The shock changes the intertemporal trade-off between consumption and bond holdings. A rise in patience can therefore increase desired saving and reduce current consumption.

We now follow the usual steps from the three-equation NK model: log-linearise and impose market clearing. The calculation differs only because Ξ_t enters marginal utility.

Proposition 539 (Output Under Discount Factor Shocks)

Output under discount factor shocks is given by:

$$y_t = -(i_t - \mathbb{E}_t[\pi_{t+1}]) + \mathbb{E}_t[y_{t+1}] - \psi_t \quad (618)$$

where we define the new term:

$$\psi_t \equiv \mathbb{E}_t \left[\log \left(\frac{\Xi_{t+1}}{\Xi_t} \right) \right].$$

Sketch. Let lower-case variables denote log deviations. Recall the log-linear approximation from the lecture on neutrality of money:

$$c_t = -(i_t - \mathbb{E}_t[\pi_{t+1}]) + \mathbb{E}_t[c_{t+1}].$$

Goods-market clearing $C_t = Y_t$ implies $c_t = y_t$, so

$$y_t = -(i_t - \mathbb{E}_t[\pi_{t+1}]) + \mathbb{E}_t[y_{t+1}].$$

Log-linearising the Euler equation adds the expected log growth of Ξ_t , because Ξ_t enters marginal utility multiplicatively. $\square$

The new term:

$$\psi_t \equiv \mathbb{E}_t \left[\log \left(\frac{\Xi_{t+1}}{\Xi_t} \right) \right]$$

is the expected log difference in the discount factor shock. This is now what we will be using as the stochastic process defining the demand shock.

Definition 540 (Discount Factor Shock Process)

The stochastic process defining the discount factor shock is:

$$\psi_t \equiv \mathbb{E}_t \left[\log \left(\frac{\Xi_{t+1}}{\Xi_t} \right) \right]. \quad (619)$$

We now rewrite this in terms of the output gap.

Proposition 541 (Output Gap Under Discount Factor Shocks)

The output gap x_t under discount factor shocks is given by:

$$x_t = -(i_t - \mathbb{E}_t[\pi_{t+1}] - r_t^*) + \mathbb{E}_t[x_{t+1}]. \quad (620)$$

where the efficient interest rate is now:

$$r_t^* = \mathbb{E}_t[y_{t+1}^* - y_t^*] - \psi_t \quad (621)$$

Sketch. Take the output:

$$y_t = -(i_t - \mathbb{E}_t[\pi_{t+1}]) + \mathbb{E}_t[y_{t+1}] - \psi_t$$

and subtract the efficient level of output y_t^* from both sides. $\square$

Notice that the efficient interest rate now depends directly on ψ_t , the expected growth rate of the discount-factor shifter.

The key point is that **the discount-factor shock now enters the efficient real interest rate directly**. A positive ψ_t lowers the efficient real interest rate and, for a given policy rate, reduces aggregate demand. Contrast this with the productivity process used in our baseline New Keynesian model:

$$a_t = \rho_a a_{t-1} + \epsilon_{at}.$$

This process feeds into efficient output y_t^* :

$$y_t^* = \frac{1 + \psi}{\sigma + \psi} a_t.$$

Efficient-output growth then determines the natural interest rate r_t^* :

$$r_t^* \equiv \sigma(\mathbb{E}_t[y_{t+1}^*] - y_t^*).$$

Generating a demand contraction through this channel would require a sufficiently large **positive** productivity shock, which is an awkward interpretation of a financial crisis. The discount-factor formulation instead lets a rise in desired saving lower the natural rate and the output gap directly.

As households become more patient, so that ψ_t rises, the efficient real interest rate r_t^* falls. Households want to save more and consume less, so the interest rate that clears the saving market must fall. The shock therefore captures a negative demand disturbance.

To isolate an impulse response to ψ_t , switch off the other shocks. Assume there are no productivity shocks, so $a_t = 0$ in every period. Efficient output is then zero:

$$y_t^* = \frac{1 + \psi}{\sigma + \psi} a_t = \frac{1 + \psi}{\sigma + \psi} 0 = 0.$$

It follows that the efficient interest rate is $r_t^* = -\psi_t$, since

$$r_t^* = \mathbb{E}_t[y_{t+1}^* - y_t^*] - \psi_t = \mathbb{E}_t[0] - 0 - \psi_t = -\psi_t.$$

Thus the efficient real interest rate maps one-for-one into the preference shock, $r_t^* = -\psi_t$, and ψ_t can itself be modelled as an AR(1) process.

We apply a persistent 25-basis-point negative demand shock, setting $\rho_\psi = 0.9$ in the AR(1) process. The responses are shown in Fig. 49.

Here, both the output gap and inflation fall when the negative demand shock occurs. Furthermore, the nominal interest rate i_t falls and, according to the unconstrained Taylor rule, would need to be below zero. This creates the zero-lower-bound problem.

To summarise, we have shown how to model financial crises as negative demand shocks. We now examine what happens when such shocks occur at the ZLB.

24.2 ZLB in the Three-Equation NK Model – Eggertsson and Woodford (2003)

We now introduce the ZLB into the three-equation New Keynesian model. We work with deviations from the zero-inflation steady state. Let i_t^{lev} denote the net nominal interest-rate level and let $\hat{i}_t \equiv i_t^{\text{lev}} - i$ denote its deviation from the steady-state rate i . (For interest rates, this linear deviation is used instead of a log deviation, because the

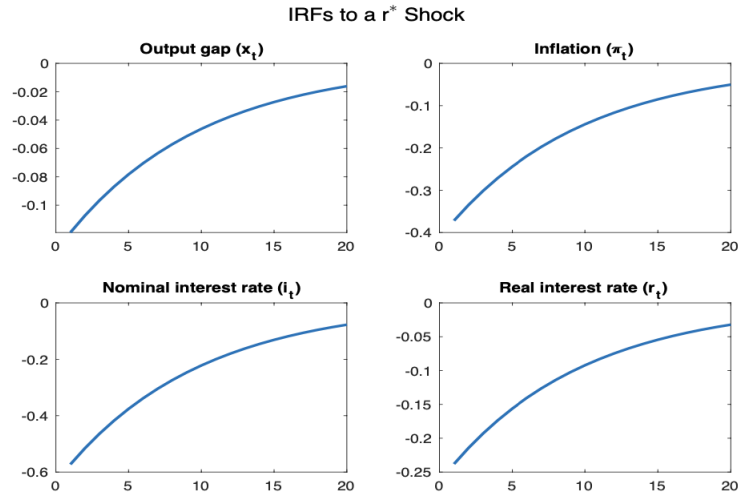


Figure 49: Impulse Response Function to a Demand shock.

logarithm is undefined at zero.) Recall that

$$i = r = \frac{1}{\beta} - 1.$$

When the nominal interest-rate level is at the ZLB, $i_t^{\text{lev}} = 0$, its deviation is therefore

$$\hat{i}_t = i_t^{\text{lev}} - i = -i.$$

Away from the ZLB, the central bank follows the Taylor rule

$$\hat{i}_t = \psi_\pi \pi_t + \psi_x x_t.$$

From now on, we drop the hat, so i_t denotes the deviation of the nominal rate from its steady-state value. Other lowercase variables continue to denote log deviations unless stated otherwise.

Definition 542 (Monetary Policy With Zero Lower Bound)

Let i_t be the deviation of the nominal interest rate from its steady-state level i . The monetary-policy rule subject to the ZLB is

$$i_t = \max\{-i, \phi_\pi \pi_t + \phi_x x_t\} \quad (622)$$

where $i = r = \frac{1}{\beta} - 1$ is the nominal interest rate in the zero-inflation steady state.

Thus the interest-rate deviation is bounded below by $-i$. In level terms, the lower bound is zero. The max operator introduces a non-linearity into the policy rule, illustrated in Fig. 50.

We now ask how to solve the three-equation NK model in the presence of this non-linearity. Recall that the efficient real interest rate maps one-for-one into the preference shock ψ_t :

$$r_t^* = -\psi_t$$

where ψ_t is no longer modelled as an AR(1) process. Recall that ψ_t is the source of our demand-driven shocks. Following Eggertsson and Woodford (2003), we instead let the efficient real interest rate take one of two values: a crisis state or a normal state.

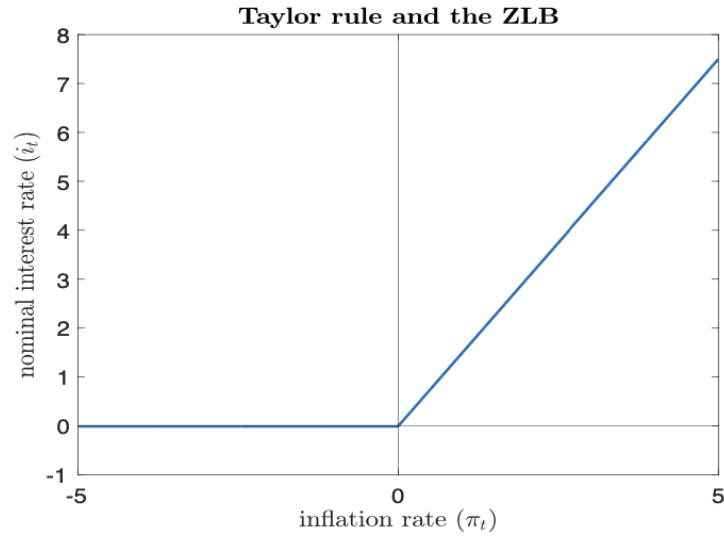


Figure 50: Interest rates with zero lower bound.

Assumption (Two-State Process for the Real Interest Rate)

At time t , the efficient real interest rate can take one of two values:

$$r_t^* = \begin{cases} 0, & \text{normal state,} \\ \tilde{r}_c < -i < 0, & \text{crisis state.} \end{cases} \quad (623)$$

In the crisis state, the efficient real rate is below $-i$, so the nominal-rate lower bound binds.

The shock to r_t^* is large enough to make the ZLB bind: the interest-rate deviation is then $i_t = -i$, corresponding to a level rate of zero. This two-state process for the efficient real rate also represents the demand shock $-\psi_t$.

At time t_0 , the economy is hit by a shock that places the efficient real interest rate at $\tilde{r}_c < 0$. The parameter μ introduced below is the probability that the crisis continues for one more period, not the probability of the initial shock.

Assumption (Economy is Hit With a Shock)

At time t_0 , the economy is hit with a shock so that the efficient real interest rate is at its crisis level:

$$r_{t_0}^* = \tilde{r}_c. \quad (624)$$

After the shock, the efficient real interest rate remains at $\tilde{r}_c$ from one period to the next with probability μ . With probability $1 - \mu$, it returns to zero and remains there. Thus zero is an **absorbing state**; this is a two-state Markov chain.

Assumption (Two-State Process for the Real Interest Rate)

Suppose that $r_{t_0}^* = \tilde{r}_c$. Conditional on still being in the crisis in period $t_0 + j - 1$, for $j > 0$,

$$\Pr(r_{t_0+j}^* = \tilde{r}_c \mid r_{t_0+j-1}^* = \tilde{r}_c) = \mu, \quad (625)$$

$$\Pr(r_{t_0+j}^* = 0 \mid r_{t_0+j-1}^* = \tilde{r}_c) = 1 - \mu. \quad (626)$$

Once r_t^* reaches zero, it remains there.

Hence, we are saying that at every period, if the real efficient interest rate is still at its crisis level $\tilde{r}_c$, there is a probability $(1 - \mu)$ that it will go back to its steady state.

Now, we want to look at the long-run and the short-run analysis of this model.

In the long-run, it is clear that at some time $t_0 + T$, the efficient real interest rate will eventually go back to its steady state and remain there:

$$r_{t_0+T}^* = 0.$$

The probability of still being in the crisis after T transitions is μ^T , which converges to zero as $T \rightarrow \infty$ when $0 \leq \mu < 1$. Once the economy exits, all variables remain at their steady-state levels, so their deviations are zero.

In the short run, we study the period between t_0 and the random exit date. The model has no endogenous state variables, so once the shock returns to zero, inflation and the output gap jump immediately to their steady-state values. Neither variable inherits a lagged value from the crisis. Because the model is linear within each regime, expectations can be formed as probability-weighted averages of the two possible next-period states.

Combining these facts, the economy either remains in the crisis or jumps back to its steady state. Expectations of the jump variables, π_t and x_t , are probability-weighted averages across these two states. During a crisis, expected future inflation and the expected future output gap therefore depend on the probability that the crisis continues.

We now want to analyse what is the aggregate demand and aggregate supply when the economy is in a crisis.

Theorem 543 (Aggregate Demand at the Zero Lower Bound)

When the economy is in a crisis, the output gap is given by:

$$(1 - \mu)x_c = r_c + \mu\pi_c \quad (627)$$

where μ is the per-period probability of remaining in the crisis, $r_c \equiv \tilde{r}_c + i < 0$ is the effective crisis demand disturbance after accounting for the lower bound, π_c is crisis inflation, and x_c is the crisis output gap.

Proof. First, we start off with the aggregate demand equation where we work with log utility so $\sigma = 1$:

$$x_t = -\left(\underbrace{i_t}_{(*)} - \underbrace{\mathbb{E}_t[\pi_{t+1}]}_{(**)} - \underbrace{r_t^*}_{(***)} + \underbrace{\mathbb{E}_t[x_{t+1}]}_{(****)} \right).$$

We want to solve for the value of the output gap during a crisis, which we denote by x_c . There are a few things we need to consider. We first write out what is the output gap during a crisis and then explain it:

$$x_c = -\left\{ \underbrace{-i}_i - \left[\underbrace{\mu \times \pi_c}_{(ii)} + \underbrace{(1 - \mu) \times 0}_{iii} \right] - \underbrace{\tilde{r}_c}_{iv} \right\} + \left[\underbrace{\mu \times x_c}_v + \underbrace{(1 - \mu) \times 0}_{vi} \right]. \quad (628)$$

We now explain where each term comes from.

(i). During a crisis, the nominal interest-rate deviation hits its lower bound, $i_t = -i$. Thus the first term inside braces in Eq. (628) is $-i$.

(ii) + (iii). This term comes from the future expected inflation (**) in the AD equation. In the next period, we know that there is a probability μ that we remain in a crisis, and in that case, inflation will remain at its crisis level π_c . There is also the scenario with probability $(1 - \mu)$ that the economy goes back to its steady state level and inflation goes back to its log-deviation steady state level of 0. Therefore, we have that:

$$(**) = \mathbb{E}_t[\pi_{t+1}] = \underbrace{\mu \times \pi_c}_{(ii)} + \underbrace{(1 - \mu) \times 0}_{iii}.$$

(iv). This corresponds to the efficient real interest rate ($***$) in the AD equation. At time t , the economy is currently in crisis so the efficient real interest is equal to its crisis level:

$$(***) = r_t^* = \tilde{r}_c.$$

(v) + (vi). This term comes from the future expected output gap ($****$) in the AD equation. In the next period, we know that there is a probability μ that we remain in a crisis, and in that case, the output gap will remain at its crisis level x_c . There is also the scenario with probability $(1 - \mu)$ that the economy goes back to its steady state level and output gap goes back to its log-deviation steady state level of 0. Therefore, we have that:

$$(***) = \mathbb{E}_t[x_{t+1}] = \underbrace{\mu \times x_c}_{(v)} + \underbrace{(1 - \mu) \times 0}_{vi}.$$

Combining all this together, we get:

$$x_c = - \left\{ \underbrace{-i}_i - \left[\underbrace{\mu \times \pi_c}_{(ii)} + \underbrace{(1 - \mu) \times 0}_{iii} \right] - \underbrace{\tilde{r}_c}_{iv} \right\} + \left[\underbrace{\mu \times x_c}_v + \underbrace{(1 - \mu) \times 0}_{vi} \right].$$

We then simplify to get:

$$x_c = - \left\{ -i - \left[\mu \times \pi_c + (1 - \mu) \times 0 \right] - \tilde{r}_c \right\} + \left[\mu \times x_c + (1 - \mu) \times 0 \right].$$

which becomes:

$$x_c = - \left\{ -i - \left[\mu \times \pi_c \right] - \tilde{r}_c \right\} + \left[\mu \times x_c \right].$$

Now, we get:

$$x_c = i + \mu \times \pi_c + \tilde{r}_c + \mu \times x_c$$

and then:

$$x_c - \mu x_c = \mu \pi_c + \tilde{r}_c + i.$$

We then define $r_c \equiv \tilde{r}_c + i < 0$ to get:

$$x_c - \mu x_c = \mu \pi_c + r_c.$$

We can then factorise to get the desired result:

$$(1 - \mu)x_c = r_c + \mu \pi_c$$

where $r_c \equiv \tilde{r}_c + i < 0$. □

Theorem 544 (Aggregate Supply at the Zero Lower Bound)

When the economy is in a crisis, inflation is given by:

$$(1 - \beta\mu)\pi_c = \kappa x_c. \quad (629)$$

where μ is the per-period probability of the economy remaining in a crisis, β is the household discount factor, κ is the slope of the Phillips curve, x_c is the output gap during the crisis and π_c is inflation during the crisis.

Proof. The argument is similar to the aggregate-demand case. Start from the aggregate-supply equation

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t[\pi_{t+1}]$$

where we switched off the cost-push shock $u_t = 0$.

We first write out the end result and explain how we derived it:

$$\pi_c = \underbrace{\kappa x_c}_* + \beta \left[\underbrace{\mu \times \pi_c + (1 - \mu) \times 0}_{(**)} \right]$$

(*). Now, during the crisis, looking at the aggregate supply equation, the output gap x_t will be the crisis level so:

$$x_t = x_c.$$

(**). Look at the aggregate supply equation, we evaluate the expectation inflation term. There is a probability of μ that the economy remains in a crisis and inflation is at its crisis level π_c . There is a probability $(1 - \mu)$ that the economy goes back to the steady state so the log-deviation of inflation goes to 0:

$$\mathbb{E}_t[\pi_{t+1}] = \mu \times \pi_c + (1 - \mu) \times 0.$$

Now, we arrived at:

$$\pi_c = \underbrace{\kappa x_c}_* + \beta \left[\underbrace{\mu \times \pi_c + (1 - \mu) \times 0}_{(**)} \right]$$

and simplify to get:

$$\pi_c = \kappa x_c + \beta \left[\mu \times \pi_c \right]$$

Then re-arrange to get the result. □

We take our expressions for the aggregate demand and supply during the crisis and notice that we have two variables, the output gap x_c and inflation π_c during a crisis, and two equations.

$$(1 - \mu)x_c = r_c + \mu\pi_c$$

$$(1 - \beta\mu)\pi_c = \kappa x_c.$$

We can therefore solve for the crisis output gap and inflation to get:

$$x_c = \frac{(1 - \beta\mu)r_c}{(1 - \mu)(1 - \beta\mu) - \kappa\mu} \quad (630)$$

$$\pi_c = \frac{\kappa r_c}{(1 - \mu)(1 - \beta\mu) - \kappa\mu}. \quad (631)$$

Finally, recall that $r_c < 0$. The numerators of both expressions are therefore negative. For x_c and π_c also to be negative, we assume that their common denominator is **positive**.

Assumption (Output Gap and Inflation Are Negative During a Crisis)

We assume that:

$$(1 - \mu)(1 - \beta\mu) - \kappa\mu > 0. \quad (632)$$

As a result, the output gap and inflation during a crisis are negative:

$$x_c < 0, \quad \pi_c < 0. \quad (633)$$

We can now represent these equations in a graphical format. In particular, we can write these equations as inflation

π_c in terms of the output gap x_c . Therefore, take our two equations:

$$\begin{aligned}(1 - \mu)x_c &= r_c + \mu\pi_c \\ (1 - \beta\mu)\pi_c &= \kappa x_c.\end{aligned}$$

and solve for them in terms of inflation as a function of the output gap.

Definition 545 (Graphical Representation of the Crisis AD and AS Curves)

For $\mu > 0$, the AD and AS curves can be represented as

$$\pi_c = -\frac{1}{\mu}r_c + \frac{1 - \mu}{\mu}x_c \quad \text{AD Curve} \quad (634)$$

$$\pi_c = \frac{\kappa}{1 - \beta\mu}x_c \quad \text{AS Curve} \quad (635)$$

These equations describe straight lines in (x_c, π_c) space. The AS curve passes through the origin, while the AD curve has a positive intercept $-r_c/\mu$ (recall that $r_c < 0$). Both curves are **positively sloped**. Equilibrium is characterised by their intersection.

Proposition 546 (AD Curve and AS Curve)

The Aggregate Demand Curve is Steeper than Aggregate Supply Curve

Proof. Recall the two linear equations:

$$\begin{aligned}\pi_c &= -\frac{1}{\mu}r_c + \frac{1 - \mu}{\mu}x_c & \text{AD} \\ \pi_c &= \frac{\kappa}{1 - \beta\mu}x_c & \text{AS}\end{aligned}$$

The AD curve is steeper than the AS curve if

$$\frac{1 - \mu}{\mu} > \frac{\kappa}{1 - \beta\mu}.$$

Notice that if you re-arrange this, you get:

$$(1 - \mu)(1 - \beta\mu) - \kappa\mu > 0.$$

This is precisely the assumption imposed above. Hence the AD curve is steeper than the AS curve. $\square$

Now, we graphically show what these curves look like:

$$\begin{aligned}\pi_c &= -\frac{1}{\mu}r_c + \frac{1 - \mu}{\mu}x_c & \text{AD} \\ \pi_c &= \frac{\kappa}{1 - \beta\mu}x_c & \text{AS.}\end{aligned}$$

In Fig. 51, the AD curve is steeper and has a positive vertical intercept, while the AS curve is flatter and passes through the origin. The two equations intersect at (x_c, π_c) , the equilibrium at the ZLB. Both the output gap and inflation are negative in this equilibrium.

In Fig. 52, set $\mu = 0$, so the economy returns to the steady state with probability one in the next period. The shock at t_0 has no persistence. It is best to use the original AD equation at $\mu = 0$, because the slope-intercept form

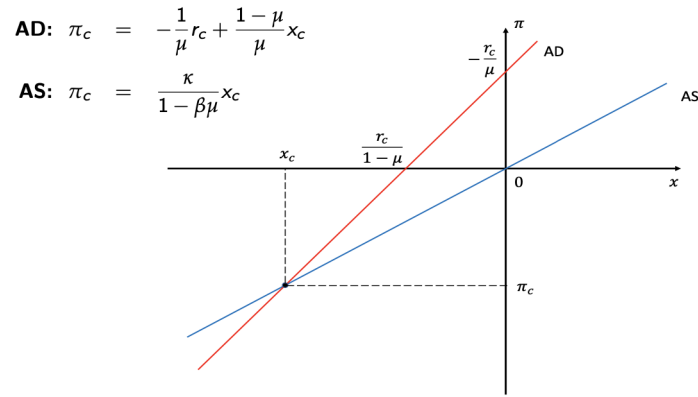


Figure 51: AD and AS curve during crisis.

No Shock Persistence ($\mu = 0$)

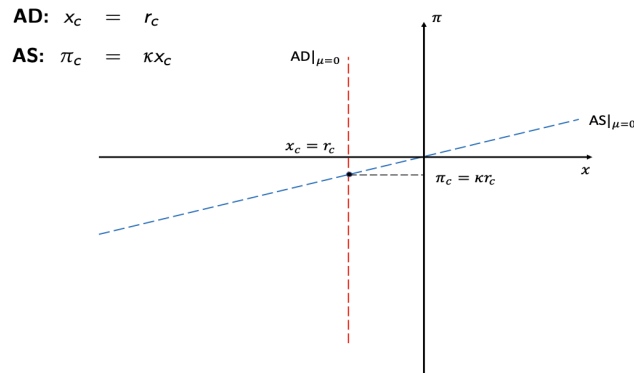


Figure 52: AD and AS curve during crisis when $\mu = 0$.

The Effect of Higher Persistence ($\mu \uparrow$)

- AS becomes steeper rotating around origin
- AD from vertical to negatively sloped
 - ▶ Horizontal intercept more negative
 - ▶ Vertical intercept decreases

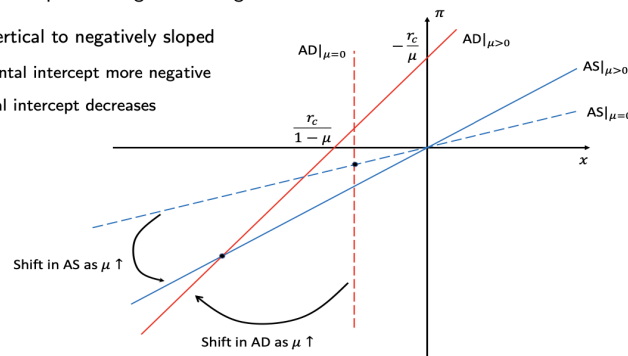


Figure 53: Crisis AD and AS curves as shock persistence μ varies.

divides by μ . The AD–AS equations become

$$\begin{array}{ll}
 x_c = r_c & \text{AD} \\
 \pi_c = \kappa x_c & \text{AS.}
 \end{array}$$

The shock r_c therefore pins down x_c , so the AD curve is vertical; the AS curve then determines inflation.

In Fig. 53, we see the effects of increasing the probability of the economy remaining in the crisis as μ increases. This can also be interpreted as saying the persistence of the demand shock increases.

Theorem 547 (Persistence of the Shock and the AS–AD Curves)

As the persistence of the shock increases, $\mu \uparrow$, the AS curve becomes steeper by rotating around the origin. The AD curve remains positively sloped but becomes flatter: its slope $(1 - \mu)/\mu$ and vertical intercept $-r_c/\mu$ both fall, while its horizontal intercept $r_c/(1 - \mu)$ becomes more negative.

The AS curve rotates around the origin, while changes in the AD curve affect both its slope and intercept. The crisis equilibrium (x_c, π_c) becomes more negative as persistence rises. Hence, a higher probability of remaining in the crisis makes the recession more severe under the parameter restriction above.

The key takeaway is that greater shock persistence worsens households' expectations and amplifies the recession.

25 Unconventional Monetary Policy

For this section, suppose that the economy is at the zero lower bound. Because the ZLB constrains conventional interest-rate policy, we consider two alternative tools: forward guidance and quantitative easing.

25.1 Forward Guidance

We first define forward guidance.

Definition 548 (Forward Guidance)

Forward guidance is a central-bank announcement about the future path of nominal interest rates.

There are two types of forward guidance:

1. Time-contingent: Announce the path of interest rates over a specified period. For example: *we will keep interest rates low for two years.*
2. State-contingent: Condition the path of rates on an economic indicator. For example: *we will keep interest rates low until unemployment reaches a specified threshold.*

Now, to see the impact of forward guidance, first recall the aggregate demand equation:

$$x_t = -\sigma^{-1}(i_t - \mathbb{E}_t[\pi_{t+1}] - r_t^*) + \mathbb{E}_t[x_{t+1}].$$

Assuming stationarity and the usual boundedness condition, we can solve the aggregate-demand equation forward to obtain

$$x_t = -\sigma^{-1}\mathbb{E}_t\left[\sum_{j=0}^{\infty}(\textcolor{red}{i}_{t+j} - \pi_{t+j+1} - r_{t+j}^*)\right]. \quad (636)$$

Expected future nominal interest rates (the term in red) affect the current output gap x_t . Hence, a credible announcement that future nominal rates will remain low stimulates current activity through aggregate demand.

To see the effect on inflation, recall the Phillips curve:

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t[\pi_{t+1}] + u_t.$$

Solving this equation forward gives

$$\pi_t = \mathbb{E}_t\left[\sum_{j=0}^{\infty} \beta^j (\kappa \textcolor{red}{x}_{t+j} + u_{t+j})\right]. \quad (637)$$

The current output gap x_t directly affects current inflation. Thus, if forward guidance raises x_t , it also raises π_t . Higher inflation further lowers the real interest rate and amplifies the increase in the output gap.

Del Negro, Giannoni, and Patterson (2012) compare the model's forward-guidance effects with the data. The standard model predicts much larger responses than are typically estimated empirically, producing the forward-guidance puzzle.

Definition 549 (Forward Guidance Puzzle)

The forward-guidance puzzle is the standard model's prediction that distant interest-rate announcements have very large effects on current output and inflation, effects that are not generally observed in the data.

Reflecting on why we have this discrepancy between the model and the data, if we look at the aggregate demand equation:

$$x_t = -\sigma^{-1} \mathbb{E}_t \left[\sum_{j=0}^{\infty} (i_{t+j} - \pi_{t+j+1} - r_{t+j}^*) \right]$$

future nominal interest rates are **not** discounted. Consequently, an announcement about the nominal rate ten years ahead can have the same marginal effect on today's output gap as an announcement about the near-term rate. This is the source of the model's implausibly strong response.

Another way to view the mechanism is through long-term interest rates.

Definition 550 (Long Term Interest Rate)

The long term interest rate is:

$$i_{N_t} = \frac{1}{N} \mathbb{E}_t \left[\sum_{j=0}^{N-1} i_{t+j} \right] \quad (638)$$

which, abstracting from term premia, is the expected average of short-term interest rates i_{t+j} .

In the model, forward guidance lowers long-term interest rates and stimulates demand. Empirically, this channel appears weaker than the benchmark model predicts.

To summarise so far, the issue is that there is a big difference between what the model would suggest would occur (really big effects) compared to the data (small effects). The natural way to resolve this is to mitigate the effect of the announcement by adding in a discounting term δ in the forward looking solution for the current output gap.

Proposition 551 (Resolution of Forward Guidance Puzzle)

A way to resolve the forward guidance puzzle is to discount future interest rates:

$$x_t = -\sigma^{-1} \mathbb{E}_t \left[\sum_{j=0}^{\infty} \delta^j (i_{t+j} - \pi_{t+j+1} - r_{t+j}^*) \right]. \quad (639)$$

where $0 < \delta < 1$.

Proof. Add a discount term into the current output gap:

$$x_t = -\sigma^{-1} (i_t - \mathbb{E}_t[\pi_{t+1}] - r_t^*) + \delta \mathbb{E}_t[x_{t+1}]$$

and iterate forward to solve the equation. □

The discount factor δ reduces the effect of distant interest-rate announcements on the current output gap. Thus forward guidance can still stimulate demand at the ZLB, but its effect is attenuated relative to the standard New Keynesian model.

25.2 Quantitative Easing

We now move onto our second tool for escaping the ZLB: Quantitative Easing.

Definition 552 (Quantitative Easing)

Quantitative easing is the central bank purchasing long-term government bonds from financial institutions.

When the central bank purchases long-term government bonds, the additional demand tends to raise their price and lower their yield. Lower long-term interest rates can stimulate consumption and investment. Abstracting from term premia, no-arbitrage also links the long-term interest rate i_{N_t} to expected short-term rates.

Definition 553 (Long Term Interest Rate)

An N-period long-term interest rate is given by:

$$i_{N_t} = \frac{1}{N} \sum_{j=0}^{N-1} \mathbb{E}_t[i_{t+j}] = \frac{i_t + \mathbb{E}_t[i_{t+1}] + \mathbb{E}_t[i_{t+2}] + \cdots + \mathbb{E}_t[i_{t+N-1}]}{N} \quad (640)$$

which is the expected average of short-term interest rates.

Forward guidance influences the right-hand side through expected short-term interest rates $\mathbb{E}_t[i_{t+j}]$. Quantitative easing instead aims to lower the long-term yield directly by raising the price of long-term bonds.

We now want to model quantitative easing. But before we proceed, we have an issue. If we tried to model quantitative easing by introducing risky and safe assets, the RBC model would show that the ownership of these assets is irrelevant. That is, in an economy, it doesn't matter who owns the risky or safe assets in particular. This is known as the Wallace irrelevance theorem.

Proposition 554 (Wallace Irrelevance)

The Euler equations of a baseline RBC model without cash give the first-order conditions

$$\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \frac{1 + i_t}{\Pi_{t+1}} \right] \quad (641)$$

$$\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \left(\alpha A_{t+1} K_t^{\alpha-1} N_{t+1}^{1-\alpha} + 1 - \delta \right) \right] \quad (642)$$

$$(1 - \alpha) A_t K_{t-1}^{\alpha} N_t^{-\alpha} = N_t^{\psi} C_t \quad (643)$$

$$A_t K_{t-1}^{\alpha} N_t^{1-\alpha} = C_t + K_t - (1 - \delta) K_{t-1} + G_t. \quad (644)$$

where capital is risky and government bonds are safe. The **quantity of safe assets and the ownership of capital are irrelevant for the equilibrium allocation**: bond quantities do not appear in these equilibrium conditions.

Proof. No need to derive this, we just need to know that this result exists. □

In a frictionless benchmark, a central-bank asset swap does not change private net wealth or the equilibrium allocation. Introducing money demand alone does not overturn this irrelevance result:

$$\frac{M_t}{P_t} = \chi \frac{1 + i_t}{i_t} C_t. \quad (645)$$

At the ZLB, money and short-term government bonds are particularly close substitutes because neither pays interest. To give QE real effects, the model therefore needs an additional friction that makes short- and long-term assets imperfect substitutes.

We now turn to a different approach to modelling quantitative easing. To proceed, we need a way to model issuing bonds since bonds can be of different maturities. To do so, we suppose that a bond that is issued is a perpetuity, which is a bond that never matures and pays an exponentially decaying coupon.

Definition 555 (Perpetuity)

A bond issued at time t pays the coupon χ^{s-1} at time $t + s$, for $s \geq 1$ and $0 \leq \chi < 1$.

The bond pays 1 in the first period, χ in the next, then χ^2 , and so on. Thus a bond bought j periods ago pays χ^j today. We can aggregate the payments promised by all outstanding perpetuities.

Definition 556 (Total Dollar Payment to Households from Holding Long-term Bonds)

The total dollar payment to households at time $t + 1$ from holding long-term bonds is:

$$\tilde{B}_{Lt} = NB_{Lt} + \chi NB_{Lt-1} + \chi^2 NB_{Lt-2} + \cdots = \sum_{j=0}^{\infty} \chi^j NB_{Lt-j} \quad (646)$$

where NB_{Lt} are purchases of long-term bonds issued at time t .

To interpret this expression,

$$\tilde{B}_{Lt} = NB_{Lt} + \chi NB_{Lt-1} + \chi^2 NB_{Lt-2} + \cdots = \sum_{j=0}^{\infty} \chi^j NB_{Lt-j}$$

the left-hand side tells us the total payment the household has received from all the bonds they hold. The first term on the right-hand side, NB_{Lt} tells us how much of a new bond at time t they have purchased. The second term, χNB_{Lt-1} tells us the amount they earn from bonds they bought a period ago, which is χ times the amount of bonds they bought in the last period. We then do this for every period in the past as the bonds are perpetuities.

Now, we can write the household's budget constraint.

Definition 557 (Household Budget Constraint)

The household budget constraint is:

$$P_t C_t + B_t^h + V_t \cdot NB_{Lt}^h = W_t N_t + (1 + i_{t-1}) B_{t-1}^h + \tilde{B}_{Lt-1}^h - T_t \quad (647)$$

where V_t is the price of a long-term bond at time t , NB_{Lt}^h is the number of newly issued bonds purchased by the household, $W_t N_t$ is labour income, and $\tilde{B}_{Lt-1}^h$ is the period- t payment on previously acquired long-term bonds. The superscript h denotes household holdings.

Now, this household budget constraint can be written.

$$NB_{Lt} = \tilde{B}_{Lt} - \chi \tilde{B}_{Lt-1}.$$

Proposition 558 (Simplified Household Budget Constraint)

The household budget constraint can be rewritten as:

$$P_t C_t + B_t^h + B_{Lt}^h = W_t N_t + (1 + i_{t-1}) B_{t-1}^h + (1 + i_{Lt}) B_{Lt-1}^h - T_t \quad (648)$$

where $B_{Lt}^h \equiv V_t \tilde{B}_{Lt}^h$ is the value of household long-term bond holdings and $1 + i_{Lt} = \frac{1 + \chi V_t}{V_{t-1}}$ is the gross return on a long-term bond.

Proof. First, take the budget constraint:

$$P_t C_t + B_t^h + V_t \cdot NB_{Lt}^h = W_t N_t + (1 + i_{t-1}) B_{t-1}^h + \tilde{B}_{Lt-1}^h - T_t.$$

Note that:

$$\tilde{B}_{Lt} - \chi \tilde{B}_{Lt-1} = \sum_{j=0}^{\infty} \chi^j NB_{Lt-j} - \chi \sum_{j=0}^{\infty} \chi^j NB_{Lt-1-j} = NB_{Lt}$$

where NB_{Lt} are purchases of long-term bonds issued at time t . Plug this into the budget constraint:

$$P_t C_t + B_t^h + V_t \tilde{B}_{Lt}^h = W_t N_t + (1 + i_{t-1}) B_{t-1}^h + (1 + \chi V_t) \tilde{B}_{Lt-1}^h - T_t.$$

Then, divide and multiply V_{t-1} on the right-hand side term:

$$P_t C_t + B_t^h + V_t \tilde{B}_{Lt}^h = W_t N_t + (1 + i_{t-1}) B_{t-1}^h + \frac{V_{t-1}}{V_{t-1}} (1 + \chi V_t) \tilde{B}_{Lt-1}^h - T_t.$$

Now, define the variables:

$$B_{Lt} \equiv V_t \tilde{B}_{Lt}^h$$

$$1 + i_{Lt} = \frac{1 + \chi V_t}{V_{t-1}}$$

and plug these in to get:

$$P_t C_t + B_t^h + B_{Lt}^h = W_t N_t + (1 + i_{t-1}) B_{t-1}^h + (1 + i_{Lt}) B_{Lt-1}^h - T_t.$$

□

We can define the utility function for the household.

Definition 559 (Household Utility Function)

The household utility function is:

$$U = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{\tilde{v}_b}{2} \left(\frac{B_t^h}{B_{Lt}^h} - \delta \right)^2 - \frac{N_t^{1+\phi}}{1+\phi} \right) \right] \quad (649)$$

where $\tilde{v}_b > 0$ and $\delta \equiv \frac{B^h}{B_L^h}$.

To interpret this, it says that households want to always hold a certain ratio of short term to long term bonds, which is reflected by the

$$\left(\frac{B_t^h}{B_{Lt}^h} - \delta \right)^2$$

term. There are many reasons why a household would want to do this, such as risk diversification.

Since we have a utility function and a budget constraint, we can derive the Euler equations for bonds.

Proposition 560 (Bond Euler Equations)

The short-term and long-term bond Euler equations are:

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[\left(C_{t+1}^{-\sigma} \frac{1+i_t}{\Pi_{t+1}} \right) \right] - \frac{\tilde{\nu}_b}{B_{Lt}^h/P_t} \left(\frac{B_t^h/P_t}{B_{Lt}^h/P_t} - \delta \right) \quad \text{Short-term Bonds} \quad (650)$$

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[\left(C_{t+1}^{-\sigma} \frac{1+i_{Lt+1}}{\Pi_{t+1}} \right) \right] + \frac{\tilde{\nu}_b}{B_{Lt}^h/P_t} \left(\frac{B_t^h/P_t}{B_{Lt}^h/P_t} - \delta \right) \frac{B_t^h/P_t}{B_{Lt}^h/P_t} \quad \text{Long-term Bonds} \quad (651)$$

One key thing to note is this new term:

$$\left(\frac{B_t^h/P_t}{B_{Lt}^h/P_t} - \delta \right)$$

This term captures how an additional unit of short-term bonds changes the household's preferred ratio of short- to long-term holdings. Central-bank purchases alter private portfolio composition and therefore enter the household Euler equations through this term.

Now, we can take a log-linear approximation of the Euler equations around the steady-state ratio of short to long term debt, which we denote by δ .

Proposition 561 (Log-Linear Approximations)

The log-linear approximations around the steady-state ratio $\frac{B_t^h}{B_{Lt}^h} = \delta$ are

$$\sigma c_t = \sigma \mathbb{E}_t[c_{t+1}] - i_t + \mathbb{E}_t[\pi_{t+1}] + \nu_b(b_t^h - b_{Lt}^h) \quad \text{Short-term Bonds} \quad (652)$$

$$\sigma c_t = \sigma \mathbb{E}_t[c_{t+1}] - \mathbb{E}_t[i_{Lt+1}] + \mathbb{E}_t[\pi_{t+1}] - \nu_b \delta (b_t^h - b_{Lt}^h) \quad \text{Long-term Bonds} \quad (653)$$

where $\nu_b \equiv \tilde{\nu}_b \delta C^\sigma / (B_L^h/P)$.

We now want to define a term that will be helpful for us in terms of our analysis in seeing the effect of QE.

Definition 562 (Term Premium)

The term premium is the difference between long-term and short-term interest rates:

$$\mathbb{E}_t[i_{Lt+1}] - i_t = \nu_b(1 + \delta)(b_{Lt}^h - b_t^h). \quad (654)$$

Here, in the term premium, the term:

$$(b_{Lt}^h - b_t^h)$$

shows that the term premium depends on private portfolio composition. We can now see how quantitative easing affects it.

Theorem 563 (Quantitative Easing Reduces the Term Premium)

Quantitative easing reduces the term premium by purchasing long-term debt in exchange for short-term liabilities.

Proof. Look at the term premium:

$$\mathbb{E}_t[i_{Lt+1}] - i_t = \nu_b(1 + \delta)(b_{Lt}^h - b_t^h).$$

Quantitative easing decreases the amount of long-term debt held by households, so b_{Lt}^h falls. The central bank finances these purchases with short-term liabilities, so b_t^h rises. Both changes reduce the term premium. $\square$

Proposition 564 (Euler Equation for Short Term Bonds with Term Premium)

The Euler equation for short term bonds can be written as:

$$c_t = \mathbb{E}_t[c_{t+1}] - \sigma^{-1} \left(\frac{\delta}{1+\delta} i_t + \frac{1}{1+\delta} \mathbb{E}_t[i_{L,t+1}] - \mathbb{E}_t[\pi_{t+1}] \right). \quad (655)$$

Proof. Start from the log-linear short-term bond Euler equation:

$$\sigma c_t = \sigma \mathbb{E}_t[c_{t+1}] - i_t + \mathbb{E}_t[\pi_{t+1}] + \nu_b(b_t^h - b_{L,t}^h).$$

The term-premium equation implies

$$\nu_b(b_t^h - b_{L,t}^h) = \frac{i_t - \mathbb{E}_t[i_{L,t+1}]}{1+\delta}.$$

Substitution and division by σ give the stated weighted-rate Euler equation. $\square$

Looking at the Euler equation for short-term bonds:

$$c_t = \mathbb{E}_t[c_{t+1}] - \sigma^{-1} \left(\frac{\delta}{1+\delta} i_t + \frac{1}{1+\delta} \mathbb{E}_t[i_{L,t+1}] - \mathbb{E}_t[\pi_{t+1}] \right)$$

both short- and long-term interest rates now affect consumption. At the ZLB, the short-term rate is zero in levels (equivalently, $-i$ under the deviation convention used above), but long-term rates need not be zero. The central bank can therefore try to lower $\mathbb{E}_t[i_{L,t+1}]$ through quantitative easing and thereby stimulate demand.

We now want to look at the government's role in this. One important thing we do now is that we will separate the central bank from the government. That is, we will now have a central bank explicitly in our model buying government debt.

Definition 565 (Government Budget Constraint and Central Bank Transfers)

The government budget constraint is:

$$B_t + V_t N B_{L,t} = (1 + i_{t-1}) B_{t-1} + \tilde{B}_{L,t-1} - Z_t - T_t \quad (656)$$

$$Z_t = (1 + i_{t-1}) B_{t-1}^{cb} + \tilde{B}_{L,t-1}^{cb} - B_t^{cb} - V_t N B_{L,t}^{cb}. \quad (657)$$

where B_t is short-term debt, $N B_{L,t}$ is newly issued long-term debt, $\tilde{B}_{L,t-1}$ is the payment to the private sector, Z_t denotes transfers from the central bank, and T_t denotes taxes. The variable B_{t-1}^{cb} is government debt held by the central bank.

As before, we can rewrite these equations.

Proposition 566 (Government Budget And Central Bank Transfer)

We can rewrite the government budget constraint as:

$$B_t + B_{L,t} = (1 + i_{t-1}) B_{t-1} + (1 + i_{L,t}) B_{L,t-1} - Z_t - T_t. \quad (658)$$

We can rewrite the central bank's transfer as:

$$Z_t = (1 + i_{t-1}) B_{t-1}^{cb} + (1 + i_{L,t}) B_{L,t-1}^{cb} - B_t^{cb} - B_{L,t}^{cb}. \quad (659)$$

Proof. See problem set. $\square$

To proceed, we need some assumptions.

Assumption (Bond Market Clearing)

Total debt equals debt owned by both the household and central bank and the stream of payments made NB_{Lt} goes to either the household or central bank:

$$B_t^h + B_t^{cb} = B_t \quad (660)$$

$$NB_{Lt} = NB_{Lt}^h + NB_{Lt}^{cb} \quad (661)$$

Assumption (Fiscal Rules)

The government keeps short and long-term debt constant in real terms:

$$\frac{B_t}{P_t} = b \quad (662)$$

$$\frac{B_{Lt}}{P_t} = \delta_f b, \quad (663)$$

where $\delta_f > 0$ is the fiscal long-to-short debt ratio. It is distinct from the household's preferred short-to-long portfolio ratio δ .

Assumption (Asset Purchase Rule)

There are no level effects on the balance sheet, only composition effects:

$$\frac{B_{Lt}^{cb}}{P_t} + \frac{B_t^{cb}}{P_t} = \mathcal{B}. \quad (664)$$

The central bank funds long-term asset purchases with short-term liabilities.

With all this, Chen, Curdia and Ferrero (2012) aimed to test the effects of QE. They do so by looking at the short-term bond Euler equation we derived earlier:

$$c_t = \mathbb{E}_t[c_{t+1}] - \sigma^{-1} \left(\frac{\delta}{1+\delta} i_t + \frac{1}{1+\delta} \mathbb{E}_t[i_{Lt+1}] - \mathbb{E}_t[\pi_{t+1}] \right). \quad (665)$$

The key takeaway from the paper is that they showed that there appears to be small marginal effects of quantitative easing. Hence, this is another puzzle as quantitative easing appears to work when we look at the empirical data, but doesn't seem to work so well in models.

26 Vector Autoregressions

In this section, we study dynamic relationships among macroeconomic variables. A VAR does not establish causality by itself; causal interpretation requires additional identifying assumptions. To see the underlying problem, suppose we observe y_t and x_t . If x_t is **exogenous** (and satisfies the relevant orthogonality condition), we can estimate

$$y_t = \alpha + \beta x_t + \epsilon_t \quad (666)$$

and obtain a consistent OLS estimator under the usual time-series assumptions. In macroeconomics, however, x_t is often endogenous. For example, output and interest rates may respond jointly to other shocks. A regression containing only these two variables can therefore be misleading. Vector autoregressions (VARs), combined with explicit identification restrictions, provide a framework for studying these interactions.

26.1 Structural VAR

We first walk through a small toy example before generalising. Suppose we are interested in the impact of monetary policy nominal interest rates mp_t on output y_t . Now, we saw that in the New Keynesian model, output (gap) depends on nominal interest rates through the aggregate demand equation and nominal interest rates are set according to the Taylor rule which depends on output (gap). For now, let us just assume that output today is affected by output yesterday, nominal interest rates today and yesterday. Likewise, let us assume that nominal interest rates will be determined by yesterday's nominal interest rate (persistence) and output today and yesterday. Therefore, we may be interested in estimating the equation:

$$\begin{aligned} y_t &= b_{10} + b_{12}mp_t + \gamma_{11}y_{t-1} + \gamma_{12}mp_{t-1} + \epsilon_{yt} \\ mp_t &= b_{20} + b_{21}y_t + \gamma_{21}y_{t-1} + \gamma_{22}mp_{t-1} + \epsilon_{mt} \end{aligned}$$

where $\epsilon_{yt}, \epsilon_{mt}$ are our usual error terms.

For notational convenience, parameterise the contemporaneous effects with minus signs:

$$\begin{aligned} y_t &= b_{10} - b_{12}mp_t + \gamma_{11}y_{t-1} + \gamma_{12}mp_{t-1} + \epsilon_{yt} \\ mp_t &= b_{20} - b_{21}y_t + \gamma_{21}y_{t-1} + \gamma_{22}mp_{t-1} + \epsilon_{mt} \end{aligned}$$

This is only a sign convention; the estimated coefficients themselves need not be positive.

These two structural equations contain a simultaneity problem. In the first equation, y_t depends on mp_t , while in the second, mp_t depends on y_t . The contemporaneous regressors are therefore jointly determined with the dependent variables, so equation-by-equation OLS is inconsistent.

First, let us move all variables at time t to the left-hand side:

$$\begin{aligned} y_t + b_{12}mp_t &= b_{10} + \gamma_{11}y_{t-1} + \gamma_{12}mp_{t-1} + \epsilon_{yt} \\ b_{21}y_t + mp_t &= b_{20} + \gamma_{21}y_{t-1} + \gamma_{22}mp_{t-1} + \epsilon_{mt} \end{aligned}$$

Now, notice that we can now write this system in terms of matrices:

$$\begin{bmatrix} 1 & b_{12} \\ b_{21} & 1 \end{bmatrix} \begin{bmatrix} y_t \\ mp_t \end{bmatrix} = \begin{bmatrix} b_{10} \\ b_{20} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ mp_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{yt} \\ \epsilon_{mt} \end{bmatrix} \quad (667)$$

This is a **structural vector autoregression** (SVAR). It specifies **contemporaneous relationships between variables**, which means that the structural coefficients cannot be identified from reduced-form regressions without additional restrictions. We also assume that the **structural innovations** ϵ_{yt} and ϵ_{mt} have mean zero and are mutually uncorrelated. This orthogonality gives the shocks an economically useful interpretation in impulse-response analysis.

Now, we impose some assumptions.

Assumption (Stationarity)

Assume that y_t and mp_t are stationary processes.

Let us define the matrix of error terms as:

$$\epsilon_t = \begin{bmatrix} \epsilon_{yt} \\ \epsilon_{mt} \end{bmatrix}.$$

Assumption (Structural Error Terms Have Mean Zero and Are Uncorrelated)

Assume that the structural error terms have mean zero:

$$\mathbb{E}[\epsilon_t] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Assume that the structural errors are mutually uncorrelated, so their covariance matrix is diagonal:

$$\Phi \equiv \mathbb{E}[\epsilon_t \epsilon_t^T] = \begin{bmatrix} \sigma_y^2 & 0 \\ 0 & \sigma_m^2 \end{bmatrix}$$

where σ_y^2, σ_m^2 are the variances of the error terms $\epsilon_{yt}, \epsilon_{mt}$ respectively.

Remark 567. For the covariance term, remember that if $\mathbb{E}[X] = 0$, then the variance is:

$$\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X^2] - 0.$$

So let's go back to our structural VAR:

$$\begin{bmatrix} 1 & b_{12} \\ b_{21} & 1 \end{bmatrix} \begin{bmatrix} y_t \\ mp_t \end{bmatrix} = \begin{bmatrix} b_{10} \\ b_{20} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ mp_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{yt} \\ \epsilon_{mt} \end{bmatrix}$$

We can define some notation for our matrices:

$$\begin{aligned} x_t &= \begin{bmatrix} y_t \\ mp_t \end{bmatrix} \\ B &\equiv \begin{bmatrix} 1 & b_{12} \\ b_{21} & 1 \end{bmatrix} \\ \Gamma_0 &= \begin{bmatrix} b_{10} \\ b_{20} \end{bmatrix} \\ \Gamma_1 &= \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \\ \epsilon_t &= \begin{bmatrix} \epsilon_{yt} \\ \epsilon_{mt} \end{bmatrix} \end{aligned}$$

Therefore, we can rewrite our system compactly as:

$$B\mathbf{x}_t = \Gamma_0 + \Gamma_1\mathbf{x}_{t-1} + \boldsymbol{\epsilon}_t.$$

This is the compact representation of our **structural VAR**. The matrix B contains the contemporaneous relationships between y_t and mp_t , a key distinguishing feature of an SVAR. The structural innovations $\boldsymbol{\epsilon}_t$ are uncorrelated by assumption, as reflected in the diagonal covariance matrix Φ .

With all this, we can now define a structural VAR in a general case. First, we have only looked at a SVAR with 2 variables and 1 lag. Clearly, we can extend this to contain n variables and k lags, which we shall do.

Definition 568 (Structural VAR)

Let $\mathbf{X}_t$ be an $n \times 1$ vector. A structural VAR with k lags is

$$B\mathbf{X}_t = \Gamma_0 + \Gamma_1\mathbf{X}_{t-1} + \Gamma_2\mathbf{X}_{t-2} + \cdots + \Gamma_k\mathbf{X}_{t-k} + \boldsymbol{\epsilon}_t \quad (668)$$

where B is an $n \times n$ matrix, Γ_0 is an $n \times 1$ intercept, Γ_j is an $n \times n$ coefficient matrix for $j = 1, \dots, k$, and $\boldsymbol{\epsilon}_t$ is the vector of mutually uncorrelated structural innovations.

The structural form captures the economic relationships of interest, but it cannot be estimated without identifying restrictions. We first transform it into a reduced-form VAR.

26.2 Reduced Form VAR

The matrix B captures contemporaneous interactions in the SVAR. Premultiplying by B^{-1} produces a reduced form with only lagged variables on the right-hand side:

$$\mathbf{X}_t = B^{-1}\Gamma_0 + B^{-1}\Gamma_1\mathbf{X}_{t-1} + B^{-1}\Gamma_2\mathbf{X}_{t-2} + \cdots + B^{-1}\Gamma_k\mathbf{X}_{t-k} + B^{-1}\boldsymbol{\epsilon}_t.$$

Let us again redefine some notation:

$$\mathbf{A}_0 \equiv B^{-1}\Gamma_0$$

$$\mathbf{A}_j \equiv B^{-1}\Gamma_j$$

$$\mathbf{e}_t \equiv B^{-1}\boldsymbol{\epsilon}_t$$

This gives us the following equation:

$$\mathbf{X}_t = \mathbf{A}_0 + \mathbf{A}_1\mathbf{X}_{t-1} + \mathbf{A}_2\mathbf{X}_{t-2} + \cdots + \mathbf{A}_k\mathbf{X}_{t-k} + \mathbf{e}_t.$$

Now, look at the mean of this new error term, $\mathbf{e}_t$, we have that:

$$\mathbb{E}[\mathbf{e}_t] = \mathbb{E}[B^{-1}\boldsymbol{\epsilon}_t] = B^{-1}\mathbb{E}[\boldsymbol{\epsilon}_t] = 0$$

as the structural error term has mean zero $\mathbb{E}[\boldsymbol{\epsilon}_t] = 0$ by assumption. Now, look at the covariance of this new error

term:

$$\begin{aligned}
\mathbb{E}[\mathbf{e}_t \mathbf{e}_t^T] &= \mathbb{E}\left[(\mathbf{B}^{-1} \boldsymbol{\epsilon}_t)(\mathbf{B}^{-1} \boldsymbol{\epsilon}_t)^T\right] \\
&= \mathbb{E}\left[\mathbf{B}^{-1} \boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t^T (\mathbf{B}^{-1})^T\right] \\
&= \mathbf{B}^{-1} \mathbb{E}\left[\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t^T\right] (\mathbf{B}^{-1})^T \\
&= \mathbf{B}^{-1} \boldsymbol{\Phi} (\mathbf{B}^{-1})^T
\end{aligned}$$

where the last line uses $\boldsymbol{\Phi} \equiv \mathbb{E}[\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t^T]$. We define the covariance matrix of the reduced-form innovations as

$$\boldsymbol{\Omega} \equiv \mathbb{E}[\mathbf{e}_t \mathbf{e}_t^T] = \mathbf{B}^{-1} \boldsymbol{\Phi} (\mathbf{B}^{-1})^T.$$

Hence, to recap what we did, we took our structural VAR which had contemporaneous relationships and removed those relationships by inverting the $\mathbf{B}$ matrix to get:

$$\mathbf{X}_t = \mathbf{A}_0 + \mathbf{A}_1 \mathbf{X}_{t-1} + \mathbf{A}_2 \mathbf{X}_{t-2} + \cdots + \mathbf{A}_k \mathbf{X}_{t-k} + \mathbf{e}_t.$$

We then showed that the mean and covariance of this new error term are

$$\mathbb{E}[\mathbf{e}_t] = 0$$

$$\boldsymbol{\Omega} \equiv \mathbb{E}[\mathbf{e}_t \mathbf{e}_t^T] = \mathbf{B}^{-1} \boldsymbol{\Phi} (\mathbf{B}^{-1})^T$$

These three things taken together are known as a **reduced form VAR**.

Definition 569 (Reduced-Form VAR)

Let $\mathbf{X}_t$ be an $n \times 1$ vector. A reduced-form VAR with k lags is

$$\mathbf{X}_t = \mathbf{A}_0 + \mathbf{A}_1 \mathbf{X}_{t-1} + \mathbf{A}_2 \mathbf{X}_{t-2} + \cdots + \mathbf{A}_k \mathbf{X}_{t-k} + \mathbf{e}_t \quad (669)$$

where $\mathbf{A}_j \equiv \mathbf{B}^{-1} \boldsymbol{\Gamma}_j$ are the coefficient matrices and $\mathbf{e}_t \equiv \mathbf{B}^{-1} \boldsymbol{\epsilon}_t$ is the reduced-form innovation vector with mean zero and covariance matrix $\boldsymbol{\Omega}$.

Now, look carefully at the reduced form VAR:

$$\mathbf{X}_t = \mathbf{A}_0 + \mathbf{A}_1 \mathbf{X}_{t-1} + \mathbf{A}_2 \mathbf{X}_{t-2} + \cdots + \mathbf{A}_k \mathbf{X}_{t-k} + \mathbf{e}_t.$$

There are no contemporaneous endogenous regressors on the right-hand side. Under stability and the usual innovation-orthogonality assumptions, we can therefore estimate each of the n reduced-form equations by OLS and obtain

$$\hat{\mathbf{A}}_j$$

and also fitted residuals:

$$\hat{\mathbf{e}}_t.$$

Now, since we know the fitted residuals $\hat{\mathbf{e}}_t$, we can then also estimate the covariance of these fitted residuals (look

at Econometrics notes if you can't recall) to get an estimate of the covariance matrix for the reduced form errors:

$$\hat{\Omega}.$$

The reduced-form estimates $\hat{A}_j$ and $\hat{\Omega}$ are not yet the structural objects of interest. We want Γ_j and the structural innovation covariance $\Phi \equiv \mathbb{E}[\epsilon_t \epsilon_t^T]$. By definition,

$$\Omega = B^{-1} \Phi (B^{-1})^T$$

where Φ is the covariance term of the structural error term. So maybe, if we can figure out what the B matrix is, and given our reduced form OLS estimate $\hat{\Omega}$, we can then back out the estimate for the covariance matrix of the structural error term:

$$\hat{\Phi} = B \hat{\Omega} B^T.$$

Likewise, we know that the coefficient of the reduced form matrix A_j is related to the coefficient of the structural form matrix Γ_j by:

$$A_j = B^{-1} \Gamma_j.$$

Again, from our reduced form regression, we have an estimate for the reduced form coefficient $\hat{A}_j$ and then if we knew what B^{-1} was, we can then back out to get an estimate for the structural form coefficient:

$$\hat{\Gamma}_j = B \hat{A}_j.$$

Thus identifying B is the key to recovering the structural VAR. This requires additional economic restrictions.

26.3 Identification

We saw that knowing what B is is critical for us to then go from our reduced form VAR to the structural form VAR. However, we now argue that identifying what B is is not so easy. First, let us go back to our toy example with 2 variables of interest rates and output:

$$\begin{bmatrix} 1 & b_{12} \\ b_{21} & 1 \end{bmatrix} \begin{bmatrix} y_t \\ mp_t \end{bmatrix} = \begin{bmatrix} b_{10} \\ b_{20} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ mp_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{yt} \\ \epsilon_{mt} \end{bmatrix}$$

Recall we can rewrite our system compactly as:

$$Bx_t = \Gamma_0 + \Gamma_1 x_{t-1} + \epsilon_t.$$

Now, to go from the structural VAR, we can invert the B matrix to get a reduced form VAR:

$$x_t = A_0 + A_1 x_{t-1} + e_t$$

where $A_j = B^{-1} \Gamma_j$ and $e_t = B^{-1} \epsilon_t$. Now, we can see that the **reduced form error terms are a linear combination of the structural error terms**:

$$e_{yt} = c_{11} \epsilon_{yt} + c_{12} \epsilon_{mt}$$

$$e_{mt} = c_{21} \epsilon_{yt} + c_{22} \epsilon_{mt}$$

where c_{ij} are the coefficients from $\mathbf{B}^{-1}$. Now, recall that the covariance matrix of the reduced form errors were:

$$\mathbf{\Omega} = \mathbf{B}^{-1}\mathbf{\Phi}(\mathbf{B}^{-1})^T.$$

The structural equations above used the normalisation $\text{diag}(\mathbf{B}) = \mathbf{1}$ and left the diagonal structural variances in $\mathbf{\Phi}$ free. For the following Cholesky illustration, switch to the alternative shock-scale normalisation $\mathbf{\Phi} = \mathbf{I}$. Under this alternative, the diagonal entries of $\mathbf{B}$ (and hence of $\mathbf{C} = \mathbf{B}^{-1}$) are not also fixed at one. These are alternative scale normalisations, not restrictions imposed simultaneously. We then have

$$\mathbf{\Omega} = \mathbf{B}^{-1}(\mathbf{B}^{-1})^T.$$

For compactness, define $\mathbf{C} \equiv \mathbf{B}^{-1}$, so that

$$\mathbf{\Omega} = \mathbf{C}\mathbf{C}^T.$$

Now, in our 2 variable toy model, we have that this equation becomes:

$$\begin{bmatrix} \sigma_y^2 & \sigma_{ym} \\ \sigma_{my} & \sigma_m^2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}^T$$

where σ_y^2 and σ_m^2 denote the variances of the reduced-form innovations and σ_{ym} their covariance. This notation is local to $\mathbf{\Omega}$ and should not be confused with the structural variances used earlier. Structural innovations are orthogonal by normalisation, but reduced-form innovations generally are correlated. Expanding the matrix product gives four entries:

$$\sigma_y^2 = c_{11}^2 + c_{12}^2$$

$$\sigma_{ym} = c_{11}c_{21} + c_{12}c_{22}$$

$$\sigma_{my} = c_{11}c_{21} + c_{12}c_{22}$$

$$\sigma_m^2 = c_{21}^2 + c_{22}^2$$

So what is the issue now? First, recall that covariance matrices are symmetric so that covariance of y with m is the same as covariance of m and y:

$$\begin{bmatrix} \sigma_y^2 & \sigma_{ym} \\ \sigma_{my} & \sigma_m^2 \end{bmatrix} = \begin{bmatrix} \sigma_y^2 & \sigma_{ym} \\ \sigma_{ym} & \sigma_m^2 \end{bmatrix}.$$

Therefore, our system of 4 equations become:

$$\sigma_y^2 = c_{11}^2 + c_{12}^2$$

$$\sigma_{ym} = c_{11}c_{21} + c_{12}c_{22}$$

$$\sigma_{ym} = c_{11}c_{21} + c_{12}c_{22}$$

$$\sigma_m^2 = c_{21}^2 + c_{22}^2$$

However, the middle two equations are the same equation so we can drop one of them:

$$\sigma_y^2 = c_{11}^2 + c_{12}^2$$

$$\sigma_{ym} = c_{11}c_{21} + c_{12}c_{22}$$

$$\sigma_m^2 = c_{21}^2 + c_{22}^2.$$

There are four unknowns, $c_{11}, c_{12}, c_{21}, c_{22}$, but only three distinct equations. Hence, there are infinitely many

matrices $\mathbf{C}$ that solve

$$\mathbf{\Omega} = \mathbf{C}\mathbf{C}^T$$

and therefore infinitely many matrices $\mathbf{B}^{-1}$ that solve

$$\mathbf{\Omega} = \mathbf{B}^{-1}(\mathbf{B}^{-1})^T$$

so $\mathbf{B}$ is not identified.

The four elements of $\mathbf{C}$ cannot be recovered from only three distinct covariance moments. We therefore need identifying restrictions. Consider again

$$\begin{aligned}\sigma_y^2 &= c_{11}^2 + c_{12}^2 \\ \sigma_{ym} &= c_{11}c_{21} + c_{12}c_{22} \\ \sigma_m^2 &= c_{21}^2 + c_{22}^2.\end{aligned}$$

Suppose we impose a **recursive (Cholesky) restriction**: the monetary-policy structural shock does not affect output contemporaneously. With output ordered first, this sets $c_{12} = 0$, so

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix}.$$

Then, we have that:

$$\begin{bmatrix} \sigma_y^2 & \sigma_{ym} \\ \sigma_{my} & \sigma_m^2 \end{bmatrix} = \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} \\ 0 & c_{22} \end{bmatrix}$$

and our system of 3 equations also become:

$$\begin{aligned}\sigma_y^2 &= c_{11}^2 \\ \sigma_{ym} &= c_{11}c_{21} \\ \sigma_m^2 &= c_{21}^2 + c_{22}^2.\end{aligned}$$

We can now solve the system because there are three distinct equations and three unknowns. The reduced-form covariance moments on the left-hand side can be estimated from the OLS residuals:

$$\begin{aligned}\hat{\sigma}_y^2 &= c_{11}^2 \\ \hat{\sigma}_{ym} &= c_{11}c_{21} \\ \hat{\sigma}_m^2 &= c_{21}^2 + c_{22}^2.\end{aligned}$$

We can now **recursively** solve for the coefficients. First, use the first equation to get:

$$\hat{\sigma}_y^2 = c_{11}^2 \Rightarrow c_{11} = \sqrt{\hat{\sigma}_y^2}.$$

Now, look at the second equation and plug in our solution for c_{11} to get:

$$\begin{aligned}\hat{\sigma}_{ym} &= c_{11}c_{21} \\ \Leftrightarrow \hat{\sigma}_{ym} &= \left(\sqrt{\hat{\sigma}_y^2}\right)c_{21} \\ \Rightarrow c_{21} &= \frac{\hat{\sigma}_{ym}}{\sqrt{\hat{\sigma}_y^2}}\end{aligned}$$

Finally, look at our last equation and plug in our results to derive c_{22} :

$$\begin{aligned}\hat{\sigma}_m^2 &= c_{21}^2 + c_{22}^2 \\ \Leftrightarrow \hat{\sigma}_m^2 &= \left(\frac{\hat{\sigma}_{ym}}{\sqrt{\hat{\sigma}_y^2}}\right)^2 + c_{22}^2 \\ \Rightarrow c_{22} &= \sqrt{\hat{\sigma}_m^2 - \left(\frac{\hat{\sigma}_{ym}}{\sqrt{\hat{\sigma}_y^2}}\right)^2}\end{aligned}$$

Choosing the conventional positive diagonal gives unique values for c_{11} and c_{22} . We have therefore recovered c_{11} , c_{21} , c_{22} and can invert $\mathbf{C}$ to obtain $\mathbf{B}$. The structural lag coefficients follow from

$$\Gamma_j = \mathbf{B}\mathbf{A}_j.$$

Here we imposed one zero restriction in a two-variable system. More generally:

Proposition 570 (Order Condition for Identification)

With n variables and the structural-shock covariance normalised to the identity, at least

$$\frac{n^2 - n}{2} \tag{670}$$

independent restrictions are required to identify the contemporaneous-impact matrix (in addition to sign normalisations).

Proof. The symmetric reduced-form covariance matrix $\mathbf{\Omega}$ supplies $n(n+1)/2$ distinct moments, while an unrestricted $n \times n$ impact matrix has n^2 elements. The difference is $n^2 - n(n+1)/2 = n(n-1)/2$. $\square$

The order condition is a **necessary** but not sufficient condition for identification. In lectures, we also saw the **rank condition** as a sufficient condition for identification.

Proposition 571 (Rank Condition)

In the corresponding simultaneous-equations formulation, a given structural equation is identified if the matrix of coefficients in the other equations on variables excluded from that equation has rank $n-1$. Equivalently, at least one relevant $(n-1) \times (n-1)$ minor must have a non-zero determinant.

Remark 572. Check slides how to apply this.

26.4 Cointegration and VECM

This material is covered in greater depth in the econometrics notes. Suppose that K economic variables may share a long-run equilibrium relationship. Collect them in

$$\mathbf{x}_t = \begin{bmatrix} x_{1t} \\ \vdots \\ x_{Kt} \end{bmatrix}.$$

Suppose their long-run equilibrium relationship is

$$\boldsymbol{\beta}^T \mathbf{x}_t = \beta_1 x_{1t} + \cdots + \beta_K x_{Kt} = 0,$$

where $\boldsymbol{\beta} = (\beta_1, \dots, \beta_K)^T$. In any given period, the deviation from this long-run relationship is

$$\boldsymbol{\beta}^T \mathbf{x}_t = z_t,$$

where z_t measures the equilibrium error. The individual variables may be $I(1)$ because they contain common stochastic trends, while a particular linear combination is stationary:

$$z_t \equiv \boldsymbol{\beta}^T \mathbf{x}_t.$$

If z_t is $I(0)$, then $\mathbf{x}_t \sim CI(1, 1)$, $\boldsymbol{\beta}$ is a **cointegrating vector**, and $\mathbf{x}_t$ is a **cointegrated process**.

Definition 573 (Cointegration of Order One)

A vector of $I(1)$ time-series variables $\mathbf{x}_t$ is **cointegrated** if there exists a non-zero vector $\boldsymbol{\beta}$ such that

$$\boldsymbol{\beta}^T \mathbf{x}_t \tag{671}$$

is $I(0)$. The vector $\boldsymbol{\beta}$ is called a cointegrating vector.

Remark 574. For comparison, a K -dimensional process $\mathbf{x}_t$ is integrated of order d , written $\mathbf{x}_t \sim I(d)$, if $\Delta^d \mathbf{x}_t$ is stationary while $\Delta^{d-1} \mathbf{x}_t$ is not.

Example 575

Suppose that $y_t \sim I(1)$ and $z_t \sim I(1)$ and

$$2y_t - 3z_t \sim I(0).$$

Then (y_t, z_t) is cointegrated with cointegration vector $\boldsymbol{\beta} = (2, -3)$.

Regressions among unrelated non-stationary variables can be spurious, but cointegration preserves meaningful long-run relations. A cointegrated VAR can be rewritten as a vector error-correction model (VECM).

Suppose that we have the K -dimensional VAR(p) process

$$\mathbf{x}_t = \mathbf{A}_1 \mathbf{x}_{t-1} + \cdots + \mathbf{A}_p \mathbf{x}_{t-p} + \mathbf{u}_t. \tag{672}$$

where each variable is either $I(1)$ or $I(0)$.

Definition 576 (Cointegration of a VAR(p))

Let $\mathbf{x}_t$ be a K -dimensional VAR(p) process:

$$\mathbf{x}_t = \mathbf{A}_1 \mathbf{x}_{t-1} + \dots + \mathbf{A}_p \mathbf{x}_{t-p} + \mathbf{u}_t. \quad (673)$$

Then, $\mathbf{x}_t$ is called **cointegrated of rank r** if

$$\mathbf{\Pi} \equiv -(\mathbf{I}_K - \mathbf{A}_1 - \dots - \mathbf{A}_p) \quad (674)$$

has rank r . Then, the matrix $\mathbf{\Pi}$ can be written as:

$$\mathbf{\Pi} = \boldsymbol{\alpha} \boldsymbol{\beta}^T \quad (675)$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are $K \times r$ matrices of rank r (so $\boldsymbol{\beta}^T$ is $r \times K$). The columns of $\boldsymbol{\beta}$ are the **cointegrating vectors**, and $\boldsymbol{\alpha}$ is the **loading matrix**.

Remark 577. *There are two special cases. When $r = 0$, there is no cointegrating relation and $\Delta \mathbf{x}_t$ has a stable VAR($p-1$) representation. When $r = K$, $\det(\mathbf{\Pi}) \neq 0$, so the VAR has no unit roots and $\mathbf{x}_t$ is stationary.*

Given our cointegrated VAR(p) model:

$$\mathbf{x}_t = \mathbf{A}_1 \mathbf{x}_{t-1} + \dots + \mathbf{A}_p \mathbf{x}_{t-p} + \mathbf{u}_t$$

we can write this VAR into a vector error correction form.

Proposition 578 (Vector Error Correction Model)

Let $\mathbf{x}_t$ be a cointegrated VAR(p) process. Then we can rewrite this cointegrated VAR(p) process as a **vector error correction model** in $\mathbf{x}_t$ by:

$$\Delta \mathbf{x}_t = \mathbf{\Pi} \mathbf{x}_{t-1} + \boldsymbol{\Gamma}_1 \Delta \mathbf{x}_{t-1} + \dots + \boldsymbol{\Gamma}_{p-1} \Delta \mathbf{x}_{t-p+1} + \mathbf{u}_t \quad (676)$$

where

$$\begin{aligned} \mathbf{\Pi} &= -(\mathbf{I}_K - \mathbf{A}_1 - \dots - \mathbf{A}_p), \\ \boldsymbol{\Gamma}_i &= -(\mathbf{A}_{i+1} + \dots + \mathbf{A}_p) \end{aligned}$$

for $i = 1, \dots, p-1$.

Remark 579. *Notice that due to the decomposition of $\mathbf{\Pi}$, we could have written the VECM as:*

$$\Delta \mathbf{x}_t = \boldsymbol{\alpha} \boldsymbol{\beta}^T \mathbf{x}_{t-1} + \boldsymbol{\Gamma}_1 \Delta \mathbf{x}_{t-1} + \dots + \boldsymbol{\Gamma}_{p-1} \Delta \mathbf{x}_{t-p+1} + \mathbf{u}_t.$$

The key point is that the VECM contains lagged differences $\Delta \mathbf{x}_{t-j}$ and a single lagged level term $\mathbf{x}_{t-1}$, with $\mathbf{\Pi}$ collecting the equilibrium-correction coefficients.

26.5 Impulse Response Function

We now want to see the impact of shocks on the system. We now look at the SVAR model without an intercept:

$$\mathbf{B} \mathbf{X}_t = \boldsymbol{\Gamma}_1 \mathbf{X}_{t-1} + \boldsymbol{\epsilon}_t.$$

We can invert $\mathbf{B}$ to get the reduced form:

$$\mathbf{X}_t = \mathbf{B}^{-1}\mathbf{\Gamma}_1\mathbf{X}_{t-1} + \mathbf{B}^{-1}\mathbf{\epsilon}_t$$

which we redefine as:

$$\mathbf{X}_t = \mathbf{A}_1\mathbf{X}_{t-1} + \mathbf{e}_t$$

where $\mathbf{A}_1 \equiv \mathbf{B}^{-1}\mathbf{\Gamma}_1$ is the coefficient matrix and $\mathbf{e}_t \equiv \mathbf{B}^{-1}\mathbf{\epsilon}_t$ is the reduced-form innovation with mean zero and covariance matrix $\mathbf{\Omega}$.

Now, rewrite this using the lag operator:

$$\mathbf{X}_t = \mathbf{A}_1L\mathbf{X}_t + \mathbf{e}_t$$

where $Lx_t = x_{t-1}$. Then re-arrange:

$$[\mathbf{I}_n - \mathbf{A}_1L]\mathbf{X}_t = \mathbf{e}_t.$$

Now, under stationarity, we can invert the expression on the left-hand side:

$$\mathbf{X}_t = [\mathbf{I}_n - \mathbf{A}_1L]^{-1}\mathbf{e}_t.$$

Now, notice that $[\mathbf{I}_n - \mathbf{A}_1L]^{-1}$ can be thought of as a geometric series. That is,

$$[\mathbf{I}_n - \mathbf{A}_1L]^{-1} = \mathbf{I}_n + \mathbf{A}_1L + (\mathbf{A}_1L)^2 + (\mathbf{A}_1L)^3 + \dots$$

So we can write our expression as:

$$\mathbf{X}_t = \left[\mathbf{I}_n + \mathbf{A}_1L + (\mathbf{A}_1L)^2 + (\mathbf{A}_1L)^3 + \dots \right] \mathbf{e}_t.$$

To see things more clearly, expand out the brackets:

$$\mathbf{X}_t = \mathbf{e}_t + \mathbf{A}_1L\mathbf{e}_t + (\mathbf{A}_1L)^2\mathbf{e}_t + (\mathbf{A}_1L)^3\mathbf{e}_t + \dots$$

and then by definition of lag operator:

$$\mathbf{X}_t = \mathbf{e}_t + \mathbf{A}_1\mathbf{e}_{t-1} + \mathbf{A}_1^2\mathbf{e}_{t-2} + \mathbf{A}_1^3\mathbf{e}_{t-3} + \dots$$

We can then write this as:

$$\mathbf{X}_t = \sum_{i=0}^{\infty} \mathbf{A}_1^i \mathbf{e}_{t-i}. \quad (677)$$

Definition 580 (Reduced-Form Vector Moving-Average Representation)

Let $\mathbf{X}_t$ follow a stationary n -variable VAR(1). Its **reduced-form** vector moving-average representation is

$$\mathbf{X}_t = \sum_{i=0}^{\infty} \mathbf{A}_1^i \mathbf{e}_{t-i}. \quad (678)$$

This VMA representation says that the variables can be expressed as an infinite sum of **reduced form errors** (just like the Wold representation theorem).

Now, if we then define:

$$\mathbf{C}(L) \equiv \left[\mathbf{I}_n + \mathbf{A}_1L + (\mathbf{A}_1L)^2 + (\mathbf{A}_1L)^3 + \dots \right]$$

we then arrive at the compact notation as well:

$$\mathbf{X}_t = C(L)\mathbf{e}_t. \quad (679)$$

Now let us go back to the VMA representation:

$$\mathbf{X}_t = \sum_{i=0}^{\infty} \mathbf{A}_1^i \mathbf{e}_{t-i}.$$

The expression on the right is in terms of the reduced form parameters $\mathbf{A}_1$ and reduced form errors $\mathbf{e}_{t-i}$, which we have already explained that we can estimate. Then, if we are able to identify the matrix $\mathbf{B}$, we can then back out the parameters in the structural form VAR $\mathbf{\Gamma}_1$ and the structural errors $\boldsymbol{\epsilon}_t$ via $\mathbf{A}_1 \equiv \mathbf{B}^{-1}\mathbf{\Gamma}_1$ and $\mathbf{e}_t \equiv \mathbf{B}^{-1}\boldsymbol{\epsilon}_t$. We can plug this back into our VMA representation:

$$\begin{aligned} \mathbf{X}_t &= \sum_{i=0}^{\infty} \mathbf{A}_1^i \mathbf{e}_{t-i} \\ \mathbf{X}_t &= \sum_{i=0}^{\infty} \left[\mathbf{B}^{-1}\mathbf{\Gamma}_1 \right]^i \mathbf{B}^{-1}\boldsymbol{\epsilon}_{t-i} \end{aligned}$$

Finally, define:

$$\boldsymbol{\Theta}_i \equiv \left[\mathbf{B}^{-1}\mathbf{\Gamma}_1 \right]^i \mathbf{B}^{-1}$$

to get:

$$\mathbf{X}_t = \sum_{i=0}^{\infty} \boldsymbol{\Theta}_i \boldsymbol{\epsilon}_{t-i}.$$

Definition 581 (Structural Vector Moving-Average Representation)

Let $\mathbf{X}_t$ follow a stationary, identified n -variable VAR(1). Its **structural** vector moving-average representation is

$$\mathbf{X}_t = \sum_{i=0}^{\infty} \boldsymbol{\Theta}_i \boldsymbol{\epsilon}_{t-i}. \quad (680)$$

An impulse response traces the effect of one structural innovation through this representation. In particular, the response at horizon h to a one-unit shock in component j at time t is $\boldsymbol{\Theta}_h \boldsymbol{\iota}_j$, where $\boldsymbol{\iota}_j$ is the j th selection vector. Because the estimated impulse-response coefficients can be nonlinear functions of the reduced-form parameters, their standard errors may be obtained using the delta method or bootstrap methods.

27 Crises

We want to look at how household debt can be connected to the business cycle. Household deleveraging can have a negative impact on output if monetary policy is constrained by the zero lower bound.

27.1 Deleveraging and Liquidity Traps - Eggertsson and Krugman (2012)

We make the following assumptions for our model.

Assumption (Closed Economy and Two Types of Households)

We have a closed economy with two types of households: patient and impatient. Furthermore, we assume that households have perfect foresight.

A key variable which we will be affecting in this model is the natural rate of interest through drops in aggregate demand as a result of deleveraging.

Definition 582 (Natural Rate of Interest)

The natural rate of interest is the real interest rate under flexible prices.

The model has patient and impatient households. Impatient households are borrowing constrained, and a deleveraging shock forces them to reduce their debt. Their spending falls, lowering aggregate demand and the natural real interest rate. With flexible prices and zero inflation, the central bank can lower the nominal rate to the natural rate and induce patient households to spend more. If this response fully offsets the fall in borrowers' demand, output is unchanged. If the required nominal rate is below zero, however, the ZLB prevents the necessary adjustment and aggregate demand is insufficient.

We shall consider two settings. First, an economy where we have flexible prices and second, an economy with nominal wage rigidities.

27.1.1 Flexible Prices Setting

We assume that prices are fully flexible and can adjust.

Assumption (Households)

We assume that savers are more patient than borrowers:

$$\beta_s > \beta_b. \quad (681)$$

We assume there are n patient households and $(1 - n)$ impatient households.

Finally, we assume that each household has $\bar{L}$ units of labour and there is no disutility from working.

Remark 583. *Since there is no disutility from working, households would want to work as much as they can and earn a wage.*

Households choose sequences of bond holdings and consumption, subject to the borrowing constraint

$$-\frac{(1 + i_t)}{P_{t+1}} B_{i,t} \leq \bar{D}$$

where $\bar{D}$ is the maximum real repayment obligation. A borrower has $B_{i,t} < 0$, which explains the minus sign. We now state the household problem.

Definition 584 (Household Problem)

A representative household chooses a sequence of consumption and bond purchases $\{C_{i,t}, B_{i,t}\}_{t=0}^{\infty}$ to maximise their utility subject to a budget constraint:

$$\begin{aligned} \max_{\{C_{i,t}, B_{i,t}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta_i^t \log(C_{i,t}) \\ \text{s.t.} \quad & B_{i,t} + P_t C_{i,t} = (1 + i_{t-1}) B_{i,t-1} + W_t L_{i,t} \\ & - \frac{(1 + i_t)}{P_{t+1}} B_{i,t} \leq \bar{D} \end{aligned} \quad (682)$$

where i denotes that a household is either a borrower b or a saver s , $\bar{D}$ is the borrowing constraint, and W_t is the labour wage.

We now derive the Euler equation and complementary slackness condition.

Proposition 585 (Euler Equation for Consumption and Complementary Slackness)

The Euler equation for consumption is:

$$\frac{1}{C_{i,t}} = \beta_i (1 + i_t) \frac{P_t}{P_{t+1}} \left(\frac{1}{C_{i,t+1}} \right) + (1 + i_t) \frac{P_t}{P_{t+1}} \mu_{i,t}. \quad (683)$$

The complementary slackness condition is:

$$\mu_{i,t} \left[\frac{(1 + i_t)}{P_{t+1}} B_{i,t} + \bar{D} \right] = 0. \quad (684)$$

Proof. Form the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta_i^t \left\{ \left[\log(C_{i,t}) \right] - \lambda_{i,t} \left[B_{i,t} + P_t C_{i,t} - (1 + i_{t-1}) B_{i,t-1} - W_t L_{i,t} \right] + \mu_{i,t} \left[\bar{D} + \frac{(1 + i_t)}{P_{t+1}} B_{i,t} \right] \right\}.$$

Take first order condition with respect to $C_{i,t}$ and $B_{i,t}$:

$$\frac{\partial \mathcal{L}}{\partial C_{i,t}} : \frac{1}{C_{i,t}} = \lambda_{i,t} P_t.$$

$$\frac{\partial \mathcal{L}}{\partial B_{i,t}} : -\lambda_{i,t} + \beta_i (1 + i_t) \lambda_{i,t+1} + \mu_{i,t} \frac{1 + i_t}{P_{t+1}} = 0$$

Substitute the first FOC into the second to get:

$$-\frac{1}{C_{i,t} P_t} + \frac{\beta_i (1 + i_t)}{C_{i,t+1} P_{t+1}} + \mu_{i,t} \frac{1 + i_t}{P_{t+1}} = 0$$

Re-arrange and multiply by P_t to get:

$$\frac{1}{C_{i,t}} = \beta_i (1 + i_t) \frac{P_t}{P_{t+1}} \left(\frac{1}{C_{i,t+1}} \right) + (1 + i_t) \frac{P_t}{P_{t+1}} \mu_{i,t}.$$

The complementary-slackness condition follows from the borrowing constraint, together with $\mu_{i,t} \geq 0$.

□

The multiplier $\mu_{i,t}$ enters the consumption Euler equation. When the borrowing constraint binds, $\mu_{i,t} > 0$, the household cannot implement the unconstrained consumption path.

We now look at the firm's problem.

Definition 586 (Firm's Problem)

A representative firm chooses labour L_t to maximise period-by-period profit subject to a linear production function:

$$\begin{aligned} \max_{L_t} \quad & P_t Y_t - W_t L_t \\ \text{s.t.} \quad & Y_t = L_t \end{aligned} \tag{685}$$

The FOC is quite easy.

Proposition 587 (Aggregate Price is Equal to Aggregate Nominal Wage)

The first order condition for the firm is:

$$P_t = W_t. \tag{686}$$

Aggregate price equals the aggregate nominal wage.

Proof. Plug the constraint into the profit function and take FOC:

$$P_t Y_t - W_t L_t = P_t L_t - W_t L_t.$$

Differentiate with respect to L_t and set it equal to zero and you are done. □

Now, we impose equilibrium conditions in the goods market, the labour market, and the credit market.

Proposition 588 (Goods Market Clearing)

Total output equals the sum of consumption by both the savers and borrowers:

$$Y_t = nC_{s,t} + (1 - n)C_{b,t}. \tag{687}$$

Proposition 589 (Labour Market Clearing)

The labour market without nominal rigidities clearing condition is:

$$Y_t = \bar{L}. \tag{688}$$

Proposition 590 (Credit Market Clearing)

The total amount saved by savers equals the total amount borrowed by borrowers:

$$nB_{s,t} = -(1 - n)B_{b,t}. \tag{689}$$

Before we proceed, we impose an assumption on the type of steady state we want to look for.

Assumption (Zero Inflation Steady State)

Consider a steady state with zero inflation:

$$\frac{P_{t+1}}{P_t} = 1 \quad (690)$$

and constant consumption so that:

$$C_{s,t} = C_s$$

$$C_{b,t} = C_b$$

in every period.

We can now look for the equilibrium steady state of this model.

Theorem 591 (Euler Equation for Consumption in Steady State)

The steady-state Euler equations for both types of households are:

$$\frac{1}{C_s} = (1+i) \left(\frac{\beta_s}{C_s} \right) + \mu_s(1+i) \quad (691)$$

$$\frac{1}{C_b} = (1+i) \left(\frac{\beta_b}{C_b} \right) + \mu_b(1+i) \quad (692)$$

Proof. Euler equation for consumption is:

$$\frac{1}{C_{i,t}} = \beta_i(1+i_t) \frac{P_t}{P_{t+1}} \left(\frac{1}{C_{i,t+1}} \right) + (1+i_t) \frac{P_t}{P_{t+1}} \mu_{i,t}.$$

Zero inflation implies that $P_t/P_{t+1} = 1$ so we then get:

$$\frac{1}{C_i} = \beta_i(1+i) \left(\frac{1}{C_i} \right) + (1+i)\mu_i.$$

□

The multipliers μ_b and μ_s indicate whether each household's borrowing constraint binds. We now show that the borrowers' constraint binds.

Theorem 592 (Borrowing Constraint for Borrowers Bind)

The borrowing constraint for the borrowers binds:

$$\mu_b > 0. \quad (693)$$

Proof. Take the two Euler equations:

$$\frac{1}{C_s} = (1+i) \left(\frac{\beta_s}{C_s} \right) + \mu_s(1+i)$$

$$\frac{1}{C_b} = (1+i) \left(\frac{\beta_b}{C_b} \right) + \mu_b(1+i)$$

Divide the first equation by the second to get:

$$\frac{C_b}{C_s} = \frac{\frac{\beta_s}{C_s} + \mu_s}{\frac{\beta_b}{C_b} + \mu_b}$$

Multiply through to get:

$$1 = \frac{\beta_s + C_s \mu_s}{\beta_b + C_b \mu_b}$$

In steady-state, we have that:

$$1 = \frac{\beta_s}{\beta_b} \left[\frac{1 + \frac{C_s \mu_s}{\beta_s}}{1 + \frac{C_b \mu_b}{\beta_b}} \right].$$

Now, suppose that $\mu_b = 0$, we then have that:

$$1 = \frac{\beta_s}{\beta_b} \left[1 + \frac{C_s \mu_s}{\beta_s} \right].$$

Since $\beta_s > \beta_b$, the first factor is strictly greater than one. The second factor is at least one because $\mu_s \geq 0$. Their product therefore cannot equal one, a contradiction. Hence

$$\mu_b > 0.$$

□

We have shown that borrowers borrow in the steady state. We now determine how much they borrow.

Proposition 593 (Borrowing Amount in Steady State)

In steady-state borrowers will borrow:

$$-\frac{(1+i)}{P} B_b = \bar{D}. \quad (694)$$

Proof. If we look at the complementary slackness condition in steady-state for borrower:

$$\mu_b \left[\frac{(1+i)}{P} B_b + \bar{D} \right] = 0.$$

Since $\mu_b > 0$ then it must be the case that:

$$\frac{(1+i)}{P} B_b + \bar{D} = 0.$$

Now, we can re-arrange to get:

$$-\frac{(1+i)}{P} B_b = \bar{D}.$$

□

To interpret this, we see that borrowers borrow up to the limit. Now, as borrowers borrow, that means that the savers are the lenders. We can figure out how much they lend.

Proposition 594 (Saving Amount in Steady State)

In steady-state savers will lend:

$$B_s = -\frac{1-n}{n} B_b. \quad (695)$$

Furthermore, savers are saving:

$$\mu_s = 0. \quad (696)$$

Proof. Go to credit market clearing condition:

$$nB_s = -(1-n)B_b.$$

Re-arrange for B_s to get:

$$B_s = -\frac{(1-n)}{n}B_b.$$

Because $B_b < 0$, market clearing implies $B_s > 0$. The savers are lenders, so their borrowing constraint is slack and complementary slackness gives $\mu_s = 0$. $\square$

Remark 595. Note that B_b is negative as borrowers are borrowing which implies that $B_s > 0$ so savers are saving.

Now, by credit market clearing condition, interest rates need to adjust. We can therefore figure out what the level of interest must be for the credit market to clear.

Proposition 596 (Steady State Interest Rate)

The discount factor of savers pins down the steady state interest rate:

$$1 + i = \frac{1}{\beta_s}. \quad (697)$$

Proof. Go to Euler equation for savers:

$$\frac{1}{C_s} = (1+i) \left(\frac{\beta_s}{C_s} \right) + \mu_s(1+i).$$

We have that $\mu_s = 0$ so then:

$$\frac{1}{C_s} = (1+i) \left(\frac{\beta_s}{C_s} \right) + 0.$$

which then becomes:

$$1 = (1+i)\beta_s.$$

Therefore we get:

$$1 + i = \frac{1}{\beta_s}.$$

$\square$

We can now determine how much borrowers consume in the steady state.

Theorem 597 (Consumption for Borrowers in Steady State)

Consumption for borrowers in steady state is:

$$C_b = \bar{L} - \frac{i}{1+i} \bar{D} \quad (698)$$

where $\bar{L}$ is their endowment of labour and $\bar{D}$ is their borrowing limit.

Proof. Look at the budget constraint in steady state:

$$B_i + PC_i = (1+i)B_i + WL_i$$

Substitute in $P = W$ from firm's FOC

$$B_i + WC_i = (1+i)B_i + WL_i.$$

Now solve for C_i :

$$C_i = \frac{(1+i)B_i}{W} - \frac{B_i}{W} + L_i.$$

Simplify to get:

$$C_i = \frac{iB_i}{P} + L_i.$$

For a borrower (household type b), this becomes

$$C_b = \frac{iB_b}{P} + L_b.$$

Plug in our expression for steady-state level of borrowing for borrower B_b :

$$-\frac{(1+i)}{P}B_b = \bar{D} \quad \leftrightarrow \quad B_b = -\frac{P\bar{D}}{(1+i)}$$

So then:

$$C_b = \frac{iB_b}{P} + L_b = \frac{i}{P} \left[-\frac{P\bar{D}}{(1+i)} \right] + L_b$$

which simplifies to:

$$C_b = L_b - \frac{i}{1+i}\bar{D}.$$

□

We can also determine how much savers consume in the steady state.

Theorem 598 (Consumption for Savers in Steady State)

Consumption for savers in steady state is:

$$C_s = \bar{L} + \frac{1-n}{n} \frac{i}{1+i} \bar{D}. \quad (699)$$

Proof. Use the savers steady-state savings:

$$B_s = -\frac{1-n}{n}B_b.$$

Plug in borrowers borrowing:

$$B_s = -\frac{1-n}{n}B_b = -\frac{1-n}{n} \left[-\frac{P\bar{D}}{(1+i)} \right] = \frac{1-n}{n} \frac{P\bar{D}}{1+i}$$

Look at the budget constraint in steady state:

$$B_i + PC_i = (1+i)B_i + WL_i$$

Substitute in $P = W$ from firm's FOC

$$B_i + WC_i = (1+i)B_i + WL_i.$$

Now solve for C_i :

$$C_i = \frac{(1+i)B_i}{W} - \frac{B_i}{W} + L_i.$$

Simplify to get:

$$C_i = \frac{iB_i}{P} + L_i.$$

For a saver (household type s), this gives

$$C_s = \frac{i}{P}B_s + \bar{L}$$

Now plug in our derivation for $B_s = \frac{1-n}{n} \frac{P\bar{D}}{1+i}$ to get:

$$\begin{aligned} C_s &= \frac{i}{P} B_s + \bar{L} \\ &= \frac{i}{P} \left[\frac{1-n}{n} \frac{P\bar{D}}{1+i} \right] + \bar{L} \\ &= \frac{1-n}{n} \frac{i}{1+i} \bar{D} + \bar{L} \end{aligned}$$

□

Borrowers consume less than their endowment and savers consume more than their endowment.

We now look at the effect of a deleveraging shock in this economy with flexible prices.

Assumption (Deleveraging Shock)

Prior to period 0, the economy is at a steady-state where $\bar{D} = D^h$. At period 0, $\bar{D}$ unexpectedly falls to D^l where $D^l < D^h$ and remains there permanently.

All adjustment takes place in a single period. From period 1 onwards, the equilibrium is given by the steady-state equations with $\bar{D} = D^l$.

We need to assume what is happening to interest rates set by the central bank during this shock.

Assumption (Central Bank Targeting Zero Inflation)

The central bank targets zero inflation. As a result, i is both the real and nominal interest rate. Finally, prices are normalised:

$$P = 1. \quad (700)$$

We can now figure out what is the consumption of borrowers during this deleveraging shock. Prior to that, they were at the steady state level of consumption.

Proposition 599 (Consumption of Borrowers in Deleveraging Shock)

Suppose a deleveraging shock occurs at $t = 0$. The consumption of borrowers at $t = 0$ is:

$$C_{b,0} = \bar{L} - D^h + \frac{D^l}{1+i_0} \quad (701)$$

where i_0 is the interest rate at time $t = 0$.

Proof. The period-0 budget constraint, with $P_0 = W_0 = 1$, is

$$B_{b,0} + C_{b,0} = (1 + i_{-1})B_{b,-1} + \bar{L}.$$

The inherited debt was issued under the old limit, so $(1 + i_{-1})B_{b,-1} = -D^h$. New borrowing is governed by the lower limit, so $B_{b,0} = -D^l/(1 + i_0)$. Substitution gives

$$C_{b,0} = \bar{L} - D^h + \frac{D^l}{1+i_0}.$$

□

We have computed the consumption level of borrowers before and during the shock. We can now show that this new level of consumption during the deleveraging shock is actually a drop in consumption.

Proposition 600 (Drop In Consumption for Borrowers During Deleveraging)

At the time of the deleveraging shock, the consumption of borrowers drop by:

$$C_b - C_{b,0} = \beta_s D^h - \frac{D^l}{1 + i_0}. \quad (702)$$

Proof. We know that the steady state consumption of borrowers before the shock is:

$$C_b = \bar{L} - (1 - \beta_s) D^h$$

and during the shock it is:

$$C_{b,0} = \bar{L} - D^h + \frac{D^l}{1 + i_0}.$$

We take the difference to get:

$$C_b - C_{b,0} = \bar{L} - (1 - \beta_s) D^h - \left[\bar{L} - D^h + \frac{D^l}{1 + i_0} \right]$$

and this gives us:

$$C_b - C_{b,0} = \beta_s D^h - \frac{D^l}{1 + i_0}.$$

□

We can now compute the savers' consumption during a deleveraging.

Proposition 601 (Consumption of Savers During Deleveraging)

Suppose a deleveraging shock occurs at $t = 0$. The consumption of savers at $t = 0$ is:

$$C_{s,0} = \bar{L} + \frac{1-n}{n} \left(D^h - \frac{D^l}{1 + i_0} \right). \quad (703)$$

Proof. First, use the market clearing condition at time $t = 0$:

$$Y_0 = n C_{s,0} + (1 - n) C_{b,0}.$$

Plug in the consumption at time $t = 0$ for borrowers $C_{b,0}$:

$$Y_0 = n C_{s,0} + (1 - n) \left[\bar{L} - D^h + \frac{D^l}{1 + i_0} \right].$$

We also know that by the firm's linear production function:

$$Y_0 = \bar{L}$$

so we have that:

$$\bar{L} = n C_{s,0} + (1 - n) \left[\bar{L} - D^h + \frac{D^l}{1 + i_0} \right].$$

Now, re-arrange for $C_{s,0}$ to get:

$$\begin{aligned} C_{s,0} &= \frac{\bar{L}}{n} - \frac{1-n}{n} \left(\bar{L} - D^h + \frac{D^l}{1+i_0} \right) \\ &= \left(\frac{\bar{L}}{n} - \frac{1-n}{n} \bar{L} \right) + \frac{1-n}{n} \left(D^h - \frac{D^l}{1+i_0} \right). \end{aligned}$$

We simplify this to get:

$$C_{s,0} = \bar{L} + \frac{1-n}{n} \left(D^h - \frac{D^l}{1+i_0} \right).$$

□

We have computed the deleveraging shock consumption for savers and their steady state consumption prior to the deleveraging. We can now compute their long-run consumption for after the shock.

Proposition 602 (Long-Run Saver's Consumption)

The long-run consumption for savers is:

$$C_s = \bar{L} + \frac{1-n}{n} (1 - \beta_s) D^l. \quad (704)$$

Furthermore,

$$C_{s,0} > C_s. \quad (705)$$

Proof. Use the steady state level of consumption for savers:

$$C_s = \bar{L} + \frac{1-n}{n} \frac{i}{1+i} D^l$$

Now, we know that interest rates are pinned down by savers' discount rate:

$$1+i = \frac{1}{\beta_s}$$

this can be re-arranged to:

$$\frac{i}{1+i} = 1 - \beta_s.$$

So we plug this in to get:

$$C_s = \bar{L} + \frac{1-n}{n} \frac{i}{1+i} D^l = \bar{L} + \frac{1-n}{n} (1 - \beta_s) D^l$$

To establish the second claim, let $q \equiv (1-n)/n$. Combining

$$C_{s,0} = \bar{L} + q \left(D^h - \frac{D^l}{1+i_0} \right) \quad \text{and} \quad C_s = \beta_s (1+i_0) C_{s,0}$$

with $C_s = \bar{L} + q(1 - \beta_s) D^l$ gives

$$1+i_0 = \frac{\bar{L} + q D^l}{\beta_s (\bar{L} + q D^h)} < \frac{1}{\beta_s},$$

because $D^l < D^h$. Hence $\beta_s(1+i_0) < 1$, and the Euler equation implies $C_{s,0} > C_s$. □

We can see that consumption drops for borrowers whilst consumption rises for savers. In the short run, savers increase their consumption to pick up the slack left by borrowers.

Proposition 603 (Saver's Short-Run Euler Equation)

The short-run saver's Euler equation is:

$$C_s = \beta_s(1 + i_0)C_{s,0} \quad (706)$$

where C_s is long-run consumption and $C_{s,0}$ is short-run consumption and $1 + i_0$ is the natural level of the interest rate.

Proof. Use the Euler equation for savers:

$$\frac{1}{C_s} = (1 + i) \left(\frac{\beta_s}{C_s} \right) + \mu_s(1 + i)$$

Plug in $C_{s,0}$ on the left-hand side, C_s on the right-hand side, $i = i_0$, and $\mu_s = 0$ as savers do not borrow to get:

$$\frac{1}{C_{s,0}} = (1 + i_0) \left(\frac{\beta_s}{C_s} \right) + 0$$

Now we can solve to get:

$$C_s = \beta_s(1 + i_0)C_{s,0}.$$

□

We can also compute what is the natural level of interest rates during the deleveraging period.

Proposition 604 (Natural Level of Interest Rate During Deleveraging)

At the time of the shock, the natural level of interest rate for the central bank to set interest rates at is:

$$i_0 = \frac{C_s}{\beta_s C_{s,0}} - 1. \quad (707)$$

Proof. Look at our saver's short-run Euler equation:

$$C_s = \beta_s(1 + i_0)C_{s,0}.$$

Solve for i_0 :

$$\frac{C_s}{\beta_s C_{s,0}} = 1 + i_0.$$

Therefore, we have that:

$$i_0 = \frac{C_s}{\beta_s C_{s,0}} - 1.$$

□

We can compare this new level of interest rates to interest rates prior to deleveraging.

Proposition 605 (Natural Level of Interest During Deleveraging is Below Steady State)

At the time of the shock, the natural level of interest is lower than the steady-state level of interest rates:

$$i_0 = \frac{C_s}{\beta_s C_{s,0}} - 1 < \frac{1}{\beta_s} - 1 = i. \quad (708)$$

Proof. Since $C_s < C_{s,0}$, we immediately have

$$i_0 < i.$$

The central bank lowers interest rates to i_0 to keep inflation at 0. Hence, deleveraging causes interest rates to fall. Hence, the central bank needs to convince savers to consume more by lowering the interest rate to i_0 so as to stimulate consumption.

Another way to see the mechanism is through the saver Euler equation, $C_{s,0} = C_s / [\beta_s(1 + i_0)]$. Holding future consumption fixed, a lower real interest rate raises desired current consumption. In equilibrium, the interest rate and both households' consumption choices are jointly determined with goods- and credit-market clearing.

As a result of lowered interest rates, the savers will consume enough to offset the decrease in consumption by borrowers. Therefore, output remains unchanged. We show this in Fig. 54

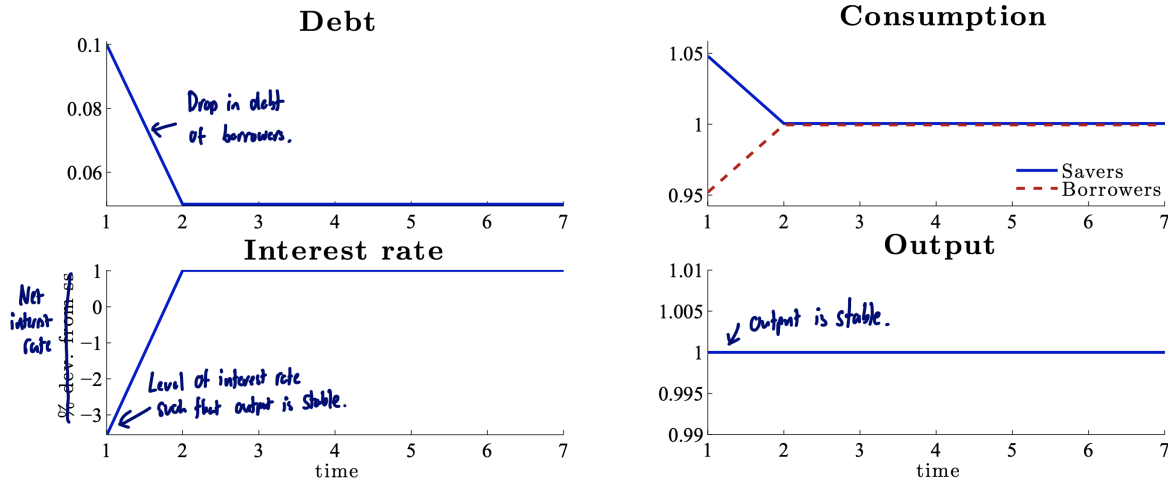


Figure 54: Deleveraging shock occurs and the zero lower bound for interest rates is not attained.

We have seen that a deleveraging shock lowers the natural interest rate. The central bank can prevent an effect on inflation or output by lowering the nominal rate. If the required rate is below zero, however, deleveraging can generate a liquidity trap and recession. To show this, we introduce nominal rigidities.

27.1.2 Nominal Wage Rigidities

We now focus on nominal wage rigidities. We make the following assumptions on the rigidity of wages.

Assumption (Nominal Wage Rigidities)

At time $t = 0$, nominal wages are fully rigid and equal to $\bar{W}$. From time $t = 1$ onwards, wages are fully flexible.

Remark 606. This is needed because if wages are rigid, then during deleveraging shocks, wages will not be able to adjust to clear the labour market. This generates unemployment since $Y_t = L_t$.

Here, we see that for $t \geq 1$, the equilibrium is the same as the economy without nominal rigidities.

We can now derive the condition under which the zero lower bound binds.

Proposition 607 (Binding ZLB)

The zero lower bound binds if

$$\frac{C_s}{\beta_s C_{s,0}^f} - 1 < 0 \quad (709)$$

where $C_{s,0}^f$ is the saver's period-0 consumption under full employment, $L_0 = \bar{L}$.

Proof. Under full employment, the flexible-price natural rate at time 0 would be

$$i_0^f = \frac{C_s}{\beta_s C_{s,0}^f} - 1.$$

The ZLB binds precisely when this required rate is negative, which gives the stated condition. $\square$

Thus the ZLB binds when the natural rate required to support full employment is negative.

We can now derive the Euler equation under the ZLB.

Proposition 608 (Euler Equation for Savers at the ZLB)

The savers' Euler equation when the ZLB binds is

$$C_s = \beta_s C_{s,0}. \quad (710)$$

Proof. The Euler equation for consumption for savers is:

$$C_s = \beta_s (1 + i_0) C_{s,0}.$$

When $i_0 = 0$, we get:

$$C_s = \beta_s C_{s,0}.$$

$\square$

We can then get the level of consumption for savers under the ZLB.

Proposition 609 (Saver Euler and Goods-Market Relation at the ZLB)

When the ZLB binds, long-run saver consumption and period-0 employment satisfy

$$C_s = \beta_s \left(L_0 + \frac{1-n}{n} (D^h - D^l) \right). \quad (711)$$

Proof. At the ZLB, period-0 goods-market clearing gives current saver consumption as

$$C_{s,0} = L_0 + \frac{1-n}{n} \left(D^h - \frac{D^l}{1+i_0} \right).$$

Combine this with the Euler equation under ZLB:

$$C_s = \beta_s C_{s,0}.$$

We therefore get:

$$C_s = \beta_s C_{s,0} = \beta_s \left[L_0 + \frac{1-n}{n} \left(D^h - \frac{D^l}{1+i_0} \right) \right].$$

Now, set $i_0 = 0$ to get:

$$C_s = \beta_s \left(L_0 + \frac{1-n}{n} (D^h - D^l) \right).$$

□

We can now derive the number of unemployed in this setting.

Theorem 610 (Unemployment under Nominal Wage Rigidities)

The level of employment under nominal wage rigidities is:

$$L_0 = \frac{C_s}{\beta_s} - \frac{1-n}{n} [D^h - D^l]. \quad (712)$$

Proof. Take the consumption for savers under ZLB:

$$C_s = \beta_s \left(L_0 + \frac{1-n}{n} (D^h - D^l) \right).$$

Divide through by β_s :

$$\frac{C_s}{\beta_s} = L_0 + \frac{1-n}{n} (D^h - D^l).$$

Now solve for L_0 to get:

$$L_0 = \frac{C_s}{\beta_s} - \frac{1-n}{n} [D^h - D^l].$$

□

Here, note that $L_0 < \bar{L}$ and hence we have unemployment. We show the results of a deleveraging shock with the ZLB binding in Fig. 55. We see as a result, interest rates are not lowered enough due to the ZLB and therefore consumption by savers is not enough to make up for the fall in consumption by borrowers. Therefore, output falls. Strangely enough, if we embed this model into a standard New Keynesian model, we see that prices also fall during a deleveraging shock with the ZLB binding, as seen in Fig. 56.

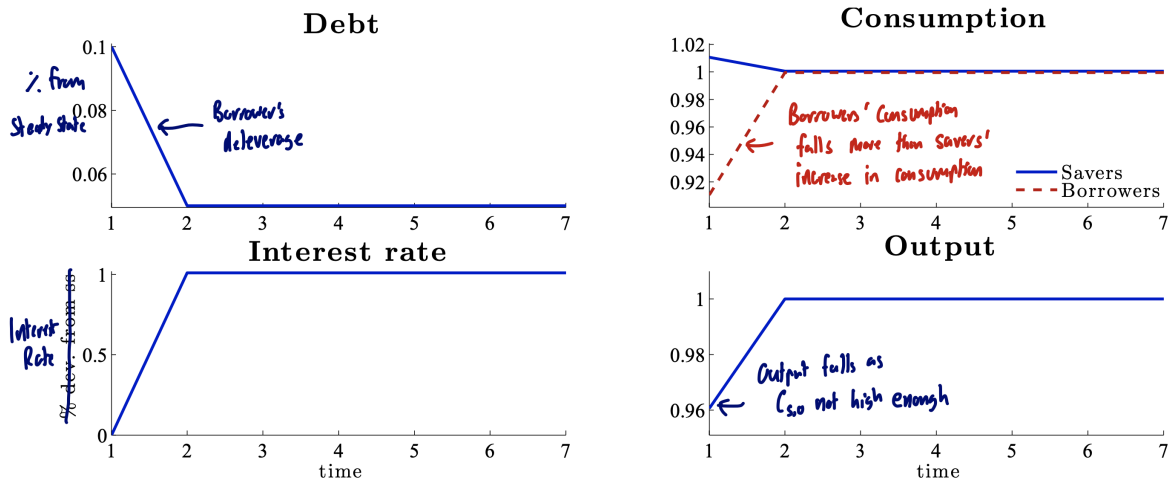


Figure 55: A deleveraging shock when the zero lower bound binds. Output falls and unemployment rises.

Falling prices can amplify the contraction through **debt deflation**. To isolate this channel, extend the preceding real-debt benchmark by letting $\mathcal{D}^h$ denote the inherited nominal repayment obligation and allowing the period-0 price

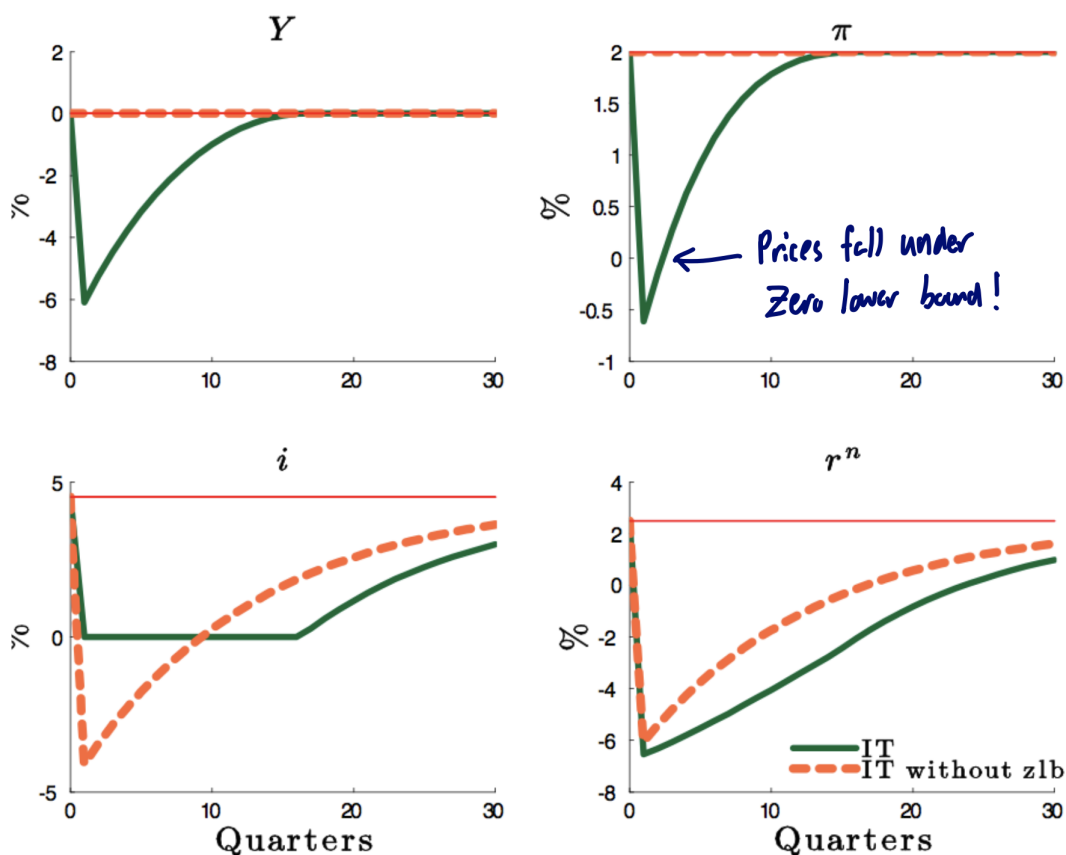


Figure 56: Deleveraging shock occurs and the zero lower bound for interest rates binds. Prices fall when the zero lower bound binds.

level to adjust. This extension does not simultaneously impose the earlier normalisation $P_0 = W_0 = 1$; with linear production and flexible product prices, $W_0 = P_0$ instead moves with the price level.

Proposition 611 (Debt Deflation)

Define P_0 as the price level in period 0. Then employment is:

$$L_0 = \frac{C_s}{\beta_s} - \frac{1-n}{n} \left[\frac{\mathcal{D}^h}{P_0} - D^l \right]. \quad (713)$$

During a liquidity trap, weaker aggregate demand can lower P_0 . This raises the real value of inherited nominal debt, $\mathcal{D}^h/P_0$, and reduces employment L_0 . The resulting decline in demand can put further downward pressure on prices. This amplification mechanism is known as **debt deflation**.

27.1.3 Policy Options

There are 6 policy options to remedy this.

1. Raise the inflation target.
2. Increase government spending financed through future taxes or taxes on savers.
3. Carry out debt relief for borrowers.

4. Use quantitative easing and distribute seigniorage revenue to borrowers.
5. Direct lending to borrowers.
6. Sustain high asset prices.

27.2 Secular Stagnation - Benigno and Fornaro (2018)

We now model the phenomenon known as **secular stagnation**. Benigno and Fornaro (2018) provide the following definition.

Definition 612 (Secular Stagnation)

Slumps characterised by high unemployment and weak growth are referred to as **secular stagnation**.

Secular stagnation is also closely tied to what they called a stagnation trap.

Definition 613 (Stagnation Trap)

A prolonged period of low growth, low interest rates, and high unemployment is called a **stagnation trap**.

The modelling approach combines a New Keynesian model with an endogenous-growth model. It has two components:

1. A New Keynesian model whereby weak aggregate demand can generate unemployment.
2. A growth model whereby profit-maximizing firms invest in technology and lead to growth.

By combining these two models, they create a two-way interaction between aggregate demand and growth.

1. A slowdown in growth reduces wealth and reduces aggregate demand.
2. A reduction in aggregate demand reduces firms' profits. Therefore, firms reduce their investment into technology. This reduces growth.

27.2.1 Household Problem

We first model the household problem. We need two assumptions.

Assumption (Household Assumptions)

There is a unit continuum of identical households that supply labour and consume goods. Households have perfect foresight.

Assumption (Household Labour Supply)

Households have no disutility from labour. The maximum amount of labour a household can supply is $\bar{L} = 1$.

The household optimisation problem is quite standard.

Definition 614 (Household Problem)

A representative household chooses sequences of consumption and bond holdings $\{C_t, b_{t+1}\}_{t=0}^{\infty}$ to maximise utility subject to a budget constraint:

$$\begin{aligned} \max_{\{C_t, b_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma} - 1}{1-\sigma} \right] \\ \text{s.t.} \quad & P_t C_t + \frac{b_{t+1}}{1+i_t} = W_t L_t + b_t + d_t \end{aligned} \quad (714)$$

where P_t is the goods price, b_{t+1} is the face value of a nominal bond purchased at price $1/(1+i_t)$, W_t is the wage, L_t is labour supplied, and d_t denotes firm dividends.

We can now solve this for the intertemporal Euler equation.

Theorem 615 (Intertemporal Euler Equation)

The intertemporal Euler consumption equation is:

$$C_t^{-\sigma} = \beta(1+i_t) \frac{C_{t+1}^{-\sigma}}{\pi_{t+1}} \quad (715)$$

where $\pi_{t+1} \equiv \frac{P_{t+1}}{P_t}$ is inflation.

Proof. First we form the current value Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma} - 1}{1-\sigma} + \lambda_t \left(W_t L_t + b_t + d_t - P_t C_t - \frac{b_{t+1}}{1+i_t} \right) \right].$$

Then, we take FOCs with respect to c_t and b_{t+1} :

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C_t} : \beta^t C_t^{-\sigma} - \beta^t \lambda_t P_t &= 0 \\ \frac{\partial \mathcal{L}}{\partial b_{t+1}} : \beta^{t+1} \lambda_{t+1} - \beta^t \frac{\lambda_t}{1+i_t} &= 0. \end{aligned}$$

Re-arrange to get:

$$\begin{aligned} \frac{C_t^{-\sigma}}{P_t} &= \lambda_t \\ \beta \lambda_{t+1} &= \frac{\lambda_t}{1+i_t}. \end{aligned}$$

Plug the consumption FOC into the bond FOC:

$$\beta \left(\frac{C_{t+1}^{-\sigma}}{P_{t+1}} \right) = \frac{\left(\frac{C_t^{-\sigma}}{P_t} \right)}{1+i_t}$$

Then you can re-arrange to get:

$$C_t^{-\sigma} = \beta(1+i_t) \frac{P_t}{P_{t+1}} C_{t+1}^{-\sigma}$$

and therefore, defining inflation as $\pi_{t+1} \equiv P_{t+1}/P_t$, we obtain

$$C_t^{-\sigma} = \beta(1+i_t) \frac{C_{t+1}^{-\sigma}}{\pi_{t+1}}$$

Let us interpret the intertemporal Euler equation for consumption:

$$C_t^{-\sigma} = \beta(1 + i_t) \frac{C_{t+1}^{-\sigma}}{\pi_{t+1}}.$$

1. Demand for consumption C_t is high if future consumption is high C_{t+1} .
2. Demand for consumption C_t is high if inflation π_{t+1} is high as households want to consume now when goods are still relatively cheap.
3. Demand for consumption C_t is high if β is low so that the agent is impatient.
4. Demand for consumption C_t is high if the return on savings $(1 + i_t)$ is low.

27.2.2 Firm Problem

We now look at the firm side. We assume that for the firm's production function, output is linear in labour only.

Assumption 616 (Production Function)

The firm's output Y_t is given by:

$$Y_t = A_t L_t \quad (716)$$

where A_t is technology and L_t is labour.

We now define the firm's problem.

Definition 617 (Firm's Problem)

A representative firm chooses labour L_t to maximise period-by-period profit subject to its production function:

$$\begin{aligned} \max_{L_t} \quad & P_t Y_t - W_t L_t \\ \text{s.t.} \quad & Y_t = A_t L_t \end{aligned} \quad (717)$$

where W_t is aggregate wage.

Remark 618. We have implicitly assumed no price rigidities, therefore firms can reset their price every period.

We can then use the firm's first order condition to derive the aggregate price level.

Proposition 619 (Aggregate Price Level)

The aggregate price level is given by:

$$P_t = \frac{W_t}{A_t} \quad (718)$$

where W_t is aggregate wage and A_t is level of technology.

Proof. Substitute the firm's production function into their profit function:

$$\begin{aligned} & P_t Y_t - W_t L_t \\ & \leftrightarrow P_t (A_t L_t) - W_t L_t \end{aligned}$$

Then, we take FOC with respect to labour to get:

$$\frac{d}{dL} : P_t A_t - W_t = 0.$$

Therefore, we conclude that:

$$P_t = \frac{W_t}{A_t}.$$

□

Now, as we want to use an endogenous growth model, we need to have a notion of productivity/technology growth in this economy.

Definition 620 (Productivity Growth)

Productivity growth is defined as:

$$g_{t+1} \equiv \frac{A_{t+1}}{A_t} \quad (719)$$

where A_t is the level of technology at time t .

We now want to impose some nominal rigidities in the model. This is so that labour markets cannot clear and we can generate unemployment in the model (otherwise wages can adjust and all workers can be hired at a lower wage).

Assumption 621 (Nominal Wage Rigidities)

We assume that nominal wages follow the exogenous growth rule

$$W_t = \pi^w W_{t-1} \quad (720)$$

where π^w is the growth rate of the wage.

Remark 622. *It is rigid even though it is growing because we are saying that the wage isn't adjusting to market conditions, but growing exogenously by itself.*

Now, we can define the level of inflation in this model by relating the growth rate of wages and the growth in technology.

Proposition 623 (Price Inflation)

Price inflation is given by:

$$\pi_t = \frac{\pi^w}{g_t} \quad (721)$$

where π^w is the wage growth rate and g_t is the productivity growth rate.

Proof. First, recall that aggregate price level is given by:

$$P_t = \frac{W_t}{A_t}.$$

Now, plug this into the definition of inflation:

$$\begin{aligned}
 \pi_t &= \frac{P_t}{P_{t-1}} \\
 &= \frac{\frac{W_t}{A_t}}{\frac{W_{t-1}}{A_{t-1}}} \\
 &= \frac{W_t}{W_{t-1}} \cdot \frac{A_{t-1}}{A_t} \\
 &= \frac{\pi^w}{g_t}
 \end{aligned}$$

where we used the definitions:

$$\begin{aligned}
 \pi^w &= \frac{W_t}{W_{t-1}} \\
 g_t &= \frac{A_t}{A_{t-1}}.
 \end{aligned}$$

□

Let us interpret the price inflation:

$$\pi_t = \frac{\pi^w}{g_t}.$$

Inflation rises when wage growth increases: higher labour costs translate into higher prices. Inflation falls when productivity growth increases, because each unit of labour produces more output.

We now want to define the natural level of output and the output gap as this will allow us to characterise equilibrium.

Definition 624 (Natural Level of Output)

The natural level of output Y_t^n is the level produced without nominal rigidities when workers supply their full labour endowment $\bar{L} = 1$:

$$Y_t^n = A_t \tag{722}$$

where A_t is the level of technology.

Remark 625. *The natural rate of output is the hypothetical level of output if we had flexible prices and workers are able to work full hours.*

Proof. Under no nominal rigidities and employees working their full amount of hours, we have that $L_t = \bar{L} = 1$. Therefore, we can use the production function to compute the natural rate of output as:

$$Y_t^n = L_t A_t = (1)A_t = A_t.$$

□

The natural rate of output is the ideal level of output. We can compute the output gap which measures the deviation of current output from this ideal level.

Definition 626 (Relative Output and the Output Gap)

Relative output is

$$\frac{Y_t}{Y_t^n} = L_t \quad (723)$$

where L_t is employment. The proportional output gap is therefore $Y_t/Y_t^n - 1 = L_t - 1$.

Proof. Use the definition of the output gap and production function of Y_t :

$$\frac{Y_t}{Y_t^n} = \frac{A_t L_t}{A_t} = L_t.$$

□

We can see that there are two possible scenarios for the economy under this output gap interpretation.

1. If $L_t = 1$, then the economy is at full employment and we have zero output gap.
2. If $L_t < 1$, then the economy has a negative output gap.

We now define the monetary policy rule that the central bank sets in this model.

Definition 627 (Monetary Policy Equation)

The monetary policy equation that the central bank sets is given by:

$$1 + i_t = \max \left\{ (1 + \bar{i}) L_t^\phi, 1 \right\} \quad (724)$$

where L_t is the level of employment, ϕ is a parameter, and 1 represents the zero lower bound.

To interpret this monetary policy equation, it says that the central bank will lower interest rates the higher the level of unemployment (low L_t). However, eventually the central bank will reach the zero lower bound where $i_t = 0$ or $(1 + i_t = 1)$. Therefore, the monetary policy is constrained by the zero lower bound $i \geq 0$.

27.2.3 Equilibrium of the Model

We now develop our three equation New Keynesian model under this setting. First, we need an assumption.

Assumption (Closed Economy and Market Clearing)

The economy is closed and the goods market clears:

$$C_t = Y_t = A_t L_t. \quad (725)$$

We first develop the IS equation.

Definition 628 (IS Equation)

The IS equation is given by:

$$L_t^\sigma = \frac{\pi_{t+1} g_{t+1}^\sigma}{\beta(1 + i_t)} L_{t+1}^\sigma \quad (726)$$

where π_{t+1} is inflation, L_t is the level of employment, g_{t+1} is the growth rate of technology, and i_t is the interest rate on bonds.

Proof. Use the Euler equation for consumption:

$$C_t^{-\sigma} = \beta(1 + i_t) \frac{C_{t+1}^{-\sigma}}{\pi_{t+1}}.$$

We have that $C_t = Y_t = A_t L_t$ by the fact that we have a closed economy. Substitute this in:

$$[A_t L_t]^{-\sigma} = \beta(1 + i_t) \frac{[A_{t+1} L_{t+1}]^{-\sigma}}{\pi_{t+1}}.$$

This then becomes:

$$[A_t L_t]^\sigma = \frac{\pi_{t+1}}{\beta(1 + i_t)} [A_{t+1} L_{t+1}]^\sigma.$$

Then we have that:

$$L_t^\sigma = \frac{\pi_{t+1}}{\beta(1 + i_t)} \frac{A_{t+1}^\sigma}{A_t^\sigma} L_{t+1}^\sigma$$

Use the definition that $g_t = \frac{A_t}{A_{t-1}}$ and conclude that:

$$L_t^\sigma = \frac{\pi_{t+1} g_{t+1}^\sigma}{\beta(1 + i_t)} L_{t+1}^\sigma.$$

□

We can now interpret the IS equation:

$$L_t^\sigma = \frac{\pi_{t+1} g_{t+1}^\sigma}{\beta(1 + i_t)} L_{t+1}^\sigma.$$

If households expect low future technology growth g_{t+1} , they reduce current consumption and aggregate demand. Output then falls because $C_t = Y_t$, and firms hire less labour because $Y_t = A_t L_t$. Hence, expectations of lower productivity growth depress aggregate demand and employment L_t .

We now move onto the second equation we need for our 3-equation NK model: the aggregate demand equation.

Definition 629 (Aggregate Demand Equation)

The aggregate demand equation is given by:

$$L_t^\sigma = \frac{\pi^w g_{t+1}^{\sigma-1}}{\beta \max \left\{ (1 + \bar{i}) L_t^\phi, 1 \right\}} L_{t+1}^\sigma \quad (727)$$

where π^w is the growth rate of wages, g_{t+1} is productivity growth and L_t is the level of employment.

Proof. Combine the IS equation:

$$L_t^\sigma = \frac{\pi_{t+1} g_{t+1}^\sigma}{\beta(1 + i_t)} L_{t+1}^\sigma$$

with the monetary policy equation:

$$1 + i_t = \max \left\{ (1 + \bar{i}) L_t^\phi, 1 \right\}$$

to get:

$$L_t^\sigma = \frac{\pi_{t+1} g_{t+1}^\sigma}{\beta \left[\max \left\{ (1 + \bar{i}) L_t^\phi, 1 \right\} \right]} L_{t+1}^\sigma.$$

Now plug in the rate of inflation:

$$\pi_{t+1} = \frac{\pi^w}{g_{t+1}}$$

to get:

$$L_t^\sigma = \frac{\left(\frac{\pi^w}{g_{t+1}}\right) g_{t+1}^\sigma}{\beta \left[\max \left\{ (1 + \bar{i}) L_t^\phi, 1 \right\} \right]} L_{t+1}^\sigma.$$

and then simplify to get:

$$L_t^\sigma = \frac{\pi^w g_{t+1}^{\sigma-1}}{\beta \max \left\{ (1 + \bar{i}) L_t^\phi, 1 \right\}} L_{t+1}^\sigma.$$

□

The third equation of the 3-equation NK model is the monetary policy rule, which we state again.

Definition 630 (Monetary Policy Equation)

The monetary policy equation that the central bank sets is given by:

$$1 + i_t = \max \left\{ (1 + \bar{i}) L_t^\phi, 1 \right\} \quad (728)$$

where L_t is the level of employment, ϕ is a parameter, and 1 represents the zero lower bound.

27.2.4 Steady State of the Model

We now want to derive the steady state level of our model. First, we can derive the steady state growth rate.

Proposition 631 (Steady State Growth)

The steady state growth rate is:

$$g^{\sigma-1} = \frac{\beta \max \left\{ (1 + \bar{i}) L^\phi, 1 \right\}}{\pi^w} \quad (729)$$

where π^w is the growth rate of wages and L is the level of employment.

Proof. First, recall the aggregate demand equation:

$$L_t^\sigma = \frac{\pi^w g_{t+1}^{\sigma-1}}{\beta \max \left\{ (1 + \bar{i}) L_t^\phi, 1 \right\}} L_{t+1}^\sigma.$$

Under steady state, $L_t = L_{t+1} = L$ and $g_{t+1} = g$. We plug this into our aggregate demand equation:

$$L^\sigma = \frac{\pi^w g^{\sigma-1}}{\beta \max \left\{ (1 + \bar{i}) L^\phi, 1 \right\}} L^\sigma.$$

Now, we re-arrange:

$$\begin{aligned}
 L^\sigma &= \frac{\pi^w g^{\sigma-1}}{\beta \max \left\{ (1 + \bar{i}) L^\phi, 1 \right\}} L^\sigma \\
 \Leftrightarrow 1 &= \frac{\pi^w g^{\sigma-1}}{\beta \max \left\{ (1 + \bar{i}) L^\phi, 1 \right\}} \cdot 1 \\
 \Rightarrow g^{\sigma-1} &= \frac{\beta \max \left\{ (1 + \bar{i}) L^\phi, 1 \right\}}{\pi^w}
 \end{aligned}$$

□

We now need to take a stand on what should the value of σ (which is the coefficient of relative risk aversion in our CRRA utility function).

Assumption (Coefficient of Relative Risk Aversion)

The coefficient of relative risk aversion σ is:

$$\sigma > 1.$$

Remark 632. *This says that we have low levels of intertemporal substitution and that the wealth effect dominates the substitution effect.*

Now, let us interpret this steady state growth rate:

$$g^{\sigma-1} = \frac{\beta \max \left\{ (1 + \bar{i}) L^\phi, 1 \right\}}{\pi^w}.$$

There are two competing effects. On the one hand, faster productivity growth g is associated with higher future wealth. This wealth effect leads households to increase their demand for current consumption in response to a rise in productivity growth. On the other hand, faster productivity growth is associated with a fall in expected inflation. Given this, the lower expected inflation increases the real interest rate inducing households to postpone consumption. This substitution effect points towards a negative relationship between productivity growth and current demand for consumption. For low levels of intertemporal substitution, i.e. for $\sigma > 1$, the wealth effect dominates and the relationship between productivity growth and demand for consumption is positive. Instead, for high levels of intertemporal substitution, i.e. for $\sigma < 1$, the substitution effect dominates and the relationship between productivity growth and demand for consumption is negative. Finally, for the special case of log utility, $\sigma = 1$, the two effects cancel out and productivity growth does not affect present demand for consumption.

Assumption (Exogenous Productivity Growth Rate)

We assume that productivity growth rate is exogenous:

$$g = g^f. \tag{730}$$

We now show the first steady state that is a *good* steady state.

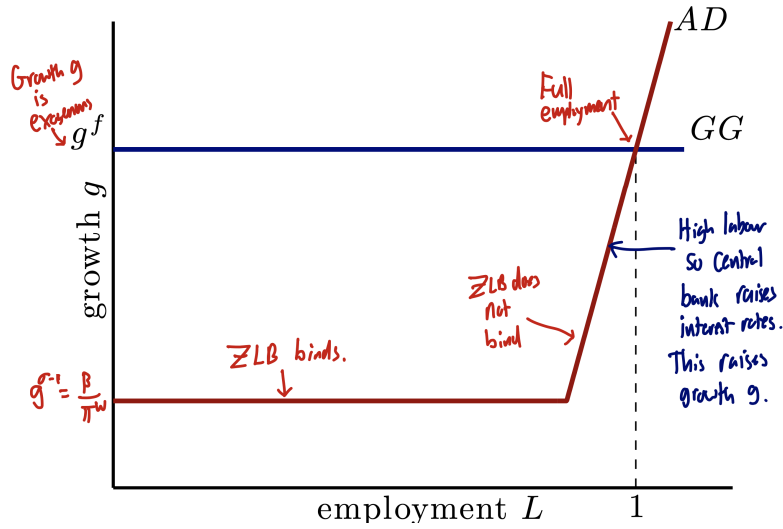


Figure 57: Aggregate demand and growth as a function of employment L .

Now, in Fig. 57, we plot both the exogenous growth rate $g = g^f$ and the steady state growth (aggregate demand). Note that when the zero lower bound binds, the growth rate is:

$$g^{\sigma-1} = \frac{\beta}{\pi^w}.$$

As the level of employment increases, the central bank raises interest rates by the monetary policy rule:

$$1 + i_t = \max \left\{ (1 + \bar{i}) L_t^\phi, 1 \right\}.$$

This then causes the steady state growth rate to increase:

$$g^{\sigma-1} = \frac{\beta \max \left\{ (1 + \bar{i}) L^\phi, 1 \right\}}{\pi^w}.$$

In the depicted calibration, the AD and GG schedules intersect at a **full-employment steady state**. It has positive growth ($g^f > 1$), a positive nominal interest rate, and full employment ($L = 1$). For $L = 1$ to be a steady state when policy is unconstrained, the parameters must satisfy $\pi^w (g^f)^{\sigma-1} = \beta(1 + \bar{i})$. The economy then operates at potential and monetary policy is not constrained by the ZLB.

We now make productivity growth endogenous.

Assumption (Endogenous Productivity Growth)

Assume that productivity growth satisfies

$$g = f(L), \quad f'(L) > 0,$$

so stronger activity and employment encourage innovation.

Remark 633. Recall that $C_t = Y_t = A_t L_t$ by market clearing. Higher demand raises output and employment. The assumption $g = f(L)$ captures the idea that stronger activity and profits encourage firms to invest more in innovation; it does not assert that growth is numerically equal to consumption.

We now give the intuition behind this endogenous productivity growth $g = f(L)$:

1. A decrease in demand leads to a fall in profits,
2. This leads firms to invest less in technology.
3. This leads to less productivity growth.
4. Then, as we argued before, aggregate demand falls in anticipation of lower productivity growth as $\sigma > 1$.

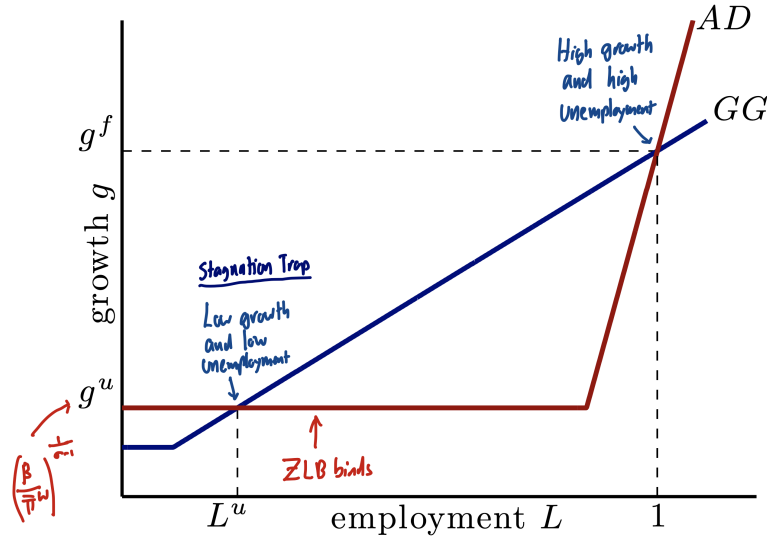


Figure 58: One bad steady state and one good steady state.

Under the shapes and parameter restrictions depicted in Fig. 58, the model can have two steady states: the full-employment steady state and a bad steady state known as a **stagnation trap**. The bad steady state is characterised by:

1. Unemployment is now present $L^u < 1 = \bar{L}$.
2. Growth is weak as it is below the full employment growth:

$$g^u = \left(\frac{\beta}{\pi^w} \right)^{\frac{1}{\sigma-1}} < g^f.$$

3. Monetary policy is constrained by the zero lower bound:

$$i^u = 0.$$

Note that this weak growth rate comes from the steady state growth rate expression:

$$g^{\sigma-1} = \frac{\beta \max \left\{ (1 + \bar{i}) L^\phi, 1 \right\}}{\pi^w}.$$

To conclude, we talk about policy implications that can be carried out.

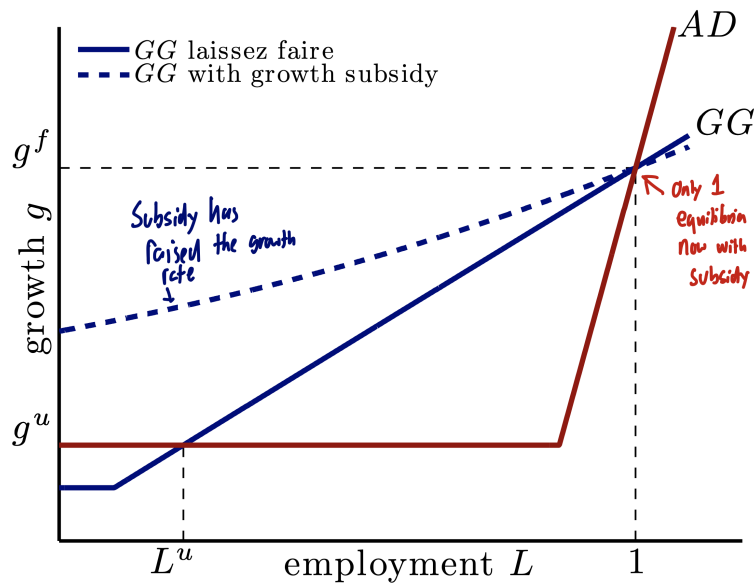


Figure 59: Only good steady state with countercyclical subsidy.

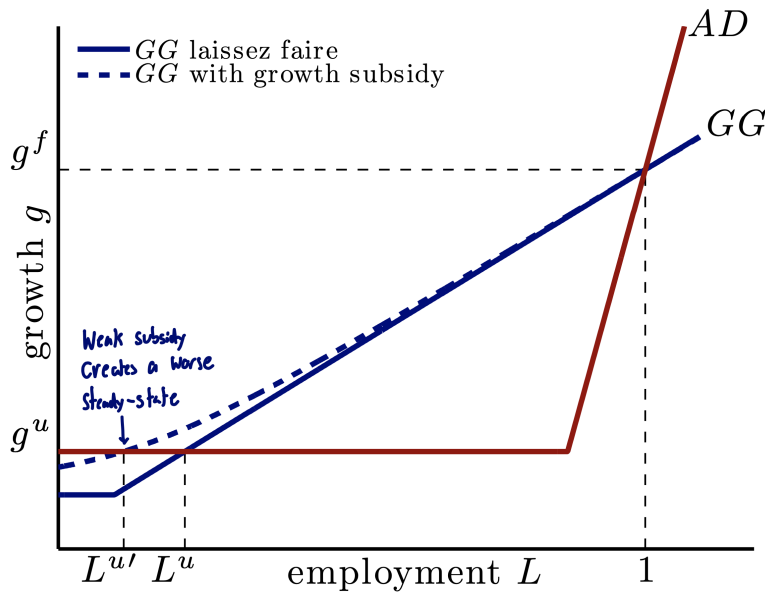


Figure 60: Two steady states again with bad steady state even worse.

Definition 634 (Countercyclical Subsidy to Innovation)

A countercyclical subsidy to innovation is:

$$g = f(L) + s(g) \quad (731)$$

where the subsidy is a decreasing function of growth $s' < 0$.

Remark 635. Note that as $s(g)$ is a decreasing function, that means we have higher subsidies when growth is low.

We see in Fig. 59 that if we have strong countercyclical subsidies, we raise the growth sufficiently enough so that we no longer have the bad steady state. Benigno and Fornaro (2018) described the intuition as follows: *Intuitively, the*

subsidy to innovation guarantees that even if firms' profits were to fall substantially, investment in innovation would still be relatively high. In turn, a high investment in innovation stimulates growth and aggregate demand, since a high future income is associated with a high present demand for consumption. By implementing a sufficiently high subsidy, the government can eliminate the possibility that aggregate demand will be low enough to make the zero lower bound constraint on the nominal interest rate bind. It is in this sense that growth policies can be thought as a tool to manage aggregate demand in our framework. Importantly, to lift the economy out of a stagnation trap the subsidy to investment has to be large enough, so as to engineer a radical regime shift in agents' expectations about future growth.

Now, something to beware of is that if we don't have subsidies that are strong enough, we can in fact make the stagnation trap worse. A small subsidy makes the GG curve rotate up, but not enough to eliminate the unemployment steady state. In fact, the small subsidy leads to lower employment in the unemployment steady state compared to the laissez faire economy. We see in Fig. 60 that if we have weak countercyclical subsidies, we create a bad steady equilibrium that is even worse than the one we saw in Fig. 58.

To conclude, we have derived a Keynesian growth model in which endogenous growth interacts with slumps driven by weak aggregate demand. Sufficiently strong supply-side policies can stimulate demand and move the economy out of stagnation.

27.3 COVID-19 Crisis - Guerrieri et al. (2022)

Guerrieri et al. (2022) model the COVID-19 pandemic and its policy implications. Rather than treating the pandemic as exclusively a supply or demand shock, the paper studies how an initial supply disruption can generate an endogenous shortfall in demand.

The model has two important ingredients. First, it contains multiple sectors because high-contact activities, such as hotels and restaurants, were affected more severely than low-contact activities. Complementarity across sectoral goods determines how a disruption in one sector propagates to the other. Second, markets may be incomplete, so agents directly affected by the shock cannot borrow enough to smooth consumption.

Assumption (Two Key Ingredients of the Model)

The two key ingredients to the Guerrieri et al. (2022) paper are:

1. Complementarities across multiple sectors.
2. Incomplete markets.

We now examine how shocks propagate through the economy, focusing on a special class of disturbances.

Definition 636 (Keynesian Supply Shocks)

A Keynesian supply shock is a negative supply shock that causes demand shortages.

A negative supply shock reduces the economy's **efficient level of production**; it need not be tied to one particular primitive, such as productivity or preferences. A demand shortage occurs when actual production and expenditure fall by **more** than the efficient response to the initial supply shock.

Supply shocks are modelled as reductions in the number of agents who can work. Demand deficiency arises when the flexible-price, full-employment natural rate falls but monetary policy cannot lower the actual real rate enough—for example, because the ZLB binds.

There are 4 types of models we want to look at.
 A diagram of the model can be seen in Fig. 61.

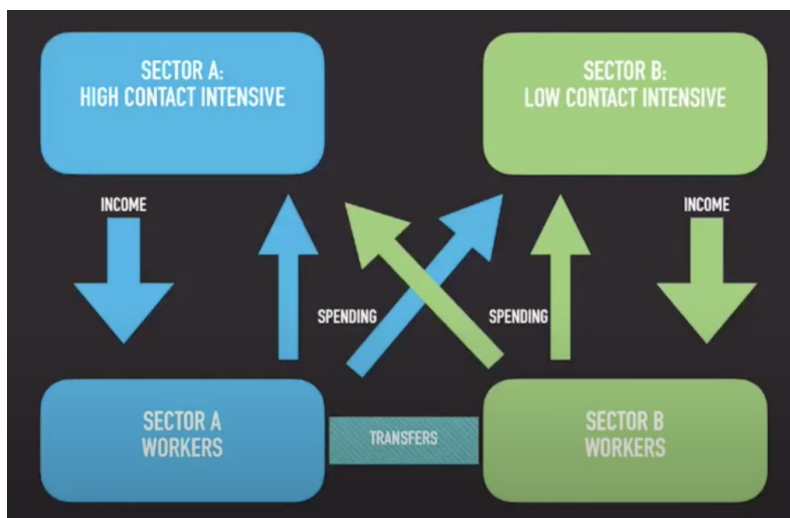


Figure 61: Two sectors with two types of workers. Sector-A workers receive sector-A income and consume goods from both sectors; the analogous pattern holds for sector-B workers. Under complete markets, the two groups can insure and borrow or lend to one another.

27.3.1 Symmetric Shock and Complete Markets

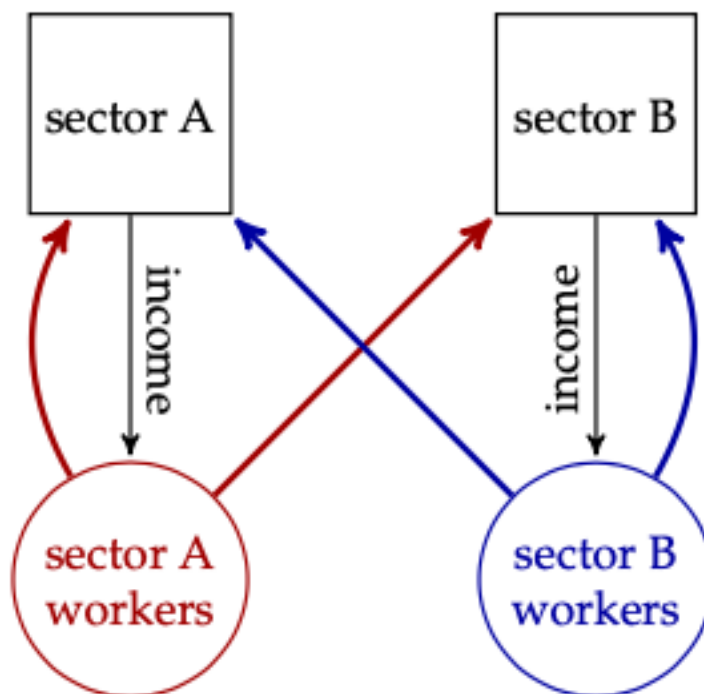


Figure 62: Two sectors with two types of workers. Sector-A workers receive sector-A income and consume goods from both sectors; the analogous pattern holds for sector-B workers. Under complete markets, the two groups can insure and borrow or lend to one another.

We now lay out the assumptions for the first model we will look at.

Assumption (One-Sector Complete-Markets Model)

There is one goods sector and one type of worker in a deterministic infinite-horizon economy with complete markets. Labour supply is inelastic and bounded by $\bar{n}_t$ units.

Under these assumptions, there is **no Keynesian supply shock**.

Definition 637 (Household Problem)

A representative household chooses a sequence of consumption and nominal assets $\{C_t, a_{t+1}\}_{t=0}^{\infty}$ to maximise utility subject to a budget constraint:

$$\begin{aligned} \max_{\{C_t, a_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t \frac{C_t^{\frac{\sigma-1}{\sigma}}}{1 - 1/\sigma} \\ \text{s.t.} \quad & P_t C_t + a_{t+1} \leq W_t n_t + (1 + i_{t-1}) a_t \end{aligned} \quad (732)$$

Proposition 638 (Euler equation for Consumption)

The Euler equation for consumption is given by:

$$1 + r_t = \frac{1}{\beta} \left(\frac{C_{t+1}}{C_t} \right)^{1/\sigma} \quad (733)$$

where $(1 + r_t) = \frac{(1+i_t)P_t}{P_{t+1}}$ is the real interest rate.

Proof. Set up the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{\frac{\sigma-1}{\sigma}}}{1 - 1/\sigma} + \lambda_t \left[W_t n_t + (1 + i_{t-1}) a_t - P_t C_t - a_{t+1} \right] \right]$$

Take first order conditions:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial a_{t+1}} : \beta^{t+1} \lambda_{t+1} (1 + i_t) - \beta^t \lambda_t &= 0 \\ \frac{\partial \mathcal{L}}{\partial C_t} : \beta^t C_t^{-\frac{1}{\sigma}} - \beta^t \lambda_t P_t &= 0 \end{aligned}$$

If we take the second FOC and re-arrange:

$$\beta^t C_t^{-\frac{1}{\sigma}} - \beta^t \lambda_t P_t = 0 \quad \Leftrightarrow \quad \frac{C_t^{-\frac{1}{\sigma}}}{P_t} = \lambda_t$$

we can then plug this into the first FOC:

$$\begin{aligned}
& \beta^{t+1} \lambda_{t+1} (1 + i_t) - \beta^t \lambda_t = 0 \\
& \Leftrightarrow \beta^{t+1} \frac{C_{t+1}^{-\frac{1}{\sigma}}}{P_{t+1}} (1 + i_t) - \beta^t \frac{C_t^{-\frac{1}{\sigma}}}{P_t} = 0 \\
& \Leftrightarrow \beta^{t+1} \frac{C_{t+1}^{-\frac{1}{\sigma}}}{P_{t+1}} (1 + i_t) = \beta^t \frac{C_t^{-\frac{1}{\sigma}}}{P_t} \\
& \Leftrightarrow \beta \frac{C_{t+1}^{-\frac{1}{\sigma}}}{P_{t+1}} (1 + i_t) = \frac{C_t^{-\frac{1}{\sigma}}}{P_t} \\
& \Leftrightarrow \frac{C_{t+1}^{-\frac{1}{\sigma}}}{P_{t+1}} (1 + i_t) = \frac{1}{\beta} \frac{C_t^{-\frac{1}{\sigma}}}{P_t} \\
& \Leftrightarrow \frac{P_t}{P_{t+1}} (1 + i_t) = \frac{1}{\beta} \frac{C_t^{-\frac{1}{\sigma}}}{C_{t+1}^{-\frac{1}{\sigma}}} \\
& \Leftrightarrow \frac{P_t}{P_{t+1}} (1 + i_t) = \frac{1}{\beta} \left(\frac{C_{t+1}}{C_t} \right)^{1/\sigma} \\
& \Leftrightarrow 1 + r_t = \frac{1}{\beta} \left(\frac{C_{t+1}}{C_t} \right)^{1/\sigma}
\end{aligned}$$

where we defined the real interest rate to be:

$$1 + r_t \equiv \frac{P_t}{P_{t+1}} (1 + i_t).$$

□

Proposition 639

With a single good (or perfectly substitutable goods), a temporary negative supply shock raises the natural interest rate. If the actual real rate is held fixed, the shock generates excess demand in both the complete- and incomplete-markets benchmarks.

Remark 640. *Intuitively, a temporary negative supply shock makes future income high relative to current income. Households therefore want to borrow today and repay tomorrow. Under incomplete markets they may be unable to borrow, but they still do not respond by increasing saving.*

If the central bank keeps the actual interest rate unchanged while the natural rate rises, desired consumption exceeds efficient production, generating excess demand.

Relative to an unchanged actual real rate, a rise in the natural rate produces excess demand, while a fall in the natural rate produces deficient demand.

Definition 641 (Firm's Problem)

A representative competitive firm chooses labour N_t and output Y_t to maximise period profit:

$$\begin{aligned} \max_{Y_t, N_t} \quad & P_t Y_t - W_t N_t \\ \text{s.t.} \quad & Y_t = N_t. \end{aligned} \quad (734)$$

For an interior allocation with positive production, the firm's condition is $P_t = W_t$.

Downward nominal wage rigidity is an equilibrium restriction, not a constraint chosen by the firm. Employment cannot exceed labour supply, and the wage cannot fall:

$$\begin{aligned} W_t &\geq W_{t-1}, \\ N_t &\leq \bar{n}_t, \\ (W_t - W_{t-1})(\bar{n}_t - N_t) &= 0. \end{aligned}$$

Thus a rising wage is compatible with full employment, while unemployment implies that the downward wage constraint binds.

Proposition 642 (Equilibrium)

In equilibrium, we have that:

1. Production is equal to consumption:

$$C_t = Y_t \quad (735)$$

2. The wage–employment complementarity condition holds:

$$(W_t - W_{t-1})(\bar{n}_t - N_t) = 0, \quad (736)$$

together with $W_t \geq W_{t-1}$ and $N_t \leq \bar{n}_t$.

3. Borrowing market clears:

$$a_{t+1} = 0. \quad (737)$$

Theorem 643 (Natural Rate under a Temporary Supply Shock)

Suppose labour supply falls temporarily at $t = 0$:

$$\bar{n}_0 = (1 - \phi)\bar{n}, \quad 0 < \phi < 1, \quad (738)$$

and returns to normal thereafter:

$$\bar{n}_t = \bar{n}, \quad t \geq 1. \quad (739)$$

Then the period-0 natural real rate exceeds its steady-state value:

$$(1 + r_0) > (1 + r). \quad (740)$$

Proof.

$$\begin{aligned}
 (1 + r_0) &= \frac{1}{\beta} \left(\frac{C_1}{C_0} \right)^{1/\sigma} \\
 &= \frac{1}{\beta} \left(\frac{\bar{n}}{(1 - \phi)\bar{n}} \right)^{1/\sigma} \\
 &= \frac{1}{\beta} \left(\frac{1}{(1 - \phi)} \right)^{1/\sigma} \\
 &> \frac{1}{\beta} = (1 + r).
 \end{aligned}$$

□

Therefore, at time $t = 0$, the central bank needs to raise interest rates from the steady state level $(1 + r)$ to the higher level at $(1 + r_0)$.

If the central bank does not raise the real rate, consumption demand exceeds the reduced efficient supply. The economy therefore experiences **excess demand**, even though the level of output may still be below its pre-shock level.

Remark 644. *A temporary negative supply shock is equivalent to good news about consumption opportunities in the future relative to today. The natural rate rises as households try to borrow against that future recovery.*

Even with excess demand, output can remain below its pre-shock level because productive capacity has fallen; the unconstrained part of the economy is operating at capacity.

Proposition 645

In a single-sector economy with complete markets, a temporary negative supply shock raises the natural interest rate and, if the policy rate is unchanged, generates excess demand.

27.3.2 Symmetric Shock and Incomplete Markets

We now consider a one-good economy with incomplete markets. There is still **no Keynesian supply shock** in this benchmark.

Proposition 646

In a single-sector economy with incomplete markets, a temporary negative supply shock raises the natural interest rate and, at an unchanged policy rate, generates excess demand.

Remark 647. *The temporary supply shock is good news about future resources relative to current resources. Agents want to borrow rather than save, although borrowing constraints may prevent them from doing so.*

27.3.3 Asymmetric Shock and Complete Markets

We now introduce two sectors and a shock that shuts down sector A, eliminating consumption and labour income in that sector. With complete markets, sector-A workers can insure against the shock and continue consuming sector-B goods. The effect on demand for sector B depends on the substitutability of the two goods. If they are complements (for example, popcorn and cinema visits), shutting down sector A also lowers demand for sector B. If they are substitutes, demand shifts toward sector B.

Here, labour is immobile whereby workers in sector A cannot work in sector B and vice versa.

Assumption (Complete Markets)

Households have access to a zero net supply of one-period bonds.

Proposition 648

When we let the elasticity of substitution $\epsilon \rightarrow \infty$ then the two goods are perfect substitutes and we recover a one-sector model.

Proposition 649

In the incomplete-markets extension below, letting the fraction of constrained agents satisfy $\mu \rightarrow 0$ recovers the complete-markets model.

Assumption (MIT Shock)

Let $s_A \in (0, 1)$ be the share of workers attached to sector A. At time 0, a temporary shock prevents these workers from working, so $n_{i0} = 0$ for sector-A workers. From $t \geq 1$, the economy returns to the normal flexible-price allocation.

Assumption

Central bank keeps interest rate unchanged.

Assumption

Downward rigid nominal wages.

When we apply a MIT shock, if there is excess demand, then this is a standard supply shock. If we get insufficient demand, then this is a Keynesian supply shock.

We now consider two goods with complete markets. A Keynesian supply shock can arise when the goods are sufficiently complementary.

Assumption (Sectoral Consumption Aggregator)

Aggregate consumption is the CES composite

$$C_t = \left[\phi^{1/\epsilon} C_{At}^{(\epsilon-1)/\epsilon} + (1-\phi)^{1/\epsilon} C_{Bt}^{(\epsilon-1)/\epsilon} \right]^{\epsilon/(\epsilon-1)}, \quad (741)$$

where ϵ is the elasticity of substitution across sectoral goods. Define the expression inside square brackets as X_t .

Proposition 650 (Marginal Utility of Sector-B Consumption)

Under the CES aggregator, the marginal utility of sector-B consumption is

$$\frac{\partial U(C_t)}{\partial C_{Bt}} = C_t^{(\sigma-\epsilon)/(\sigma\epsilon)} (1-\phi)^{1/\epsilon} C_{Bt}^{-1/\epsilon}. \quad (742)$$

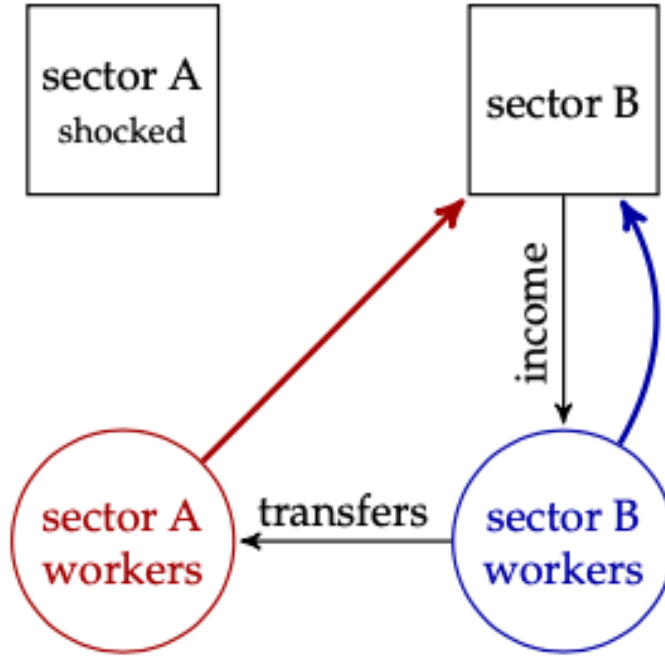


Figure 63: Two sectors with complete markets. Sector A is shut down, so its workers lose their sectoral income but can insure or borrow. During the shutdown, consumption is concentrated in sector B.

Proof.

$$\begin{aligned}
 \frac{\partial U(C_t)}{\partial C_{Bt}} &: \frac{\sigma}{\sigma-1} \left[\frac{\epsilon}{\epsilon-1} \frac{\sigma-1}{\sigma} \right] \left[X_t^{\frac{\epsilon}{\epsilon-1} \cdot \frac{\sigma-1}{\sigma} - 1} \right] \frac{\epsilon-1}{\epsilon} (1-\phi)^{\frac{1}{\epsilon}} C_{Bt}^{\frac{\epsilon-1}{\epsilon} - 1} \\
 &= \left[X_t^{\frac{\epsilon}{\epsilon-1} \cdot \frac{\sigma-1}{\sigma} - 1} \right] (1-\phi)^{\frac{1}{\epsilon}} C_{Bt}^{\frac{\epsilon-1}{\epsilon} - 1} \\
 &= \left[X_t^{\frac{\epsilon}{\epsilon-1} \cdot \frac{\sigma-1}{\sigma} - 1} \right] (1-\phi)^{\frac{1}{\epsilon}} C_{Bt}^{-\frac{1}{\epsilon}} \\
 &= \left[X_t^{\frac{\sigma-\epsilon}{\sigma(\epsilon-1)}} \right] (1-\phi)^{\frac{1}{\epsilon}} C_{Bt}^{-\frac{1}{\epsilon}}
 \end{aligned}$$

We have

$$X_t^{\frac{\epsilon}{\epsilon-1}} = C_t \quad \Rightarrow \quad X_t = C_t^{\frac{\epsilon-1}{\epsilon}}$$

We plug this in to get:

$$\begin{aligned}
 &= \left[X_t^{\frac{\sigma-\epsilon}{\sigma(\epsilon-1)}} \right] (1-\phi)^{\frac{1}{\epsilon}} C_{Bt}^{-\frac{1}{\epsilon}} \\
 &= \left[C_t^{\frac{\sigma-\epsilon}{\sigma(\epsilon-1)} \cdot \frac{\epsilon-1}{\epsilon}} \right] (1-\phi)^{\frac{1}{\epsilon}} C_{Bt}^{-\frac{1}{\epsilon}} \\
 &= \left[C_t^{\frac{\sigma-\epsilon}{\sigma\epsilon}} \right] (1-\phi)^{\frac{1}{\epsilon}} C_{Bt}^{-\frac{1}{\epsilon}}
 \end{aligned}$$

□

Theorem 651 (Condition for a Keynesian Supply Shock)

In the two-sector complete-markets model, the supply shock is Keynesian if

$$\sigma > \epsilon \quad (743)$$

The elasticity ϵ governs substitution across sectors, so a low ϵ means that the goods are complements. When sector A shuts down, demand for sector-B goods then falls as well. The parameter σ is the elasticity of intertemporal substitution. A high σ makes households more willing to postpone consumption until sector-A goods become available again, further depressing current demand for sector-B goods. The condition $\sigma > \epsilon$ captures the case in which this intertemporal response dominates within-period substitution toward sector B.

At the time of the shock, the amount of sector B goods being produced is:

$$(1 - s_A)\bar{n}.$$

27.3.4 Asymmetric Shock and Incomplete Markets

We now look at a two good sector with incomplete markets. We will see that there **will be an even stronger Keynesian supply shock** compared to the setting where we had multiple sectors with complete markets.

Under complete markets, cross-sector complementarity determines how the shutdown affects demand for good B. Incomplete markets amplify the contraction because sector-A workers cannot fully insure or borrow against their lost income. In the extreme, they have no resources with which to purchase good B. Thus, even when the goods are relatively substitutable and sector-B workers increase their own demand, total demand for good B may fall because unemployed, credit-constrained sector-A workers cut spending sharply.

The set of (ϵ, σ) combinations that generate Keynesian supply shocks therefore expands under incomplete markets.

27.4 Policy

In this stripped-down model, the government-spending multiplier is one. Spending cannot restore employment in sector A while that sector is shut down, because its workers cannot move to sector B; policy is therefore more effective when it supports demand and income in the sectors and households that remain able to transact.

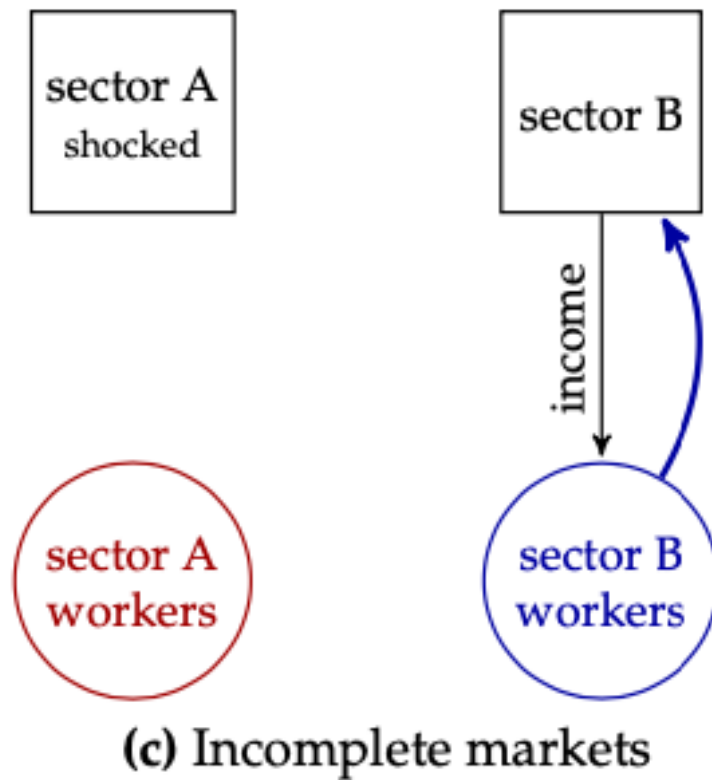


Figure 64: Two sectors with incomplete markets. Sector A is shut down, and its workers lose income and cannot borrow from sector-B workers. Sector-B workers are therefore the principal consumers of good B.